Example 4.4.1

$$Q(\vec{x}) = 6x_1^2 + 2x_2^2 - 6x_1x_2 - x_1 - 2x_2 \rightarrow \min_{x_1, x_2}$$

$\underset{\sim}{A}, \vec{b}, c = ?$

$$\nabla Q = \begin{bmatrix} 12x_1 - 6x_2 - 1 \\ 4x_2 - 6x_1 - 2 \end{bmatrix} \Rightarrow \nabla\nabla Q = \begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix} = \underset{\sim}{A}$$

$\vec{b}$ = vector coeff of linear terms $= \begin{bmatrix} -1 \\ -2 \end{bmatrix}$, $c = 0$

Prove that $\underline{A}$ is p.d.

Assume $\underset{\sim}{B}$ is $\triangledown$: $\underset{\sim}{B} = \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix}$

$$\Rightarrow \underset{\sim}{B}^T \underset{\sim}{B} = \begin{bmatrix} b_{11} & 0 \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix} = \begin{bmatrix} b_{11}^2 & b_{11}b_{12} \\ b_{11}b_{12} & b_{12}^2 + b_{22}^2 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix} \Rightarrow \begin{cases} b_{11}^2 = 12 \Rightarrow b_{11} = 2\sqrt{3} \ (\text{or} -2\sqrt{3}) \\ b_{11}b_{12} = -6 \Rightarrow b_{12} = -\dfrac{6}{2\sqrt{3}} = -\sqrt{3} \\ b_{12}^2 + b_{22}^2 = 4 \Rightarrow b_{22}^2 = 4 - 3 = 1 \end{cases}$$

$$\Rightarrow b_{22} = 1$$

$\Rightarrow \underline{B} = \begin{bmatrix} 2\sqrt{3} & -\sqrt{3} \\ 0 & 1 \end{bmatrix}$, $\underset{\sim}{B}^{-1} = ?$ $\quad \det(\underset{\sim}{B}) = 2\sqrt{3}$

$$\Rightarrow \underset{\sim}{B}^{-1} = \frac{1}{2\sqrt{3}} \begin{bmatrix} 1 & \sqrt{3} \\ 0 & 2\sqrt{3} \end{bmatrix} = \frac{\sqrt{3}}{6} \begin{bmatrix} 1 & \sqrt{3} \\ 0 & 2\sqrt{3} \end{bmatrix} = \frac{1}{6} \begin{bmatrix} \sqrt{3} & 3 \\ 0 & 6 \end{bmatrix}$$

$$\Rightarrow \underset{\sim}{B}^{-T} \vec{b} = \frac{1}{6} \begin{bmatrix} \sqrt{3} & 0 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -\sqrt{3} \\ -15 \end{bmatrix} = -\vec{c}$$

$$\Rightarrow Q(\vec{y}) = \frac{1}{2}(y_1^2 + y_2^2) - \frac{\sqrt{3}}{6} y_1 - \frac{15}{6} y_2 = Q_0$$

$\Rightarrow$ a family of concentric circles with centre at

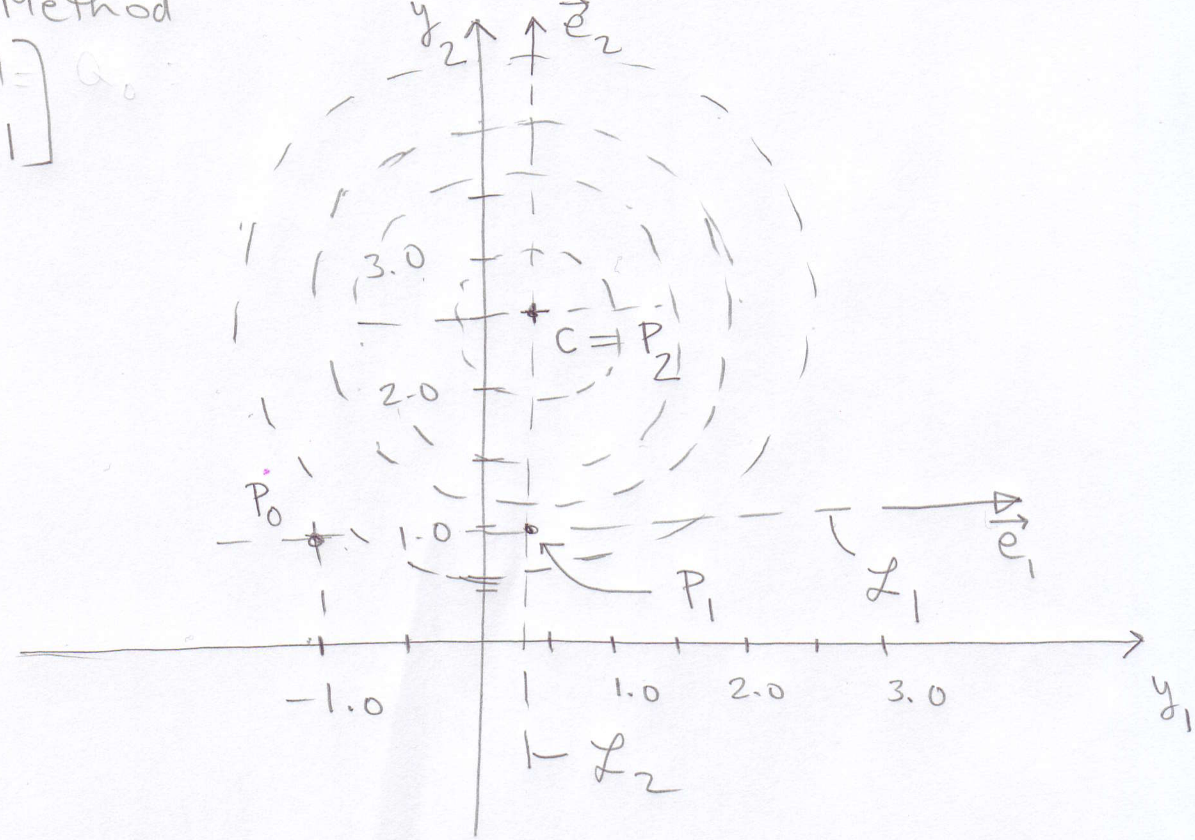
$$\vec{c} = -\underset{\sim}{B}^T \vec{b} = \frac{1}{6}\begin{bmatrix} \sqrt{3} \\ 15 \end{bmatrix} \Rightarrow \vec{y}_{opt} = \vec{c} = \frac{1}{6}\begin{bmatrix} \sqrt{3} \\ 15 \end{bmatrix}$$

Check:

$$\nabla Q(\vec{y})\Big|_{\vec{y}=\vec{c}} = \begin{bmatrix} y_1 - \sqrt{3}/6 \\ y_2 - 15/6 \end{bmatrix}\Bigg|_{y_1=\frac{\sqrt{3}}{6};\, y_2=\frac{15}{6}} = \begin{bmatrix} \frac{\sqrt{3}}{6} - \frac{\sqrt{3}}{6} \\ \frac{15}{6} - \frac{15}{6} \end{bmatrix} = \vec{0}_2 \quad \underline{\text{OK}}$$

Powell's Method

$$\vec{y}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



Search along $\mathcal{L}_1$:

$$\vec{y}_1 = \vec{y}_0 + \lambda \vec{e}_1 = \begin{bmatrix} -1+\lambda \\ 1 \end{bmatrix} \Rightarrow Q(\vec{y}_1) = \frac{1}{2}\left[(-1+\lambda)^2 + 1^2 - \frac{\sqrt{3}}{3}(-1+\lambda) - 5\right] \to \min_{\lambda}$$

$$\frac{dQ}{d\lambda} = (-1+\lambda) - \frac{\sqrt{3}}{6} = 0 \Rightarrow \lambda = 1 + \frac{\sqrt{3}}{6} \Rightarrow y_1 = \begin{bmatrix} \sqrt{3}/6 \\ 1 \end{bmatrix}$$

Search along $\mathcal{L}_2$: $\vec{y}_2 = \vec{y}_1 + \lambda \vec{e}_2 = \begin{bmatrix} \sqrt{3}/6 \\ 1+\lambda \end{bmatrix}$

$$\Rightarrow Q(\vec{y}_2) = \frac{1}{2}\left[\left(\frac{\sqrt{3}}{6}\right)^2 + (1+\lambda)^2 - \frac{\sqrt{3}}{3}\frac{\sqrt{3}}{6} - 5(1+\lambda)\right] \to \min_{\lambda}$$

$$\frac{dQ}{d\lambda} = 1+\lambda - \frac{5}{2} = 0 \Rightarrow \lambda = \frac{3}{2} \Rightarrow \vec{y}_2 = \begin{bmatrix} \sqrt{3}/6 \\ 5/2 \end{bmatrix} = \vec{c}$$

$$\Rightarrow \vec{x}_{opt} = \underset{\sim}{B}^{-1}\vec{c} = \frac{1}{6}\begin{bmatrix} \sqrt{3} & 3 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} \sqrt{3}/6 \\ 5/2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 8 \\ 15 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 5/2 \end{bmatrix}$$