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The Qualitative Synthesis of Parallel Manipulators

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Abstract: *As shown in this paper, when designing parallel manipulators for tasks involving less than six degrees of freedom, the topology can be laid out by resorting to qualitative reasoning. More specifically, the paper focuses on cases whereby the manipulation tasks pertain to displacements with the algebraic structure of a group. Besides the well-known planar and spherical displacements, this is the case of displacements involving: rotation about a given axis and translation in the direction of the same axis (cylindrical subgroup); translation in two and three dimensions (two- and three-dimensional translation subgroups); three independent translations and rotation about an axis of fixed direction, what is known as the* Schönflies subgroup*; and similar to the Schönflies subgroup, but with the rotation and the translation about the axis of rotation replaced by a screw displacement. For completeness, the fundamental concepts of motion representation and groups of displacements, as pertaining to rigid bodies, are first recalled. Finally, the concept of* Π*-joint, introduced elsewhere, is generalized to two and three degrees of freedom, thereby ending up with the* Π^2*- and the* Π^3*-joints, respectively.*

li·ai·son 1: a binding or thickening agent used in cooking
2a) a close bond or connection : INTERRELATIONSHIP
b): an illicit sexual relationship : AFFAIR

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Qui pourrait ne pas frémir en songeant aux malheurs
que peut causer une seule liaison dangereuse!

Lettre CLXXV. Madame DE VOLANGES
à Madame DE ROSEMONDE (de Laclos, 1782)

1 Introduction

As robot designers realized the immense possibilities offered by parallel manipulators, the variety of designs has not ceased to grow. It would not be exaggerated to say that we are living an era of robot design comparable to the Cambrian period of natural history. Indeed, the number of novel designs either published in conference proceedings and archival journals or disclosed in patent files is too rich to be recorded exhaustively. We thus not aim here at a comprehensive account of all work currently known, but rather a representative sample of this work. The motivation behind the intense work in parallel manipulator design is equally rich, mostly pertaining to applications areas such as: a) machine-tool design, whereby parallel manipulators are termed *Parallel-Kinematics Machines* (PKM); b) robot-assisted surgery; c) surveillance; d) telescope design; and e) motion simulation. In virtually all these areas it has been realized that a full six-degree-of-freedom capability is not necessary; in some tances, all six degrees of freedom are sometimes even undesirable. This is the case, for example, in the assembly of electronic boards, with an essentially planar geometry, whereby any rotation about an axis lying in the plane of the board is to be prevented; else, the assembly will not be successful and the board can be even damaged.

Work on parallel robot design outside six-degree-of-freedom (six-dof) systems can be traced back to Hunt (1983). Later work focused mostly on planar and spherical manipulators (Gosselin and Angeles, 1987; Craver, 1989). An architecture that received special attention involves three legs of the revolute-prismatic-spherical type, producing two rotations and one translation of its moving platform (Lee and Shah, 1987; Lee and Arjuman, 1991; Agrawal, 1991; Pfreundschuh, Kumar and Sugar, 1991).

Of special interest is the design of three- and four-dof manipulators for the production of either pure translations or displacements of the Schönflies type, respectively. The latter consist of three independent translations and one rotation about an axis of fixed direction. A first manipulator of the pure-translation type was disclosed in (Clavel 1990). Although the foregoing patent file claims a possible rotation about an axis of fixed direction, for the production of Schönflies displacements, this rotation is achieved by means of a motor mounted on top of the moving platform, and hence, the device is not a Schönflies-motion generator, properly speaking. Brogårdh (2001) discloses a parallel array that is capable of three-dof translations and can produce Schönflies motions by the addition of one motor at its end plate, similar to Clavel's. Other instances of three-dof translational manipulators are those proposed in (Hervé and Sparacino, 1992; Arai, Hervé and Tanikawa, 1996). A fully-parallel Schönflies-motion generator was disclosed fairly recently in (Company, Pierrot, Shibukawa and Koji, 2001).

2 Kinematics Background

The concepts, and to a great extent the notation and nomenclature that follow, are taken from (Hervé, 1978).

The kinematics of machines is studied via their underlying *kinematic chains*. A kinematic chain is the result of the coupling of rigid bodies, called *links*. Upon coupling *two* links, a kinematic pair is obtained. When the coupling takes place in such a way that the two links share a common surface, a *lower kinematic pair* results; when the coupling takes place along a common line or a common point of the two links, a *higher kinematic pair* is obtained.

Lower kinematic pairs deserve special attention for various reasons: One is that they model fairly well the mechanical couplings in a variety of machines; one more is that they are known to occur in exactly six types, to be described presently. Higher kinematic pairs occur in the coupling by cam-follower mechanisms and by gears, in which contact occurs along common lines or common points of the coupled bodies.

We shall denote with lower-case boldfaces all vectors; with upper-case boldfaces all matrices. Sets will be denoted with caligraphic fonts, e.g., $\mathcal{A}$, $\mathcal{B}$, etc., while lower kinematic pairs are denoted with *sans-serif* upper cases: R, P, H, C, E, and S denote the six pairs of interest (Denrit and Hartenberg, 1964), which are recalled below:

(i) The *revolute pair* R allows a relative rotation through an angle ϕ about one axis $\mathcal{A}$ passing through a point A of position vector **a** and parallel to the unit vector **e**;

(ii) The *prismatic pair* P allows a relative translation u in the direction of a unit vector **e**;

(iii) The *screw pair* H allows both a relative rotation through an angle ϕ about an axis $\mathcal{A}$ passing through a point A of position vector **a** and parallel to the unit vector **e**, and a relative translation u in the direction of **e**. However, the rotation and the translation are not independent, for they are related by the *pitch* p of the pair: $u = p\phi$;

(iv) The *cylindrical pair* C allows both a relative rotation through an angle ϕ about an axis $\mathcal{A}$ passing through a point A of position vector **a** and parallel to the unit vector **e**, and a relative translation in the direction of **e**, with rotation and translation being independent;

(v) The *planar pair* E allows two independent translations t_u and t_v in the directions of the *distinct* unit vectors **u** and **v**, respectively, and a rotation ϕ about an axis normal to the plane of these two vectors; and

(vi) The *spherical pair* S, allowing one independent rotation about each of three noncoplanar axes concurrent at a point O. The relative motions allowed by S are thus characterized by point O, and are associated with an axis parallel to the unit vector **e** and with the angle of rotation ϕ about this axis, as per Euler's Theorem (Angeles, 1982).

Remark: While the R, H and C pairs are characterized by an axis, the P pair is characterized by a *direction* alone; not by an axis!

3 Groups of Displacements

In the sequel, we shall resort to the algebraic concept of *group*. A group is a set $\mathcal{G}$ of elements related by a *binary operation* $\star$ with four properties:

(a) if a and $b \in \mathcal{G}$, then $a \star b \in \mathcal{G}$;

(b) if a, b, and $c \in \mathcal{G}$, then $a \star (b \star c) = (a \star b) \star c$;

(c) $\mathcal{G}$ contains an element ι called the *identity* of $\mathcal{G}$ under $\star$, such that $a \star \iota = \iota \star a = a$; and

(d) for every $a \in \mathcal{G}$, there exists an element a^{-1}, called the *inverse of a under* $\star$ such that $a \star a^{-1} = a^{-1} \star a = \iota$.

If the elements of a set $\mathcal{D}$ are the displacements undergone by a rigid body, then we can define a binary operation $\odot$—read "o-dot"—of displacements as the *composition* of displacements: As the body undergoes first a displacement d_a and then a displacement d_b, taking the body, successively, from pose $\mathcal{B}_0$ to pose $\mathcal{B}_a$, and then to pose $\mathcal{B}_b$, it is intuitively apparent that the composition of the two displacements, $d_a \odot d_b$, is in turn a rigid-body displacement. Hence,

(a) $d_a \odot d_b \in \mathcal{D}$;

(b) given d_a and d_b as introduced above, we define a third displacement d_c carrying $\mathcal{B}$ from pose $\mathcal{B}_b$ to pose $\mathcal{B}_c$. Then, $d_a \odot (d_b \odot d_c) = (d_a \odot d_b) \odot d_c$;

(c) under no motion, any pose $\mathcal{B}$ of a rigid body is preserved, the motion undergone by the body then being represented by a displacement ι that can be defined as the *identity element* of $\mathcal{D}$, such that, for any displacement d, $d \odot \iota = \iota \odot d = d$; and

(d) for any displacement d carrying the body from pose $\mathcal{B}_0$ to pose $\mathcal{B}$, the *inverse displacement* d^{-1} is defined as that bringing back the body from $\mathcal{B}$ to $\mathcal{B}_0$, and hence, $d \odot d^{-1} = d^{-1} \odot d = \iota$.

From the foregoing discussion it is apparent that the set of rigid-body displacements $\mathcal{D}$ has the algebraic structure of a group. Henceforth, we refer to the set of displacements of a rigid body as *group* $\mathcal{D}$. The interest in studying rigid-body displacements as algebraic groups lies in that, on the one hand, $\mathcal{D}$ includes interesting and practical subgroups that find relevant applications in the design of production-automation and prosthetic devices. On the other hand, the same subgroups can be *combined* to produce novel mechanical layouts that would be insurmountably difficult to produce by sheer intuition. The combination of subgroups, in general, can take place via the standard set operations of *union* and *intersection*. As we shall see, however, the set defined as that comprising the elements of two displacement subgroups is not necessarily a subgroup, and hence, one cannot speak of the union of displacement subgroups. On the contrary, the intersection of two displacement subgroups is always a subgroup itself, and hence, the *intersection of displacement subgroups* is a valid group operation.

Rather than the union of groups, what we have is the *product* of groups (Macdonald, 1968). Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two groups defined over the same binary operation $\star$; if $g_1 \in \mathcal{G}_1$ and $g_2 \in \mathcal{G}_2$, then the product of these two groups, represented by $\mathcal{G}_1 \bullet \mathcal{G}_2$, is the *set* of elements of the form $g_1 \star g_2$, where the order is important, for commutativity is not to be taken for granted in group theory. That is, in general, $\mathcal{G}_1 \bullet \mathcal{G}_2 \neq \mathcal{G}_2 \bullet \mathcal{G}_1$.

The intersection of the two foregoing groups, represented by the usual set-theoretic symbol $\cap$, i.e., $\mathcal{G}_1 \cap \mathcal{G}_2$, is the group of elements g belonging to both $\mathcal{G}_1$ and $\mathcal{G}_2$, and hence, the order is not important. Thus, $\mathcal{G}_1 \cap \mathcal{G}_2 = \mathcal{G}_2 \cap \mathcal{G}_1$.

3.1 Displacement Subgroups

A *subgroup* $\mathcal{G}_s$ of a given group $\mathcal{G}$ is a set with two properties: (a) its elements belong to $\mathcal{G}$, although some elements of $\mathcal{G}$ may not belong to $\mathcal{G}_s$, and (b) $\mathcal{G}_s$ has the algebraic structure of a group. Therefore, a subgroup $\mathcal{D}_s$ of the group of rigid-body displacements $\mathcal{D}$ is itself a group of displacements, but may lack some rigid-body displacements.

The six lower kinematic pairs can be regarded as *generators* of displacement subgroups. We thus have:

(i) The revolute pair R of axis $\mathcal{A}$ generates the subgroup $\mathcal{R}(\mathcal{A})$ of rotations about $\mathcal{A}$. Each element of this group is characterized by the angle ϕ of the corresponding rotation;

(ii) the prismatic pair in the direction **e** generates the subgroup $\mathcal{P}(\mathbf{e})$ of translations along **e**. Each element of $\mathcal{P}(\mathbf{e})$ is characterized by the translation u along **e**;

(iii) the screw pair of axis $\mathcal{A}$ and pitch p generates the subgroup $\mathcal{H}(\mathcal{A}, p)$ of rotations ϕ about $\mathcal{A}$ and translations u along the direction of the same axis, translations and rotations being related by the pitch p in the form $u = p\phi$, as described when the screw pair was introduced. Each element of $\mathcal{H}(\mathcal{A}, p)$ can thus be characterized either by u or by ϕ;

(iv) the cylindrical pair of axis $\mathcal{A}$ generates the subgroup $\mathcal{C}(\mathcal{A})$ of independent rotations about and translations along $\mathcal{A}$. Each element of $\mathcal{C}(\mathcal{A})$ is thus characterized by both the displacement u and the rotation ϕ;

(v) the planar pair generates the subgroup $\mathcal{F}(\mathbf{u}, \mathbf{v})$ of two independent translations in the directions of the *distinct* unit vectors **u** and **v**, and one rotation about an axis normal to both **u** and **v**. Each element of $\mathcal{F}(\mathbf{u}, \mathbf{v})$ is thus characterized by the two translations t_u, t_v and the rotation ϕ;

(vi) the spherical pair generates the subgroup $\mathcal{S}(O)$ of rotations about point O. Each element of $\mathcal{S}(O)$ is characterized by the axis of rotation passing through O in the direction of a unit vector **e** and through an angle ϕ. Alternatively, each element can be characterized by the independent rotations about three designated axes, e.g., the well-known Euler angles.

Besides the six foregoing subgroups, we can define six more, namely,

(vii) The *identity subgroup* $\mathcal{I}$, whose single element is the identity displacement ι introduced above;

($viii$) the *planar-translation subgroup* $\mathcal{T}_2(\mathbf{u}, \mathbf{v})$ of translations in the directions of the two distinct unit vectors **u** and **v**. Each element of this group is thus characterized by two translations, t_u and t_v;

(ix) the *translation subgroup* $\mathcal{T}_3$ of translations in $\mathcal{E}$, each element of which is characterized by three independent translations t_u, t_v, and t_w;

(x) the subgroup $\mathcal{Y}(\mathbf{e}, p)$ of motions allowed by a screw of pitch p and axis parallel to **e** undergoing arbitrary translations in a direction normal to **e**. Each element of this subgroup is thus characterized by the two independent translations t_u, t_v of the axis, and either the rotation ϕ about this axis or the translation t_w along the axis. Faute-de mieux, we shall call this subgroup the *translating-screw group*;

(xi) the subgroup $\mathcal{X}(\mathbf{e}) = \mathcal{F}(\mathbf{e}) \bullet \mathcal{P}(\mathbf{e})$, resulting of the product of the planar subgroup of plane normal to **e** and the prismatic subgroup of direction **e**. Each element of this subgroup is thus characterized by the two translations t_u, t_v and the angle ϕ of the planar subgroup plus the translation t_w in the direction of **e**. Moreover, note that $\mathcal{F}(\mathbf{e}) \bullet \mathcal{P}(\mathbf{e}) = \mathcal{P}(\mathbf{e}) \bullet \mathcal{F}(\mathbf{e})$. This subgroup is known as the *Schönflies subgroup*, and is generated most commonly by what is known as SCARA systems, for *Selective-Compliance Assembly Robot Arm*;

(xii) the group $\mathcal{D}$ itself. Each element of this subgroup is characterized by three independent translations and three independent rotations.

It is thus apparent that each subgroup includes a set of displacements with a specific degree of freedom. We shall need below an extension of the concept of dof, for which reason we term the dof of each subgroup its *dimension*, and denote the dimension of any subgroup $\mathcal{G}_s$ by $\dim[\mathcal{G}_s]$. Thus,

$$\dim[\mathcal{I}] = 0 \tag{1a}$$
$$\dim[\mathcal{R}(\mathcal{A})] = \dim[\mathcal{P}(\mathbf{e})] = \dim[\mathcal{H}(\mathcal{A}, p)] = 1 \tag{1b}$$
$$\dim[\mathcal{T}_2(\mathbf{u}, \mathbf{v})] = \dim[\mathcal{C}(\mathcal{A})] = 2 \tag{1c}$$
$$\dim[\mathcal{T}_3] = \dim[\mathcal{F}(\mathbf{e})] = \dim[\mathcal{S}(O)] = \dim[\mathcal{Y}(\mathbf{e}, p)] = 3 \tag{1d}$$
$$\dim[\mathcal{X}(\mathbf{e})] = 4 \tag{1e}$$
$$\dim[\mathcal{D}] = 6 \tag{1f}$$

The foregoing list of displacement subgroups is *exhaustive*. The reader may wonder whether displacement products are missing from the list that might be subgroups. However, any displacement product not appearing in the list *is not a subgroup*, e.g.,

(a) $\mathcal{P}(\mathbf{e}) \bullet \mathcal{R}(\mathcal{A})$, with $\mathcal{A}$ defined as a line normal to the unit vector **e** and passing through a point O, is not a subgroup. This set of displacements is characterized by a translation t_e in the direction of **e** and a rotation about $\mathcal{A}$ through an angle θ, as depicted in Fig. 1. It should be apparent from this figure that this set of displacements does not form a group.

(b) $\mathcal{R}(\mathcal{A}) \bullet \mathcal{R}(\mathcal{A}')$ is not a subgroup, unless $\mathcal{A}$ and $\mathcal{A}'$ coincide. The reason here is that, assuming for example, that these two axes intersect at a point O, the composition $\mathcal{R}(\mathcal{A}) \bullet \mathcal{R}(\mathcal{A}')$ is, in general, equivalent to a new rotation, according to Euler's Theorem, about a third axis $\mathcal{A}''$, different from any of the first two axes, although still passing through O.

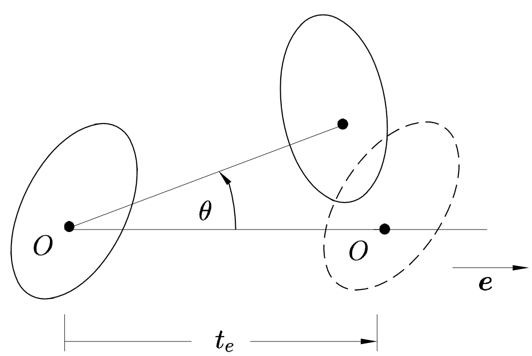


Figure 1: An instance of the $\mathcal{P}(\mathbf{e}) \bullet \mathcal{R}(\mathcal{A})$ set of displacements not constituting a group

4 Kinematic Bonds

Displacement subgroups can be combined to produce new sets of displacements that may or may not be displacement subgroups themselves. To combine subgroups, we resort to the group operations of product ($\bullet$) and intersection ($\cap$).

Now we recall the concept of *kinematic bond*, which is a generalization of kinematic pair, as first proposed by Hervé (1978), who called this concept *liaison cinématique* in French. This concept has been termed *kinematic liaison* (Angeles, 1982) or *mechanical connection* (Hervé, 1997) in English. Since "liaison" in English is usually applied to human relations, the term "bond" seems more appropriate, and hence, is adopted here.

We illustrate the concept with an example: Let us assume three links, numbered from 1 to 3, and coupled by two kinematic pairs generating the two subgroups $\mathcal{G}_1$ and $\mathcal{G}_2$, where these two subgroups are instanced by specific displacement subgroups below. We then have

$$\mathcal{G}_1 \bullet \mathcal{G}_2 = \mathcal{R}(\mathcal{A}) \bullet \mathcal{P}(\mathbf{e}) = \mathcal{C}(\mathcal{A}), \quad \mathbf{e} \parallel \mathcal{A} \tag{2a}$$
$$\mathcal{G}_1 \bullet \mathcal{G}_2 = \mathcal{R}(\mathcal{A}) \bullet \mathcal{T}_2(\mathbf{u}, \mathbf{v}) = \mathcal{F}(\mathbf{e}), \quad \mathbf{e} \text{ and } \mathcal{A} \perp \mathbf{u} \text{ and } \mathbf{v} \tag{2b}$$
$$\mathcal{G}_1 \bullet \mathcal{G}_2 = \mathcal{R}(\mathcal{A}) \bullet \mathcal{R}(\mathcal{B}) = \mathcal{L}(1,3) \tag{2c}$$
$$\mathcal{G}_1 \cap \mathcal{G}_2 = \mathcal{R}(\mathcal{A}) \cap \mathcal{C}(\mathcal{A}) = \mathcal{R}(\mathcal{A}) \tag{2d}$$
$$\mathcal{G}_1 \cap \mathcal{G}_2 = \mathcal{R}(\mathcal{A}) \cap \mathcal{S}(O) = \mathcal{R}(\mathcal{A}), \quad O \in \mathcal{A} \tag{2e}$$
$$\mathcal{G}_1 \cap \mathcal{G}_2 = \mathcal{R}(\mathcal{A}) \cap \mathcal{P}(\mathbf{e}) = \mathcal{I}, \quad \forall \quad \mathcal{A} \text{ and } \mathbf{e} \tag{2f}$$

All of the above examples, except for the third one, amount to a displacement subgroup. This is why no subgroup symbol is attached to that set. Instead, we have used the symbol $\mathcal{L}(1,3)$ to denote the kinematic bond between the first and third links of the chain. In general, a kinematic bond between links i and n of a kinematic chain, when no ambiguity is possible, is denoted by $\mathcal{L}(i,n)$. When the chain connecting these two links is not unique, such as in a closed chain, where these two links can be regarded as being connected by two possible *paths*, a subscript will be used, e.g., $\mathcal{L}_I(i,j)$, $\mathcal{L}_{II}(i,j)$, etc. A kinematic bond is, thus, a set of displacements, as stemming from a product of displacement subgroups, although the bond itself need not be a subgroup. Obviously, the 12 subgroups described above are themselves special cases of kinematic bonds.

The kinematic bond between links i and n can be conceptualized as the product of the various subgroups associated with the kinematic pairs between the ith and the nth links. To keep the discussion general enough, we shall denote the subgroup associated with the kinematic pair coupling links i and $i+1$ as $\mathcal{L}(i, i+1)$, with a similar notation for all other kinematic-pair subgroups, i.e.,

$$\mathcal{L}(i,n) = \mathcal{L}(i,i+1) \bullet \mathcal{L}(i+1,i+2) \bullet \cdots \bullet \mathcal{L}(n-1,n) \tag{3}$$

Thus, in a six-axis serial manipulator, we can set $i = 1, n = 7$, all six kinematic pairs in-between being revolutes of skew axes $\mathcal{R}(\mathcal{A}_1)$, $\mathcal{R}(\mathcal{A}_2)$, …, $\mathcal{R}(\mathcal{A}_6)$. Then,

$$\mathcal{L}(1,7) = \mathcal{D}$$

That is, the 6R manipulator is a generator of the general six-dimensional group of rigid-body displacements $\mathcal{D}$.

As an example of group-intersection, let us consider the *Sarrus mechanism*, which is depicted in Fig. 2.

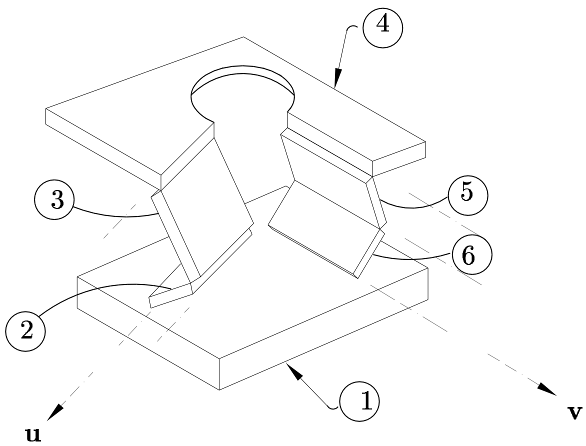


Figure 2: The Sarrus mechanism

In the Sarrus mechanism, we have six links, coupled by six revolute pairs. Moreover, the revolute pairs occur in two triplets, each on one leg of the mechanism. The axes of the three revolute pairs of each leg are parallel to each other. The bond $\mathcal{L}(1,4)$, apparently, is not unique, for it can be defined by traversing any of the two legs. Let the leg of links 1, 2, 3 and 4, coupled by revolutes of axes parallel to the unit vector **u**, be labelled I; the other leg, of links 4, 5, 6 and 1, coupled by revolutes of axes parallel to the unit vector **v**, is labelled II. It is apparent that, upon traversing leg I, we obtain

$$\mathcal{L}_I(1,4) = \mathcal{F}(\mathbf{u})$$

while, upon traversing leg II,

$$\mathcal{L}_{II}(1,4) = \mathcal{F}(\mathbf{v})$$

That is, leg I is a generator of the planar subgroup $\mathcal{F}$ of plane normal to vector **u**, while leg II is that of the subgroup $\mathcal{F}$ of plane normal to vector **v**. Therefore, $\mathcal{L}_I(1,4) \cap \mathcal{L}_{II}(1,4)$ is the set of displacements common to the two $\mathcal{F}$-subgroups, namely, the prismatic subgroup of translations in the direction $\mathbf{w} = \mathbf{v} \times \mathbf{u}$, i.e.,

$$\mathcal{L}_I(1,6) \cap \mathcal{L}_{II}(1,6) = \mathcal{P}(\mathbf{w})$$

The Sarrus mechanism is thus a revolute realization of the prismatic joint.

5 The Π Joint and Its Generalizations

The foregoing concepts are now applied to the *qualitative* synthesis of parallel robotic architectures. By qualitative we mean the determination of the topology of the kinematic chain, not including the corresponding dimensions. These dimensions are found at a later stage, by means of methods of *quantitative synthesis*. The full determination of the kinematic chain, including dimensions, yields what is known as the *architecture* of the robotic system at hand. Prior to the discussion of interest, we recall the Π joint, first introduced by Hervé and Sparacino (1992).

5.1 The Π Joint

A four-bar linkage with its opposite links of the same length is known as a *parallelogram*. In the standard terminology, and referring to Fig. 3, the linkage is composed of: a) one fixed link, labelled 1; b) one input link, labelled 2; c) one coupler link, labelled 3; and one output link, labelled 4. In a parallelogram, the opposite links move with a relative pure translation, each point of one link describing a circular trajectory onto the other link. The linkage, shown in Fig. 3, thus provides a kinematic pair of the coupler link 3 with respect to the fixed link 1, which Hervé and Sparacino (1992) termed a Π joint. Notice that the four R joints of the parallelogram linkage can be paired so that each of the two pairs is either (a) fixed to one single coupled link or (b) fixed to different coupled links. In the linkage of Fig. 3, the pairs (R_1, R_4) and (R_2, R_3) are of the *first kind*. Correspondingly, (R_1, R_2) and (R_3, R_4) are of the *second kind*. Likewise, we distinguish two kinds of links, namely, the *coupled links* 1 and 3, and the *coupling links* 2 and 4.

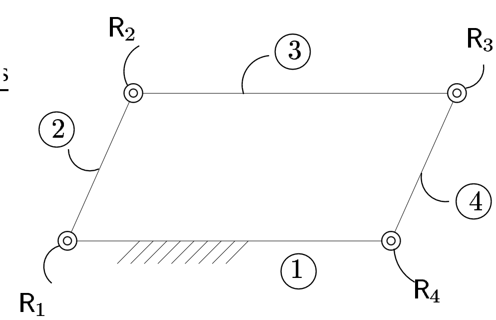


Figure 3: The Π joint, a four-bar parallelogram coupling links 1 and 3

Moreover, notice that the Π joint does not belong to any of the two classes of lower and higher kinematic pairs. Indeed, it couples two adjacent links by means of an infinity of circular cylindrical surfaces of the same radius, but with axes at different locations normal to the plane of the parallelogram linkage. The Π joint is thus characterized by the unit vector **e** normal to the plane of the parallelogram and the radius R of its family of cylindrical surfaces. Therefore, R is nothing but the common length of the coupling links. In summary, the Π joint couples two links while allowing a relative translation along a circular trajectory.

While the Π joint is neither a lower nor a higher pair, we can associate a kinematic bond to it. When combined with other kinematic pairs or other kinematic bonds, the Π joint can generate both $\mathcal{T}_3$ and Schönflies subgroups, as discussed in Section 6.

Below we introduce some extensions of the Π joint.

5.2 The R-Π Joint

Two kinds of joints are possible when a Π joint is concatenated with a R joint of axis lying in the plane of the Π joint. The difference lies on whether the axis $\mathcal{A}$ of the new R joint is the common normal to the axes of two parallelogram joints of the first kind or is normal to the plane of these joints, as depicted in Figs. 4a and b, respectively.

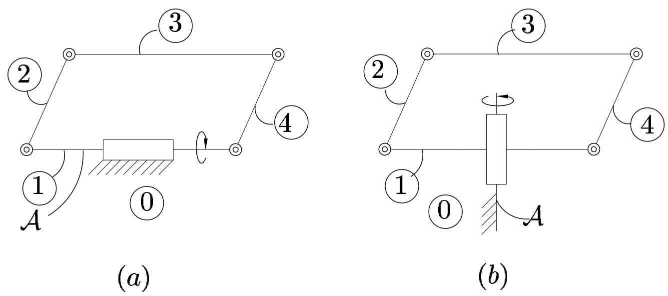


Figure 4: The two kinds of R-Π joints: (a) new R-axis lying along the common normal to two parallelogram R axes and (b) new R-axis normal to the plane of the two parallelogram axes of the first kind

We shall call the composite joints of Figs. 4a and b R-Π_a and R-Π_b, respectively. When two links, 0 and 3, are coupled by means of a R-Π_a or a R-$\Pi_b$joint, their points generate a family of tori of main axis $\mathcal{A}$.

5.3 The Π^2 Joint

It is apparent that, if the four joints of the parallelogram linkage of Fig. 3 are replaced by universal (U) joints, then the plane of the parallelogram undergoes a rotation about axis $\mathcal{A}$ of Fig. 4a. We recall here that a universal joint is the concatenation of two R joints intersecting at right angles. Moreover, by properly constraining the motion of link 3 with respect to link 1, it is possible to have link 3 still move with respect to link 1 with pure translation. In this case, the points of link 3 describe spheres of identical radii R equal to the length of the coupling links of the parallelogram.

Apparently, the constraint needed to produce the foregoing motion can be realized by coupling two identical parallelograms of parallel planes and sharing the same base link 1 and the same moving link 3. The result is displayed in Fig. 5.

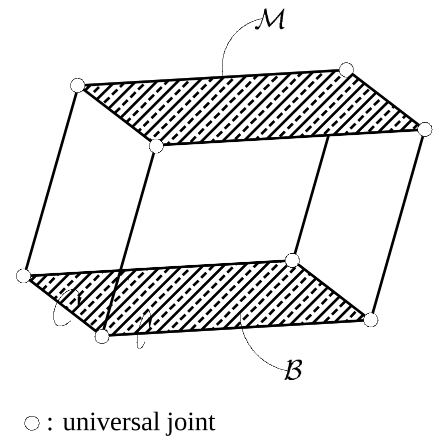


Figure 5: A Π^2 joint

The Π^2 joint thus generates two-dof displacements that can be represented by the bond $\mathcal{L}(\mathcal{B},\mathcal{M})$. Moreover, these displacements are elements of the $\mathcal{T}_2$ subgroup characterized by the unit vector **e** normal to the plane of the R joints of the first kind of the given parallelograms. However, $\mathcal{L}(\mathcal{B},\mathcal{M})$ does not constitute a subgroup.

5.4 The Π^3 Joint

We can go one more step and allow the coupled links of a Π^2 joint to undergo an additional pure translation in the direction of its coupling-link axes, as depicted in Fig. 6. In this figure, the four coupling links of the Π^2 joint of Fig. 5 have been replaced by P joints. By means of a suitable constraint, all four P joints can be made to undergo identical translations. We call this a Π^3 joint.

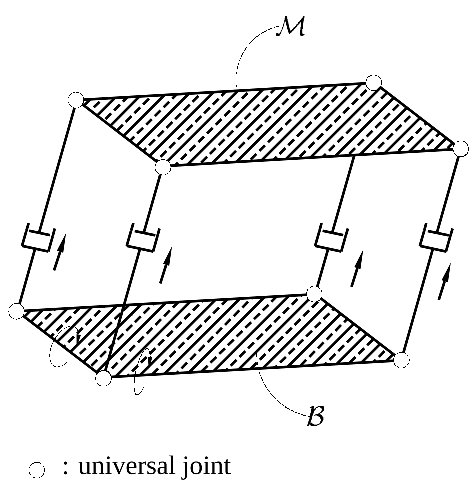
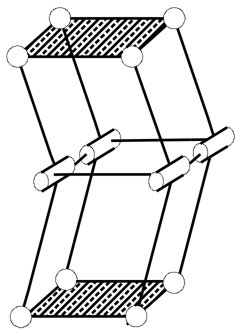


Figure 6: A Π^3 joint

The Π^3 joint thus generates the subgroup $\mathcal{T}_3$ of three-dof pure translations. Notice that a simple means of implementing a Π^3 joint is by cascading two Π^2 joints, upon attaching rigidly the moving link of one with the fixed link of the second one, as depicted in Fig. 7. In this layout, the two Π^2 joints are coupled by means of four R joints of parallel axes.



○: universal joint

Figure 7: A possible realization of a Π^3 joint

We illustrate below how the foregoing ideas can be used in the synthesis of parallel—and serial—manipulators.

6 The Synthesis of Serial and Parallel Robotic Architectures

The first parallel architecture with Π pairs was proposed by Clavel (1988), in what he called the *Delta Robot*. The kinematic chain of this robot is displayed in Fig. 8. This robot is a generator of the $\mathcal{T}_3(\mathbf{u})$ displacement subgroup. Delta is thus capable of three-dof translations.

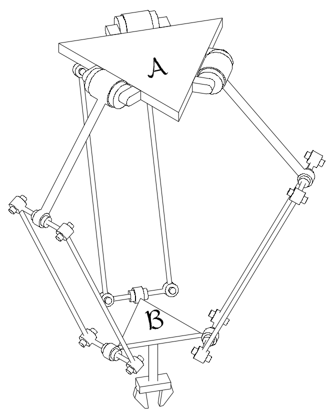


Figure 8: Kinematic chain of the Clavel Delta robot

The kinematic chain of the Delta robot is composed of two triangular plates, the top ($\mathcal{A}$) and the bottom ($\mathcal{B}$) plates. The top plate supports the three (direct-drive) motors, the bottom plate the gripper, and hence, constitutes the end-effector (EE) of the robot. The EE is capable of translating in 3D space with respect to the upper plate, which is considered fixed. The two plates are coupled by means of three legs, each with a RRΠR chain.

While Clavel did not cite any group-theoretical reasoning behind his ingenious design, an analysis in this framework will readily explain the principle of operation of the Delta robot.

The ith leg is a generator of the Schönflies $\mathcal{X}(\mathbf{e}_i)$ subgroup, with $\mathbf{e}_i$ denoting the unit vector parallel to the axis of the ith motor. That is, the ith leg generates a Schönflies subgroup of displacements comprising translations in 3D space and one rotation about an axis parallel to $\mathbf{e}_i$. The subset of EE displacements is thus the intersection of the three subgroups $\mathcal{X}(\mathbf{e}_i)$, for $i = 1, 2, 3$, i.e., the subgroup $\mathcal{T}_3$. Therefore, the EE is capable of pure translations in 3D space.

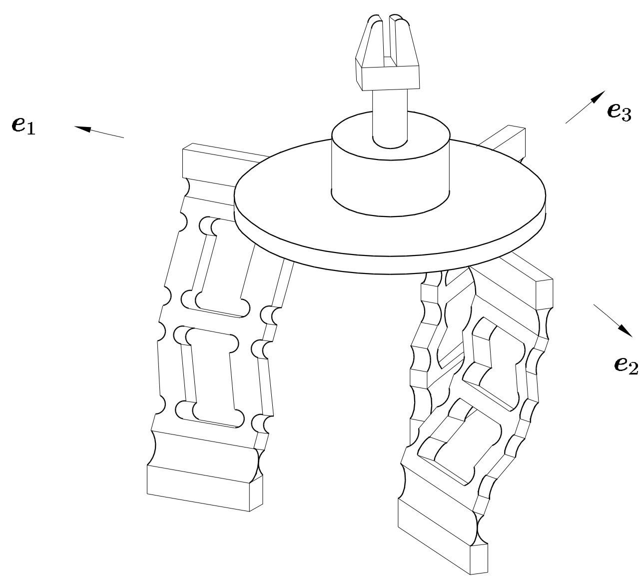


Figure 9: The MEL microfinger

One second applications example is the microfinger of Japan's Mechanical Engineering Laboratory (MEL) at Tsukuba (Arai, Hervé and Takinawa, 1996), as displayed in Fig. 9. In the MEL design, the inventors use a structure consisting of two plates that translate with respect to each other by means of three legs coupling the plates. The ith leg entails a R$\Pi\Pi$R chain, shown in Fig. 10, that generates the Schönflies subgroup in the direction of a unit vector $\mathbf{e}_i$, for $i = 1, 2, 3$. The three unit vectors, moreover, are coplanar and make angles of 120°. The motion of the moving plate is thus the result of the intersection of these three subgroups, which is, in turn, the subgroup $\mathcal{T}_3$. Moreover, the kinematic chain of each leg is made of an elastic material in one single piece, in order to allow for micrometric displacements.

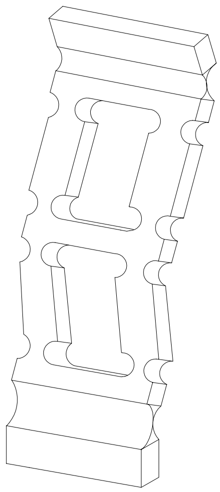


Figure 10: The ith leg of the MEL microfinger

Yet another example is the Y-Tristar robot, developed at Ecole Centrale de Paris by Hervé and Sparacino (1992). Interestingly, the above instances of parallel manipulators using R-Π joints are capable of generating only the $\mathcal{T}_3$ subgroup. However, they are intended for Schönflies displacements. The inventors of these architectures have solved the problem of Schönflies-motion generation by cascading a fourth actuated axis to the parallel manipulator, thereby obtaining a hybrid parallel-serial one. An architecture realizing a Schönflies-motion generator is the linkage of Fig. 11. The Schönflies displacement subgroup is $\mathcal{X}(\mathbf{e})$, with $\mathbf{e}$ parallel to $\mathcal{L}$.

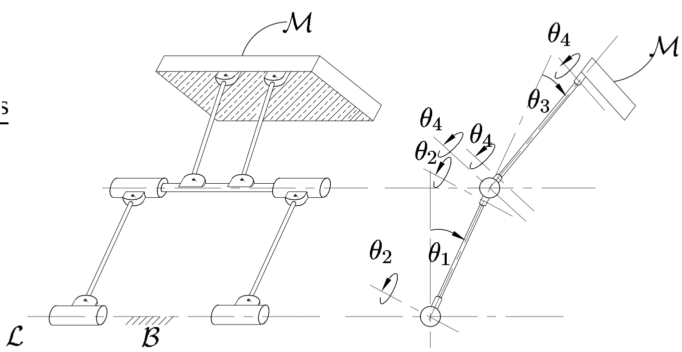


Figure 11: RΠRΠ kinematic bond generating Schönflies-motions

One more aplication of the same concepts is the serial Schönflies-motion generator proposed by Angeles, Morozov, and Navarro (2000). This robot entails a kinematic chain of the RΠRΠ type, as displayed in Fig. 12. In fact, the chain is the concatenation of two pan-tilt generating chains, each constituted by a R-Π_b joint, as displayed in Fig. 4b.

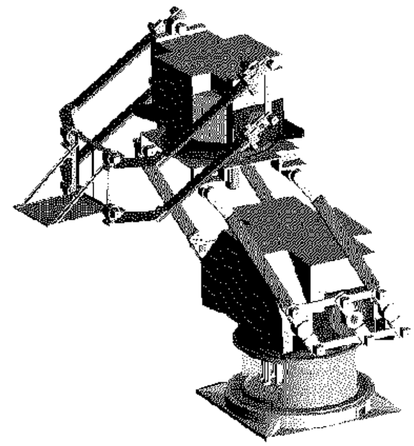


Figure 12: A serial Schönflies-motion generator

To be true, Company, Pierrot, Shibukawa and Koji (2001) dsiclosed a fully-parallel Schönflies-motion generator in European Patent 1084802. This robot comprises four legs, each being a Schönflies-displacement generator. Besides this manipulator, no other one is known with the same motion capabilities and a fully-parallel architecture. Nevertheless, the parallel robot of EP1084802 does not make proper use of Schönflies-motion generators. Indeed, this robot is the result of coupling two two-leg parallel manipulators, each leg being, in turn, the parallel array of two Schönflies-motion generators identical to those of the legs of the Delta robot. Such a parrallel array is displayed in Fig. 14, where it is apparent that two RRΠR legs generating Schönflies subgroups $\mathcal{X}(\mathbf{u})$ and $\mathcal{X}(\mathbf{v})$ are coupled by means of the end-effector 42, using the same notation as in the foregoing patent. Link 42, thus, undergoes the set of motions resulting from the intersection of the two Schönflies subgroups, namely,

$$\mathcal{X}(\mathbf{u}) \cap \mathcal{X}(\mathbf{v}) = \mathcal{T}_3$$

Thus, link 42 undergoes pure translations in three-dimensional space. However, the parallel array is supplied with only two actuators, one per leg, and hence, one translation is left uncontrolled, but this uncontrolled motion is exploited in producing Schönflies motions, as explained below.

What Company and his co-inventors did in order to produce the Schönflies subgroup was to couple the end-effectors of two identical parallel arrays like that displayed in Fig. 13 by means of revolutes of parallel axes, one normal to the $\mathbf{u}$ and $\mathbf{v}$ unit vectors, the other normal to the $\mathbf{u}'$ and $\mathbf{v}'$ vectors. Such a coupling is displayed in Fig. 14. In this coupling, the parallel axes of the rev-

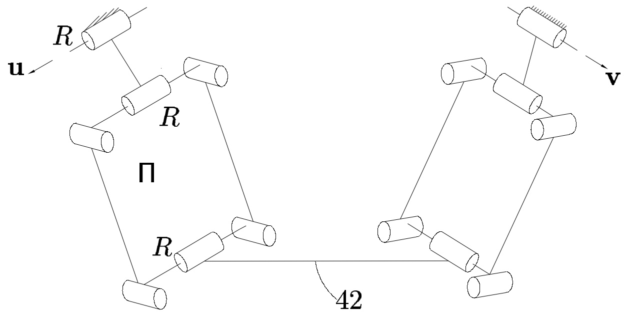


Figure 13: The coupling of two identical $\mathcal{T}_3$ generators

olutes of the end-effector 41 can translate freely in space, while keeping their parallel orientations. Such a motion is controlled by the actuators driving each of the two legs. Furthermore, the concatenation of one of the two revolutes of 41 with each $\mathcal{T}_3$ generator yields a hybrid parallel-serial generator of the Schönflies subgroup $\mathcal{X}(\mathbf{w})$, where $\mathbf{w} \equiv \mathbf{u} \times \mathbf{v} = \mathbf{u}' \times \mathbf{v}'$. The coupling of the two Schönflies-motion generators thus yields a set of displacements lying in the intersection of the two Schönflies subgroups, i.e.,

$$\mathcal{X}(\mathbf{w}) \cap \mathcal{X}(\mathbf{w}) = \mathcal{X}(\mathbf{w})$$

That is, the intersection of the two identical Schönflies subgroups is the same Schönflies subgroup.

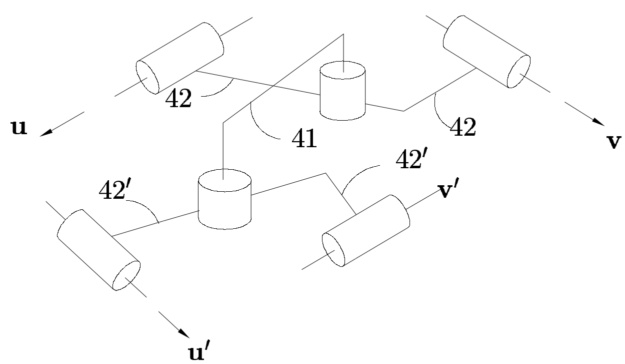


Figure 14: A parallel array of two Schönflies-motion generators

7 Conclusions

The use of qualitative reasoning in the synthesis of the topology of parallel manipulators was highlighted in this paper. The motivation behind is the design of parallel manipulators with three and four dof. To this end, the theory of groups, as first proposed by Hervé in 1978, was used extensively, and the concepts associated with kinematic chains in the same context were discussed. In this vein, various Schönflies-motion generators were recalled, and new kinematic bonds producing these were proposed. The concepts were illustrated with various examples.

Acknowledgements

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