MECH 577 Optimum Design

Project # 2 (Unconstrained Optimization) Optimum Design of a Quasi-homokinetic Joint

Assigned: October 4, 2010 Due: November 1st, 2010

Background

Homokinetic joints are used to transmit torque and motion between two shafts of intersecting axes with a 1:1 ratio. Conventional applications of homokinetic joints, e.g., in the automotive industry, call for both an unlimited rotatability of the two shafts and variations of the angle that they make. In robotics applications, however, rotatability requirements are confined to fractions of a full turn and a constant angle between shafts. Unlimited rotatability can be achieved with double universal joints or other, more compact transmissions, like Rzeppa joints. Moreover, in robotics applications an exact ratio of 1:1 is only desirable, but not required. Indeed, small deviations from a constant ratio can be accommodated by means of computer control.

In the design of what is known as parallel robots, i.e., robots that carry the payload by means of an assembly of limbs in parallel, all motors can be grounded, which is not the case in serial robots—robots with the morphology of the human arm, of concatenated joints. Grounding the motors, however, requires, sometimes, transmitting torque and motion from a vertical (horizontal) shaft to a horizontal (vertical) shaft, the two axes of the shafts normally intersecting at 90°. One common industrial solution to the problem of motion transmission between two shafts intersecting at right angles, to produce a constant ratio, is bevel gears. The problem with these is that: a) they are too noisy, unless expensive gears with spiral teeth are used; b) no matter how expensive, bevel gears always entail Coulomb friction between meshing teeth; and c) they exhibit backlash, which mars the effectiveness of the control system. For this reason, an alternative is sought in this project.

Project Statement

As an alternative to bevel gears, *Intelligent Robotics Inc.* (IRI) would like to use a spherical four-bar linkage, like the one shown in Fig. 1, to produce an *approximately* constant transmission ratio of 1:1. As homokineticity is not fully achieved, this linkage is termed *quasihomokinetic*. In the same figure, Z_1 and Z_2 are the axes of the *output* and the *input* shafts, respectively, ψ and ϕ being, correspondingly, the input and output angles.

From the kinematics of spherical four-bar linkages, it is known that the input and output angles are related by (Chiang, 1988)

$$F(\psi, \phi) \equiv k_1 + k_2 \cos\psi + k_3 \cos\psi \cos\phi - k_4 \cos\phi + \sin\psi \sin\phi = 0 \quad (1)$$

which is known as the IO equation, and whose coefficients k_i, for $i = 1, \ldots, 4$, are the Freudenstein parameters (FP) of the linkage, related to the linkage dimensions—arc lengths—shown in Fig. 1 by

$$\mathrm{c}\alpha_1 - k_3 = 0,\ \mathrm{c}\alpha_4 \mathrm{s}\alpha_1 - k_2 \mathrm{s}\alpha_4 = 0,\ \mathrm{c}\alpha_2 \mathrm{s}\alpha_1 - k_4 \mathrm{s}\alpha_2 = 0,\ \mathrm{c}\alpha_1 \mathrm{c}\alpha_2 \mathrm{c}\alpha_4 - \mathrm{c}\alpha_3 - k_1 \mathrm{s}\alpha_2 \mathrm{s}\alpha_4 = 0 \quad (2)$$

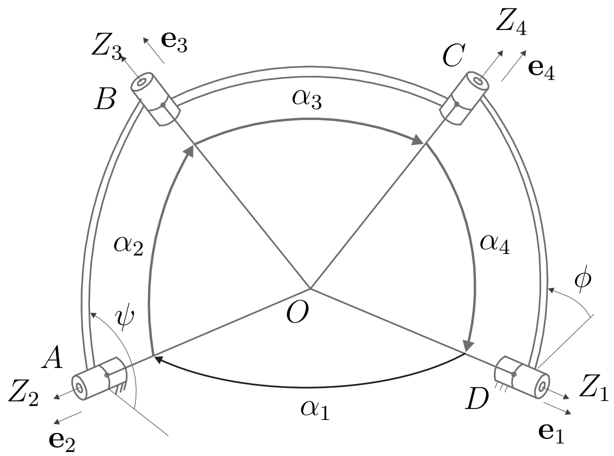

Figure 1: The kinematic chain of a four-bar spherical linkage

where $\mathrm{c}(\,\cdot\,) \equiv \cos(\,\cdot\,)$ and $\mathrm{s}(\,\cdot\,) \equiv \sin(\,\cdot\,)$.

Notice that the IO equation is linear in the four FP, and hence, a spherical four-bar linkage can be readily synthesized, i.e., its dimensions can be found, if four IO pairs $(\psi_i,\ \phi_i)$ are prescribed. Indeed, if these pairs are substituted into the IO equation, four equations linear in the FP can be found whose unique solution provides the numerical values required by the linkage at hand to meet the four prescribed IO relations. With the FP known, the linkage dimensions can be obtained from eqs.(2), as long as these equations yield real values for the dimensions sought[1]. In the case at hand, however, $\alpha_1 = 90°$, which leads to $k_3 = 0$, thereby reducing the number of FP to only three. Moreover, a clever IRI engineer has noted that, since *form follows function*, as she learned in school, and given the required ratio, the desired linkage should be symmetric—the input and output shafts can exchange their roles—which means that $\alpha_2 = \alpha_4$, and hence, $k_4 = k_2$, thereby reducing the number of independent FP to only two, k_1 and k_2.

If m IO pairs $\{\,\psi_i,\ \phi_i\,\}_1^m$ are assigned, then m synthesis equations linear in the 2-dimensional vector of FP are obtained, that can be cast in the standard form

$$\mathbf{S}\mathbf{k} = \mathbf{b}, \quad \mathbf{S} = \begin{bmatrix} 1 & \cos\psi_1 - \cos\phi_1 \\ 1 & \cos\psi_2 - \cos\phi_2 \\ \vdots & \vdots \\ 1 & \cos\psi_m - \cos\phi_m \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -\sin\psi_1\sin\phi_1 \\ -\sin\psi_2\sin\phi_2 \\ \vdots \\ -\sin\psi_m\sin\phi_m \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \tag{3}$$

For starters, set $m = 2$ and compute the linkage parameters thus resulting. Moreover, plot the output angle $\phi(\psi)$ in the range $0 \le \psi \le 2\pi/3$. To this end, notice that, if what is known as the *tan-half identities*, namely,

$$\cos\phi \equiv \frac{1 - T^2}{1 + T^2}, \quad \sin\phi \equiv \frac{2T}{1 + T^2}, \quad T \equiv \tan\left(\frac{\phi}{2}\right) \tag{4}$$

are introduced in the IO equation (1), then this equation becomes quadratic in T:

$$D(\psi)T^2 + 2E(\psi)T + F(\psi) = 0 \tag{5a}$$

[1]Note that, if, e.g., $|k_3| > 1.0$, then the first of eqs.(2) will yield a complex α_1.

with coefficients that are given below:

$$D(\psi) \equiv k_1 + (k_2 - k_3)\cos\psi + k_4,\ E(\psi) \equiv \sin\psi,\ F(\psi) \equiv k_1 + (k_2 + k_3)\cos\psi - k_4 \quad (5b)$$

Now, plotting the output angle ϕ vs. ψ reduces to solving the quadratic equation for ϕ, when the FP and ψ are given. Notice that two branches are possible. Moreover, plot the deviation $e(\psi)$ from the 1:1 ratio vs. ψ in the same interval: $0 \leq \psi \leq 2\pi/3$. The deviation should be zero at the two prescribed IO values.

The deviation can be reduced further if $m > 2$ IO pairs are prescribed, but this will result in an overdetermined system of linear equations (3), which can be satisfied only with an error $\mathbf{e} \equiv \mathbf{b} - \mathbf{Sk}$, termed the **design-error vector**. Solve this system for $m = 11$ and $m = 101$, while recording the weighted Euclidean norm of $\mathbf{e}$, with weighting matrix $(1/m)\mathbf{1}$, and $\mathbf{1}$ denoting the $m \times m$ identity matrix. This norm is termed the *design error* e, its square denoted by $f(\mathbf{k})$, the objective function to be minimized, i.e.,

$$f(\mathbf{k}) \equiv \frac{1}{m}\|\mathbf{e}\|_2^2 \quad \rightarrow \quad \min_{\mathbf{k}} \tag{6}$$

In order to validate the design-error vector of minimum weighted Euclidean norm, make sure that it verifies the first-order normality conditions (FONC), i.e., the gradient of f with respect to $\mathbf{k}$ must vanish.

The same clever engineer pointed out that there is no reason why, for the application envisaged, the input and output angles be measured from the Z_1-Z_2 plane. She suggests to measure these angles from zeros located on the dials of the input and output axes at ψ_0 and ϕ_0, respectively, thereby ending up with new values of IO pairs: $\{\, \psi_i^*,\, \phi_i^* \,\}_1^m$, ψ_i^* and ϕ_i^* being defined as $\psi_i^* = \psi_0 + \psi_i$ and $\phi_i^* = \phi_0 + \phi_i$, thereby gaining two additional parameters, the *shift angles* ψ_0 and ϕ_0, that should help the design team reduce further the design error.

Now, in order to find optimum values of the shift angles, two additional FONC must be verified: the gradient of f with respect to the shift-angle vector $\mathbf{l} = [\,\psi_0,\, \phi_0\,]^T$ must also vanish. The new vector of design variables, denoted by $\mathbf{x}$, can now be defined as a four-dimensional array, namely,

$$\mathbf{x} = \begin{bmatrix} \mathbf{k}^T & \mathbf{l}^T \end{bmatrix}^T \tag{7}$$

and the objective function now being $f(\mathbf{x})$, the optimization problem thus reading:

$$f(\mathbf{x}) \equiv \frac{1}{m}\|\mathbf{e}\|_2^2 \quad \rightarrow \quad \min_{\mathbf{x}} \tag{8}$$

Again, find optimum values of $\mathbf{x}$ that will minimize f, for $m = 11,\ 101$ pairs of prescribed input-output values, and validate the optimum thus obtained.

The project report should include an analysis of the results and a CAD rendering of the optimum linkage, preferably with an animation to verify its mobility.

Reference

Chiang, C.H., 1988, *Kinematics of Spherical Mechanisms*, Cambridge University Press, Cambridge, UK.