

# **CS-417 INTRODUCTION TO ROBOTICS AND INTELLIGENT SYSTEMS**

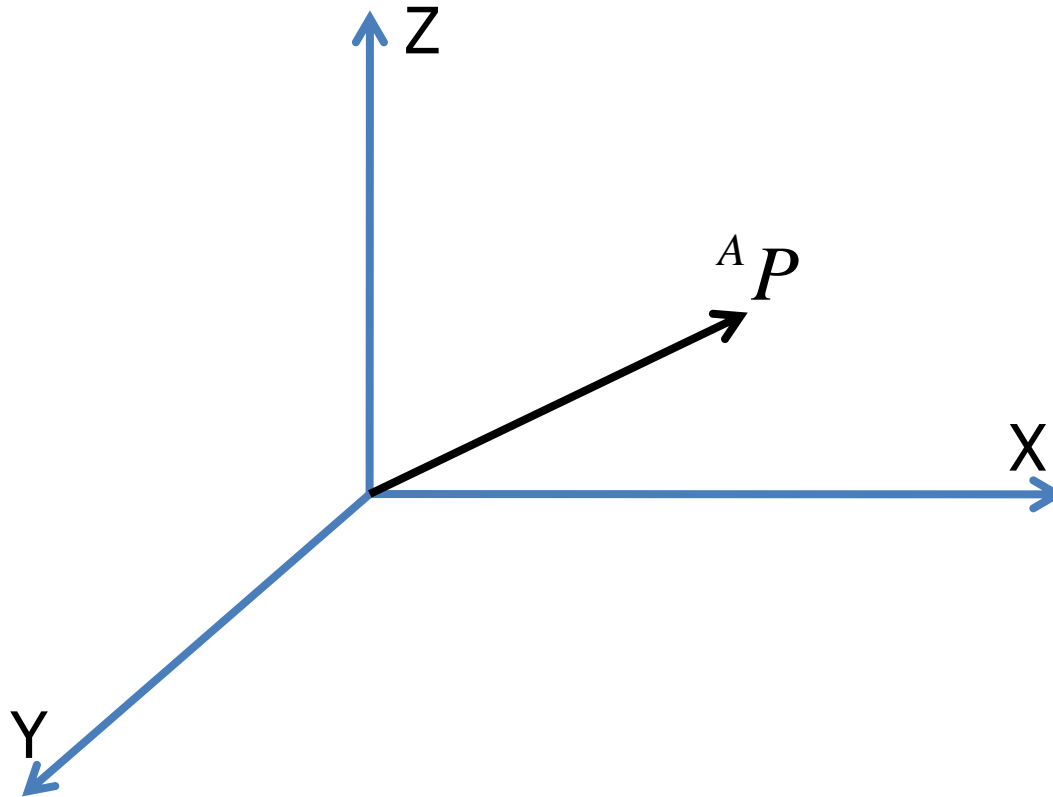
## **Coordinate Systems**

# Position Representation

- Position representation

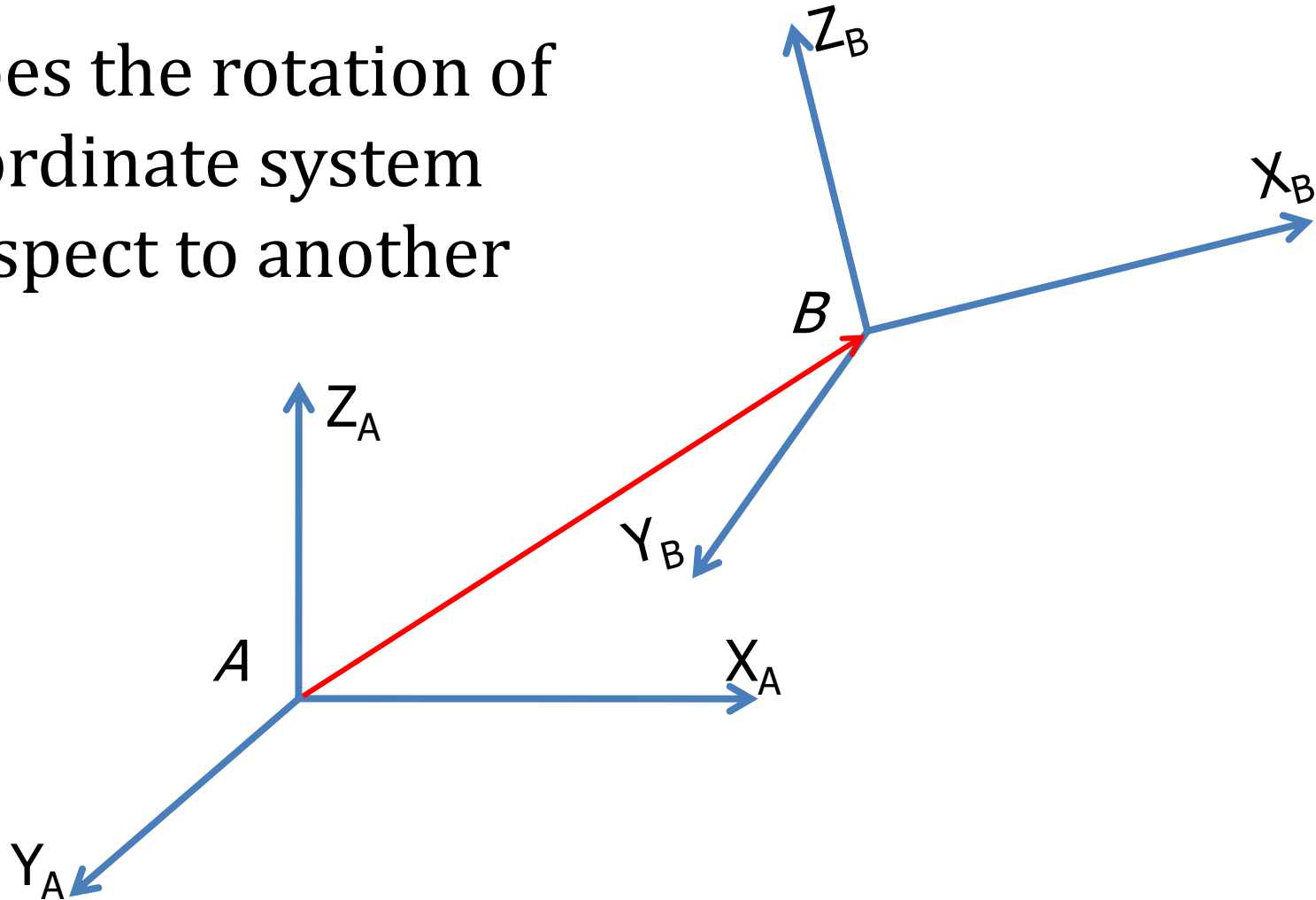
is:

$${}^A P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$



# Orientation Representations

- Describes the rotation of one coordinate system with respect to another

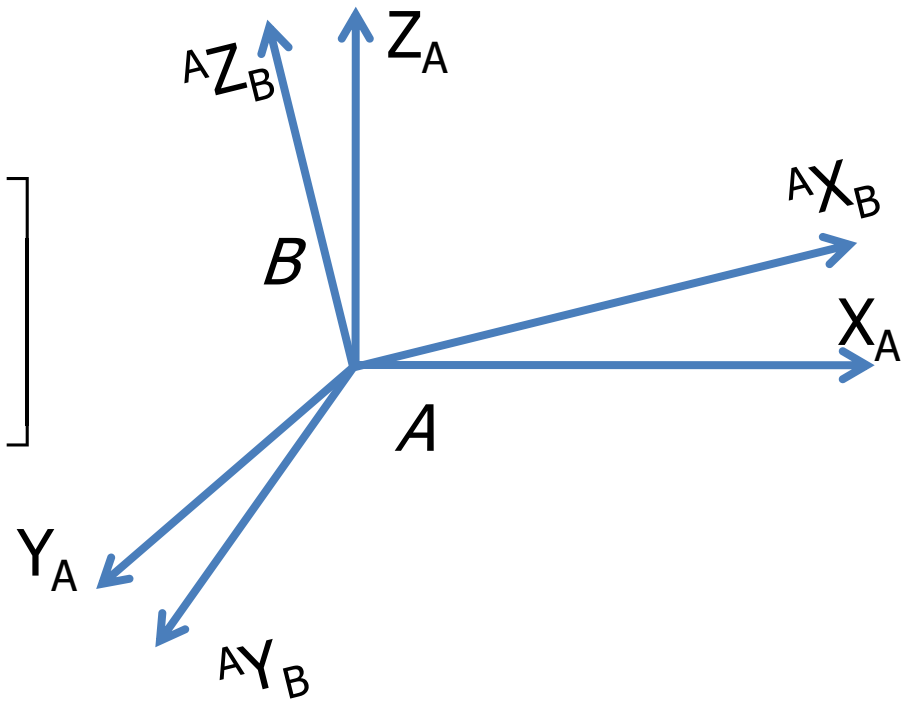


# Rotation Matrix

- Write the unit vectors of  $B$  in the coordinate system of  $A$ .
- Rotation Matrix:

$${}^A_B R = \begin{bmatrix} {}^A \hat{X}_B & {}^A \hat{Y}_B & {}^A \hat{Z}_B \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}$$



# Properties of Rotation Matrix

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$${}^B R_A = {}^A R_B^T$$

$${}^A R_B^T {}^A R_B = I_3$$

$${}^A R_B = {}^B R_A^{-1} = {}^B R_A^T$$



# Coordinate System Transformation

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$$M = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R & T \\ \mathbf{0}_{3 \times 1} & 1 \end{bmatrix}$$

where  $R$  is the rotation matrix and  $T$  is the translation vector



# Rotation Matrix

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- The rotation matrix consists of 9 variables, but there are many constraints. The minimum number of variables needed to describe a rotation is three.



# Rotation Matrix-Single Axis

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$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$





# Fixed Angles

- One simple method is to perform three rotations about the axis of the original coordinate frame:
  - X-Y-Z fixed angles

$${}^A_B R(\theta, \phi, \psi) = R_z(\psi)R_y(\phi)R_x(\theta)$$

$$= \begin{bmatrix} \cos(\psi)\cos(\phi) & \cos(\psi)\sin(\phi)\sin(\theta) - \sin(\psi)\cos(\theta) & \cos(\psi)\sin(\phi)\cos(\theta) + \sin(\psi)\sin(\theta) \\ \sin(\psi)\cos(\phi) & \sin(\psi)\sin(\phi)\sin(\theta) + \cos(\psi)\cos(\theta) & \sin(\psi)\sin(\phi)\cos(\theta) + \cos(\psi)\sin(\theta) \\ -\sin(\phi) & \cos(\phi)\sin(\theta) & \cos(\theta)\cos(\psi) \end{bmatrix}$$

- There are 12 different combinations



# Inverse Problem

- From a Rotation matrix find the fixed angle rotations:

$${}^A_B R(\theta, \phi, \psi) = {}^A_B R \Rightarrow$$
$$\begin{bmatrix} \cos(\psi)\cos(\phi) & \cos(\psi)\sin(\phi)\sin(\theta) - \sin(\psi)\cos(\theta) & \cos(\psi)\sin(\phi)\cos(\theta) + \sin(\psi)\sin(\theta) \\ \sin(\psi)\cos(\phi) & \sin(\psi)\sin(\phi)\sin(\theta) + \cos(\psi)\cos(\theta) & \sin(\psi)\sin(\phi)\cos(\theta) + \cos(\psi)\sin(\theta) \\ -\sin(\phi) & \cos(\phi)\sin(\theta) & \cos(\theta)\cos(\psi) \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

thus:

$$\phi = A \tan 2\left(-r_{31}, \sqrt{(r_{11}^2 + r_{21}^2)}\right)$$

$$\psi = A \tan 2\left(\frac{r_{21}}{\cos(\phi)}, \frac{r_{11}}{\cos(\phi)}\right)$$

$$\theta = A \tan 2\left(\frac{r_{32}}{\cos(\phi)}, \frac{r_{33}}{\cos(\phi)}\right)$$



# Euler Angles

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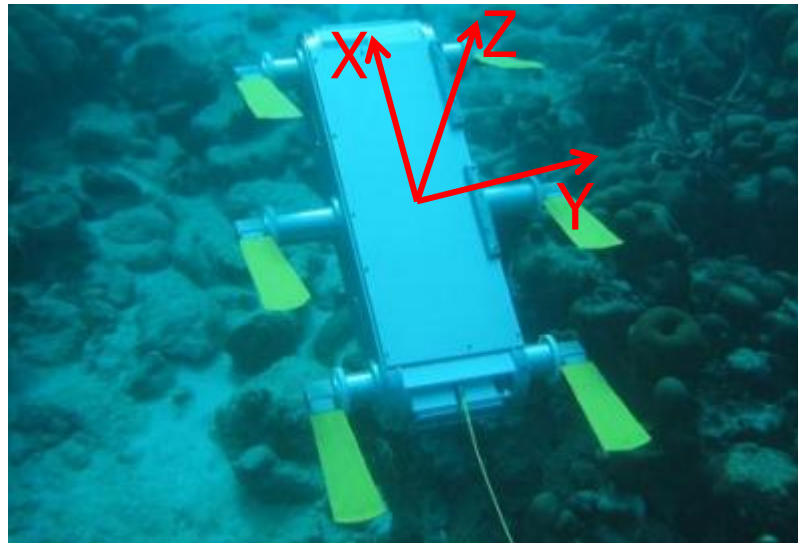
- **ZYX:** Starting with the two frames aligned, first rotate about the  $Z_B$  axis, then by the  $Y_B$  axis and then by the  $X_B$  axis. The results are the same as with using XYZ fixed angle rotation.
- There are 12 different combination of Euler Angle representations



# Euler Angles

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- Traditionally the three angles along the axis are called Roll, Pitch, and Yaw

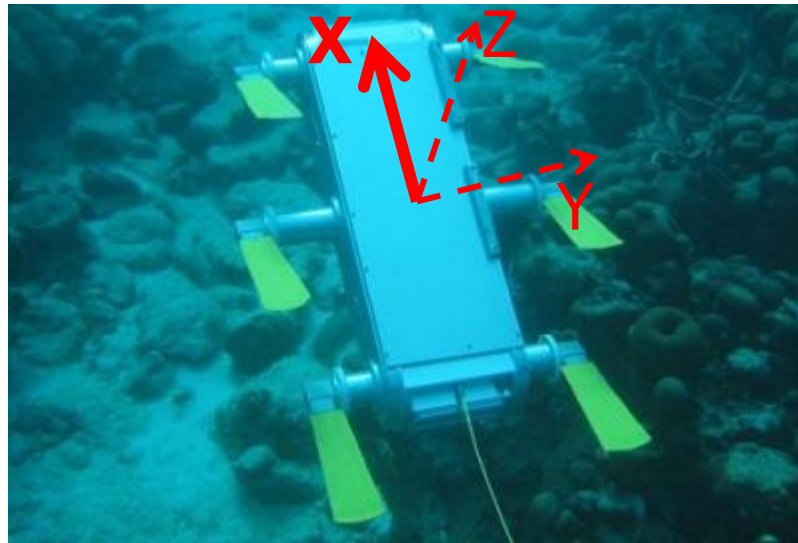


# Euler Angles

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- Traditionally the three angles along the axis are called Roll, Pitch, and Yaw

**Roll**

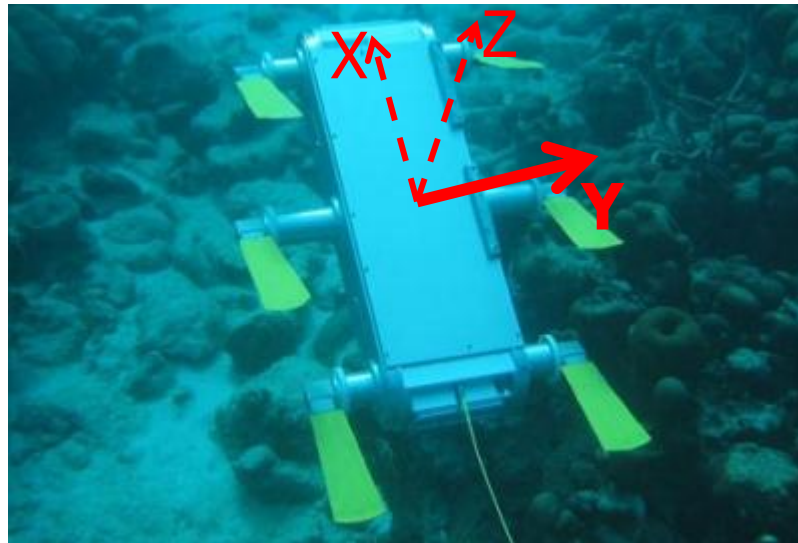


# Euler Angles

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- Traditionally the three angles along the axis are called Roll, Pitch, and Yaw

**Pitch**

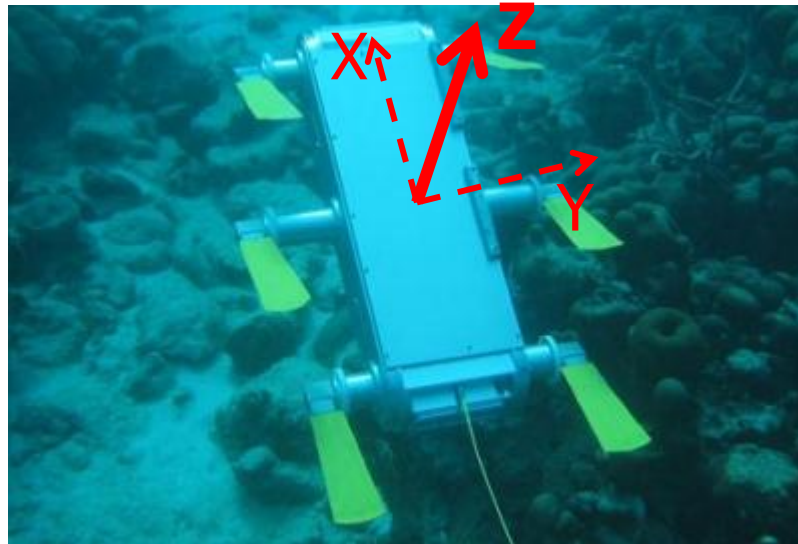


# Euler Angles

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- Traditionally the three angles along the axis are called Roll, Pitch, and Yaw

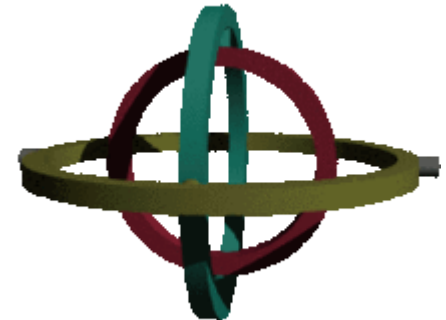
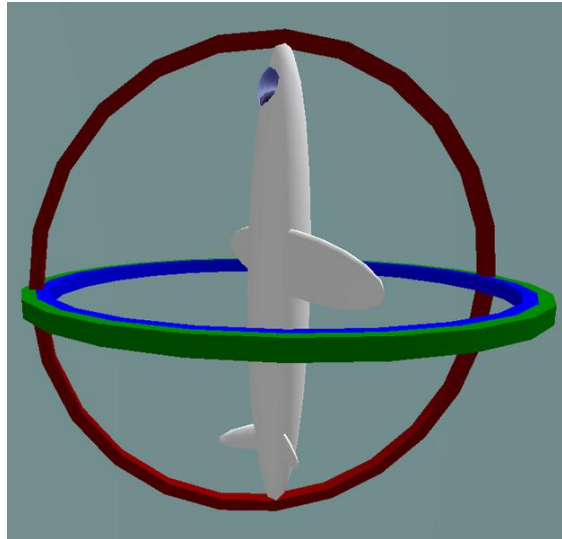
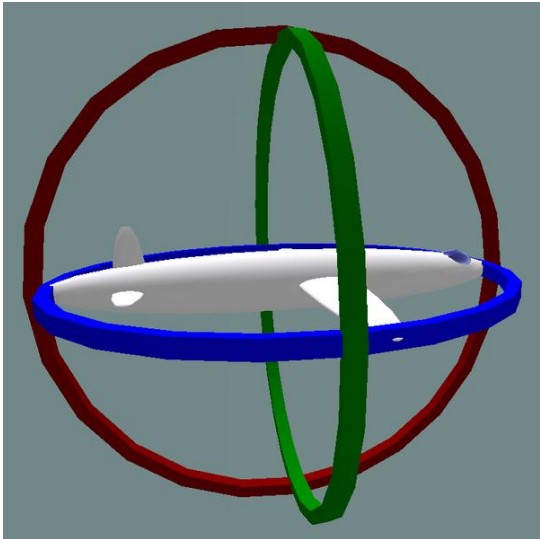
**Yaw**



# Euler Angle concerns: Gimbal Lock

Using the **ZYZ** convention

- $(90^\circ, 45^\circ, -105^\circ) \equiv (-270^\circ, -315^\circ, 255^\circ)$  multiples of  $360^\circ$
- $(72^\circ, 0^\circ, 0^\circ) \equiv (40^\circ, 0^\circ, 32^\circ)$  singular alignment (Gimbal lock)
- $(45^\circ, 60^\circ, -30^\circ) \equiv (-135^\circ, -60^\circ, 150^\circ)$  bistable flip

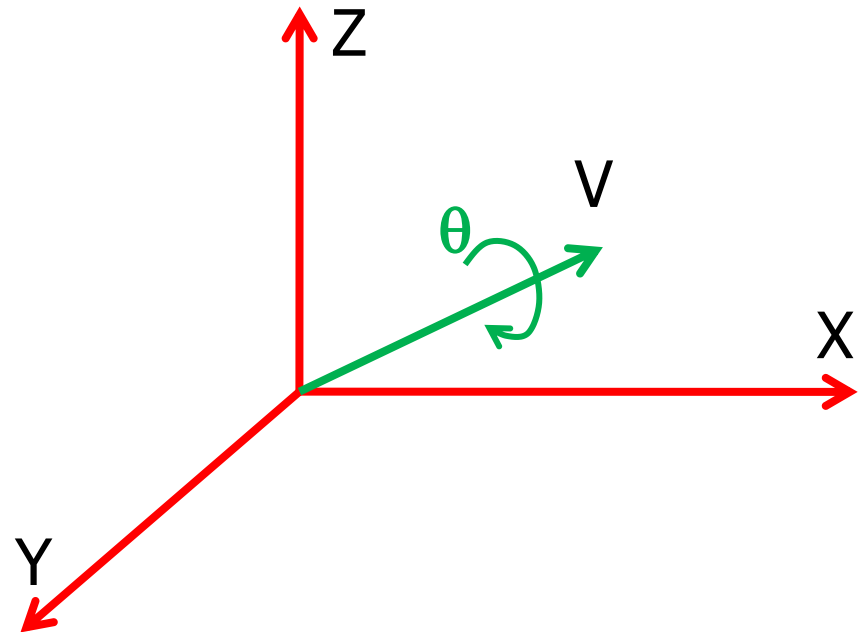




# Axis-Angle Representation

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- Represent an arbitrary rotation as a combination of a vector and an angle



# Quaternions

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- Are similar to axis-angle representation
- Two formulations
  - Classical
  - Based on JPL's standards

W. G. Breckenridge, "Quaternions - Proposed Standard Conventions," JPL, Tech. Rep. INTEROFFICE MEMORANDUM IOM 343-79-1199, 1999.
- Avoids Gimbal lock
- See also: M. D. Shuster, "A survey of attitude representations," *Journal of the Astronautical Sciences*, vol. 41, no. 4, pp. 439–517, Oct.–Dec. 1993.



# Quaternions

		Classic notation	JPL-based
		$\bar{q} = q_4 + q_1i + q_2j + q_3k$	$\bar{q} = q_4 + q_1i + q_2j + q_3k$
		$i^2 = j^2 = k^2 = ijk = -1$	$i^2 = j^2 = k^2 = -1$
		$ij = -ji = k, jk = -kj = i, ki = -ik = j$	$-ij = ji = k, -jk = kj = i, -ki = ik = j$
Vector Notation		$\bar{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}, q_0 = \cos(\theta/2), \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \sin(\theta/2)\cos(\beta_x) \\ \sin(\theta/2)\cos(\beta_y) \\ \sin(\theta/2)\cos(\beta_z) \end{bmatrix}$	$\bar{q} = \begin{bmatrix} \mathbf{q} \\ q_4 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} k_x \sin(\theta/2) \\ k_y \sin(\theta/2) \\ k_z \sin(\theta/2) \end{bmatrix}, q_4 = \cos(\theta/2)$
			$\ \bar{q}\  = 1, \bar{q} \otimes \bar{p}, \mathbf{q} \times \mathbf{p}, \bar{q}_I, \lfloor \mathbf{q} \times \rfloor$

See also: N. Trawny and S. I. Roumeliotis, "Indirect Kalman Filter for 3D Attitude Estimation," University of Minnesota, Dept. of Comp. Sci. & Eng., Tech. Rep. 2005-002, March 2005.

