The physical meaning of the proposed optimality criterion is explained below: *When a structure is subject to a load with a given magnitude but applied in an arbitrary direction, the possible displacement field is defined by the stiffness ellipsoid of the stiffness matrix.* Therefore, the maximum displacement lies on the largest semi-axis of the ellipsoid, i.e., in the direction of the eigenvector associated with the minimum eigenvalue. Therefore, the minimax problem pertaining to the minimization of the maximum displacement thus leads to the problem of maximizing the minimum eigenvalue of the stiffness matrix.

Notice that the load **p** of Eq. (1) is of a *generalized* nature, for it lies in $\mathbb{R}^m$. In a finite element model, **p**, the nodal force, includes all Cartesian forces acting on the structure, some of which are constant, and some of which are either deterministic but variable, or stochastic.

3.1 Condensation. The concept of the proposed optimality criterion is extended further into a broader range of applications when combined with a *condensation* process.

Instances occur in a structural optimization problem where only the stiffness over a certain region is critical, whereas the stiffness elsewhere in the structure is unrelated to the design objective. For example, in the design of the Speed-o-Cam roller carrying disk, displacements of the nodes associated with the location of the roller pins affect the positioning of the rollers and, hence, affect the stiffness of the entire mechanism, whereas the displacements elsewhere of the disk are not. *Condensation* is a process that reduces the dimension of the stiffness matrix from the overall number of nodes to a reduced number of nodes, i.e., those associated with the critical region. Not only the dimension of the optimization problem is reduced significantly, but also the optimization problem can be formulated more accurately in following this approach.

Let $q<n$ denote the number of nodes at which the loads are applied; $\boldsymbol{\delta}_2$ the displacement vector of the loaded nodes; and $\boldsymbol{\delta}_1$ the displacement vector of the remaining nodes. Moreover, let $\mathbf{K}_{11}$, $\mathbf{K}_{12}$, and $\mathbf{K}_{22}$ be $(m-\nu q)\times(m-\nu q)$, $(m-\nu q)\times \nu q$, and $\nu q\times\nu q$ blocks of the stiffness matrix, respectively. In the foregoing relations, $\nu=2$ or 3, depending on whether the structure is planar or solid. Then, Eq. (1) can be expressed as

$$\begin{bmatrix}\mathbf{K}_{11} & \mathbf{K}_{12}\\ \mathbf{K}_{12}^T & \mathbf{K}_{22}\end{bmatrix}\begin{bmatrix}\boldsymbol{\delta}_1\\ \boldsymbol{\delta}_2\end{bmatrix}=\begin{bmatrix}\mathbf{f}_1\\ \mathbf{f}_2\end{bmatrix} \tag{7}$$

Equation (7), when expanded, becomes

$$\mathbf{K}_{11}\boldsymbol{\delta}_1+\mathbf{K}_{12}\boldsymbol{\delta}_2=\mathbf{f}_1$$

$$\mathbf{K}_{12}^T\boldsymbol{\delta}_1+\mathbf{K}_{22}\boldsymbol{\delta}_2=\mathbf{f}_2$$

Upon elimination of $\boldsymbol{\delta}_1$,

$$(\mathbf{K}_{22}-\mathbf{K}_{12}^T\mathbf{K}_{11}^{-1}\mathbf{K}_{12})\boldsymbol{\delta}_2+\mathbf{K}_{12}^T\mathbf{K}_{11}^{-1}\mathbf{f}_1=\mathbf{f}_2 \tag{8}$$

Since $\mathbf{f}_1=\mathbf{0}$, Eq. (8) reduces to

$$\mathbf{K}_c^*\boldsymbol{\delta}_2=\mathbf{f}_2 \tag{9}$$

where $\mathbf{K}_c^*$ is the $\nu q\times\nu q$ *condensed stiffness matrix* defined as

$$\mathbf{K}_c^*=\mathbf{K}_{22}-\mathbf{K}_{12}^T\mathbf{K}_{11}^{-1}\mathbf{K}_{12} \tag{10}$$

In normalized form, the above matrix becomes the *normalized condensed (NC) stiffness matrix* $\tilde{\mathbf{K}}_c^*$ given by

$$\tilde{\mathbf{K}}_c^*=q\mathbf{K}_c^* \tag{11}$$

for the number of nodes around the critical region is q. Equation (9) thus becomes

$$\tilde{\mathbf{K}}_c^*\tilde{\boldsymbol{\delta}}_2=\mathbf{f}_2 \tag{12}$$

Assuming that $\|\mathbf{f}_2\|$ is constant, we have

$$\tilde{\boldsymbol{\delta}}_2^T\tilde{\mathbf{K}}_c^{*T}\tilde{\mathbf{K}}_c^*\tilde{\boldsymbol{\delta}}_2=\|\mathbf{f}_2\|^2 \tag{13}$$

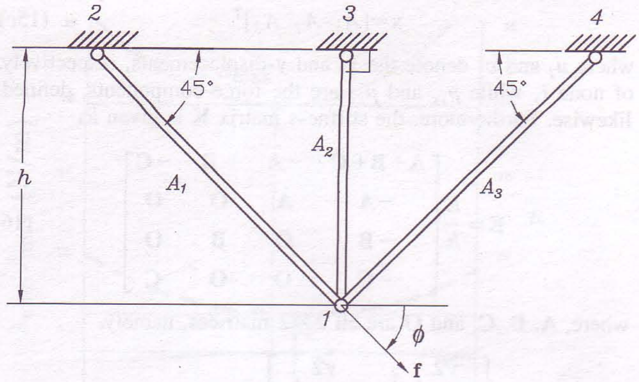

Fig. 3 A three-bar truss subject to a force of bounded magnitude but of arbitrary orientation ϕ

where, for a given level of loading, $\|f_2\|^2=$const. The above equation represents an ellipsoid in the space of the normalized critical displacement $\tilde{\boldsymbol{\delta}}_2$. Any point on the boundary of the ellipsoid denotes the *root-mean-square* of the components of the possible nodal displacements along the critical region. We can now state the criterion below:

Criterion 3.2: For a given amount of material and an uncertain load whose Euclidean norm is of a known bound, but applied in an arbitrary direction, the structure with the maximum stiffness at a specific region is the one whose **normalized condensed stiffness matrix** *has the maximum smallest eigenvalue.*

The problem is thus reformulated as finding a structure of stiffness matrix $\tilde{\mathbf{K}}_c^*$, with the largest possible minimum eigenvalue $k_{\min}$. Moreover, this layout leads to a minimum deformation at the points of load application.

4 Examples

We illustrate the application of the proposed design concept with two examples, a three-bar truss and two serially coupled springs, both subject to uncertain loading conditions and isometry constraints. In the first example, the optimum solution occurs when the minimum eigenvalue equals the maximum eigenvalue, thus leading to an isotropic stiffness matrix. However, in the second example, the optimum solution does not yield an isotropic stiffness matrix.

Notice that, since the number of nodes in both examples is fixed, it is not necessary in these examples to normalize either the nodal displacement vectors nor the stiffness matrices.

4.1 Example 1: A Truss Under Uncertain Loading and an Isometry Constraint. Consider the truss of Fig. 3. The lengths of the three bars are $\sqrt{2}h$, h, and $\sqrt{2}h$, respectively. The structure is subject to a load $\mathbf{f}=[f_x\ f_y]^T$ with a magnitude $F=\sqrt{f_x^2+f_y^2}$, applied at the free node, but oriented at an uncertain angle ϕ.

For conciseness, let the cross sections of the 1-2, 1-3, and 1-4 bars be all circular—other shapes can be equally considered—of areas A_1, A_2, and A_3, respectively, these areas playing the role of the design variables.

By fixing the amount of material, the structure is subject to the isometry condition

$$\sqrt{2}A_1+A_2+\sqrt{2}A_3=A=\text{const} \tag{14}$$

The design aims at determining the optimum values of A_1, A_2, and A_3 that yield a minimum deformation under *any* direction of the load. The nodal displacement vector $\boldsymbol{\delta}$, the nodal force vector **p**, and the design-variable vector **x** are defined in this case as

$$\boldsymbol{\delta}=[u_1\ v_1\ u_2\ v_2\ u_3\ v_3\ u_4\ v_4]^T \tag{15a}$$

$$\mathbf{p}=[p_{1x}\ p_{1y}\ p_{2x}\ p_{2y}\ p_{3x}\ p_{3y}\ p_{4x}\ p_{4y}]^T \tag{15b}$$

$$\mathbf{K}=\frac{E}{h}\begin{bmatrix}\mathbf{A}+\mathbf{B}+\mathbf{C} & -\mathbf{A} & -\mathbf{B} & -\mathbf{C}\\ -\mathbf{A} & \mathbf{A} & \mathbf{O} & \mathbf{O}\\ -\mathbf{B} & \mathbf{O} & \mathbf{B} & \mathbf{O}\\ -\mathbf{C} & \mathbf{O} & \mathbf{O} & \mathbf{C}\end{bmatrix} \tag{16}$$

where, **A**, **B**, **C**, and **O** are all 2×2 matrices, namely,

$$\mathbf{A}=\begin{bmatrix}\frac{\sqrt{2}}{4}A_1 & -\frac{\sqrt{2}}{4}A_1\\ -\frac{\sqrt{2}}{4}A_1 & \frac{\sqrt{2}}{4}A_1\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}0 & 0\\ 0 & A_2\end{bmatrix},$$

$$\mathbf{C}=\begin{bmatrix}\frac{\sqrt{2}}{4}A_3 & \frac{\sqrt{2}}{4}A_3\\ \frac{\sqrt{2}}{4}A_3 & \frac{\sqrt{2}}{4}A_3\end{bmatrix},\quad \mathbf{O}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \tag{17}$$

In addition, the applied force at node 1 and the displacements at nodes 2, 3, and 4 are prescribed as

$$p_{1x}=f_x,\quad p_{1y}=f_y,\quad u_2=v_2=u_3=v_3=v_4=u_4=0 \tag{18}$$

Upon substituting the above conditions into Eq. (18), the equilibrium equation becomes

$$\mathbf{K}_r\boldsymbol{\delta}=\mathbf{p}_r \tag{19}$$

with $\mathbf{K}_r$ defined as

$$\mathbf{K}_r=\frac{E}{h}\begin{bmatrix}\mathbf{A}+\mathbf{B}+\mathbf{C} & \mathbf{O}_{2\times 6}\\ \mathbf{O}_{2\times 6}^T & \mathbf{1}_{6\times 6}\end{bmatrix}$$

in which $\mathbf{O}_{2\times 6}$ is the 2×6 zero matrix and $\mathbf{1}_{6\times 6}$ is the 6×6 identity matrix, while $\mathbf{p}_r$ is defined as

$$\mathbf{p}_r=[\mathbf{f}^T\ \ \mathbf{0}_6^T]^T$$

where $\mathbf{0}_6$ is the six-dimensional zero vector. Therefore, the system of simultaneous equations defined in Eq. (19) can be condensed into a system of two equations in two unknowns, i.e.,

$$\mathbf{K}_c\boldsymbol{\delta}_c=\mathbf{p}_c \tag{20}$$

In Eq. (20), $\mathbf{K}_c$ and $\mathbf{p}_c$ represent the 2×2 condensed stiffness matrix and the two-dimensional condensed nodal force vector, while $\boldsymbol{\delta}_c$ is the condensed two-dimensional displacement vector, all these items being displayed below:

$$\mathbf{K}_c=\frac{E}{h}\begin{bmatrix}\frac{\sqrt{2}}{4}A_1+\frac{\sqrt{2}}{4}A_3 & -\frac{\sqrt{2}}{4}A_1+\frac{\sqrt{2}}{4}A_3\\ -\frac{\sqrt{2}}{4}A_1+\frac{\sqrt{2}}{4}A_3 & \frac{\sqrt{2}}{4}A_1+A_2+\frac{\sqrt{2}}{4}A_3\end{bmatrix}$$

$$\mathbf{p}_c\equiv\mathbf{f}=\begin{bmatrix}f_x\\ f_y\end{bmatrix},\quad \boldsymbol{\delta}_c=\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

Since $\|\mathbf{f}\|\equiv F$ is constant, we have

$$\boldsymbol{\delta}_c^T\mathbf{K}_c^T\mathbf{K}_c\boldsymbol{\delta}_c=\boldsymbol{\delta}_c^T\mathbf{K}_c^2\boldsymbol{\delta}_c\equiv F^2 \tag{21}$$

Equation (21) represents an ellipse in the space of $\boldsymbol{\delta}_c=[u_1,\ u_2]^T$. Hence, the maximum value of $\sqrt{u_1^2+u_2^2}$ occurs when $[u_1,\ u_2]^T$ is aligned with the largest semi-axis of the ellipse, i.e., when $[u_1,\ u_2]^T$ is directed along the eigenvector corresponding to the minimum eigenvalue of $\mathbf{K}_c^2$ or, correspondingly, of $\mathbf{K}_c$.

Fig. 4 A two-spring system

Let λ_1 and λ_2, with $\lambda_1\leq\lambda_2$, denote the eigenvalues of $\mathbf{K}_c^2$ and k_1 and k_2 with $k_1\leq k_2$ those of $\mathbf{K}_c$. We thus have $\lambda_1=k_1^2$ and $\lambda_2=k_2^2$. Now, Eq. (14) allows us to express A_2 in terms of A_1 and A_3, which then leads to the eigenvalues of $\mathbf{K}_c$ in terms of independent variables. Indeed, upon resorting to computer algebra, we obtain closed-form expressions for the eigenvalues of $\mathbf{K}_c$:

$$k_1=\frac{\sqrt{2}E}{8h}(-2A_1-2A_3+2\sqrt{2}A-2\sqrt{B})$$

$$k_2=\frac{\sqrt{2}E}{8h}(-2A_1-2A_3+2\sqrt{2}A+2\sqrt{B})$$

where A_2 has been eliminated by means of Eq. (14) and

$$B=5A_1^2+6A_1A_3-4\sqrt{2}AA_1+5A_3^2-4\sqrt{2}AA_3+2A^2$$

Since $\mathbf{K}_c$ is symmetric and positive-definite, k_1 and k_2 are bound to be real and positive. Hence, $B>0$, and

$$0<k_1\leq k_2$$

The *unconstrained* optimization problem at hand consists in determining the values of A_1 and A_3 that render $1/k_1$ a minimum, i.e.,

$$\min_{A_1,A_3}\frac{1}{k_1}\quad\text{or}\quad\max_{A_1,A_3}k_1 \tag{22}$$

The optimum solution must thus satisfy the first-order conditions

$$\frac{\partial k_1}{\partial A_1}=0,\quad \frac{\partial k_1}{\partial A_3}=0$$

whence the optimum values of A_1 and A_3 are found to be

$$A_1=A_3=\frac{\sqrt{2}}{4}A \tag{23}$$

which leads to $A_2=0$, while the stiffness matrix of the optimum structure takes the form

$$\mathbf{K}=k\mathbf{1} \tag{24}$$

with $\mathbf{1}$ denoting the 2×2 identity matrix and k given as

$$k=\frac{1}{4}\frac{EA}{h} \tag{25}$$

This result shows that at the optimum solution, the maximum and the minimum eigenvalues are equal, i.e., $\lambda_{\min}=\lambda_{\max}$. The stiffness matrix of the optimum structure in the presence of an uncertain load orientation is therefore *isotropic*. The optimum solution thus occurs, in this case, when the condition number κ of **K**, defined as the ratio $k_{\max}/k_{\min}$, attains its minimum value of unity.

4.2 Example 2: Two Serially Coupled Springs Under Uncertain Loading and an Isometry Constraint.

As shown in Fig. 4, two springs with spring constants denoted by K_1 and K_2, respectively, are serially coupled. Moreover, the system is subject to external forces f_1 and f_2, applied at nodes 1 and 2, respectively.

We assume that (i) the magnitude of the nodal force vector lies within a known bound P and (ii) the amount of material in the