McGill

Book Number /Cahier d'examen n° _____ de /of _____

EXAMINATION BOOK/CAHIER D'EXAMEN

Last Name/*Nom* : CLASS TEST 3

First Name/*Prénom* :

McGill ID/*N° de matricule McGill* :

Date of Exam/*Date de l'examen* : 2010/12/03 (yyyy/mm/dd) *(année/mois/jour)*

Subject and Course Code/*Sujet et code du cours* : MECH577 Optimum Design

Section/*Section* : Room/*Salle* : Row and Seat Number/*Rangée et siège* :

INSTRUCTIONS:

1. Fill in the above carefully.
2. Write your answers on the right-hand side of the exam book. Use the left-hand side for rough work and calculations.
3. Do not write in the margins.
4. If a page is accidentally left blank, write "P.T.O." on it.
5. Do not tear pages from the exam book.
6. At the time of the examination, you must not have in your possession any cellphones, books, calculators, dictionaries, notes or any other extraneous material unless otherwise indicated on the Exam Paper Cover instructions.
7. Put additional books inside the first book when submitting your exam.
8. **This book cannot be taken from the examination room.**

DIRECTIVES :

1. Remplissez soigneusement la section ci-dessus.
2. Écrivez vos réponses dans la section de droite du cahier d'examen. Utilisez la section de gauche pour l'ébauche et le calcul.
3. N'écrivez pas dans les marges.
4. Si vous avez laissé involontairement une page blanche, veuillez y inscrire « voir page suivante ».
5. Aucune page du cahier d'examen ne doit être retirée.
6. Durant l'examen, vous ne pouvez avoir en votre possession de cellulaire, livre, calculatrice, dictionnaire, note ou tout matériel superflu à moins d'indication contraire dans les directives indiquées sur la couverture de l'examen.
7. Veuillez insérer les cahiers additionnels à l'intérieur du premier cahier au moment de remettre votre examen.
8. **Le présent cahier doit demeurer dans la salle d'examen.**

For Examiner's Use Only Section réservée à l'examinateur	
1.	60
2.	40
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McGill University values academic integrity, which entails mutual respect, honesty, trust, fairness, and responsibility[1]. A healthy academic community can flourish only with intellectual and personal honesty during examinations. Thus, it is essential that the work submitted on examinations reflects one's own honest efforts.

L'Université McGill accorde beaucoup d'importance à l'intégrité universitaire, laquelle repose sur le respect mutuel, l'honnêteté, la confiance, l'équité et la responsabilité[1]. Un milieu universitaire sain ne peut prospérer que si les participants aux examens font preuve d'honnêteté intellectuelle et personnelle. Il est donc essentiel que les examens reflètent l'effort personnel de chacun.

[1] Center for Academic Integrity. *The Fundamental Values of Academic Integrity.* Raleigh, North Carolina: Duke University, 1999.

1 (a) $\vec{p}_{opt} = \underset{\sim}{C}^{+} \vec{d}$

$$\underset{\sim}{C}^{+} = \underset{\sim}{C}^{T} (\underset{\sim}{C} \underset{\sim}{C}^{T})^{-1} \vec{d}$$

$$\underset{\sim}{C} \underset{\sim}{C}^{T} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\Delta = \det(\underset{\sim}{C} \underset{\sim}{C}^{T}) = 9 - 1 = 8$$

$$\Rightarrow (\underset{\sim}{C} \underset{\sim}{C}^{T})^{-1} = \frac{1}{8} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

$$\Rightarrow \vec{p}_{opt} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{8} \underbrace{\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\begin{bmatrix} 4 \\ -4 \end{bmatrix}} = \frac{4}{8} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \vec{p}_{opt} = \frac{1}{2} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$ Ans.

(b) $$\underbrace{\underset{\sim}{C} \underset{\sim}{H}^{T}}_{(\underset{\sim}{H} \underset{\sim}{C}^{T})} \underbrace{\underset{\sim}{H} \vec{p}}_{\vec{q}} = \vec{d} \qquad (1)$$

$$\underset{\sim}{H} \underset{\sim}{C}^{T} = \begin{bmatrix} -\sqrt{3}/3 & -\sqrt{3}/3 & -\sqrt{3}/3 \\ -\sqrt{6}/6 & \sqrt{6}/3 & -\sqrt{6}/6 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\sqrt{3}}{3} \times 3 & -\frac{2\sqrt{3}}{3} + \frac{\sqrt{3}}{3} \\ -\frac{2\sqrt{6}}{6} + \frac{\sqrt{6}}{3} & -\frac{2\sqrt{6}}{6} - \frac{\sqrt{6}}{3} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{3} & -\sqrt{3}/3 \\ 0 & -2\sqrt{6}/3 \\ 0 & 0 \end{bmatrix}$$

Let $\vec{q} = \begin{bmatrix} \vec{q}_U \\ \vec{q}_L \end{bmatrix}$, $\vec{q}_U = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$, $\vec{q}_L = [q_3]$

$$(1) \Rightarrow \begin{bmatrix} -\sqrt{3} & 0 & 0 \\ -\sqrt{3}/3 & -2\sqrt{6}/3 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow q_1 = -1/\sqrt{3} = -\sqrt{3}/3$$

$$-\frac{\sqrt{3}}{3} q_1 - \frac{2\sqrt{6}}{3} q_2 = -1 \Rightarrow -\frac{2\sqrt{6}}{3} q_2 = -1 + \overbrace{\frac{\sqrt{3}}{3}\left(-\frac{\sqrt{3}}{3}\right)}^{-\frac{1}{3}}$$

$$\Rightarrow q_2 = -\frac{3}{2\sqrt{6}}\left(-1 - \frac{1}{3}\right) = -\frac{\cancel{3}\sqrt{6}}{\cancel{2}\times 6}\left(-\frac{\cancel{4}^{\,2}}{\cancel{3}}\right) = \frac{2\sqrt{6}}{6} = \frac{\sqrt{6}}{3}$$

$q_3 = 0$, for $\vec{q}$ of minimum Euclidean norm.

$$\Rightarrow \vec{q} = \frac{1}{3} \begin{bmatrix} -\sqrt{3} \\ \sqrt{6} \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{p}_{opt} = \underset{\sim}{H}^T \vec{q} = \begin{bmatrix} -\sqrt{3}/3 & -\sqrt{6}/6 & \sqrt{2}/2 \\ -\sqrt{3}/3 & \sqrt{6}/3 & 0 \\ -\sqrt{3}/3 & -\sqrt{6}/6 & -\sqrt{2}/2 \end{bmatrix} \frac{1}{3} \begin{bmatrix} -\sqrt{3} \\ \sqrt{6} \\ 0 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 - 1 \\ 1 + 2 \\ 1 - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \underline{\text{Ans}} \quad \text{OK}$$

2(a) The unconstrained minimum lies outside of the feasible region $\Rightarrow$ inequality constraint is active $\Rightarrow$ $g(\vec{x}) = 0$

$\Rightarrow$ Problem reduces to one of the equality-constrained type

KKT conds:

$$\begin{cases} \nabla f + \underset{\sim}{G}^T \vec{\mu} = \vec{0} & (1) \\ g(\vec{x}_{opt}) = x_2 - x_1 + \frac{1}{2} = 0 & (2) \\ \underset{\sim}{G} = [-1 \quad 1], \quad \vec{\mu} = [\mu], \quad \mu > 0 & (3) \end{cases}$$

$$\nabla f \Big|_{\vec{x}_{opt}} = 2 \begin{bmatrix} -200(x_2 - x_1^2)x_1 - 1 + x_1 \\ 100(\underbrace{x_2 - x_1^2}_{-0.250096}) \end{bmatrix}_{\vec{x}_{opt}}$$

$$= 2 \begin{bmatrix} -200\overbrace{(0.009800 - 0.509800^2)}^{-0.250096}0.509800 - 1 + 0.509800 \\ 100\underbrace{(0.009800 - 0.509800^2)}_{-0.250096} \end{bmatrix}$$

$$= 2 \begin{bmatrix} 25.0096 \\ -25.0096 \end{bmatrix} = \begin{bmatrix} 50.0192 \\ -50.0192 \end{bmatrix} \qquad (4)$$

$$(3) \Rightarrow \underset{\sim}{G}^T \vec{\mu} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} [\mu] = \begin{bmatrix} -\mu \\ \mu \end{bmatrix} \qquad (5)$$

$$(4) \,\&\, (5) \text{ into } (1) \Rightarrow \begin{cases} 50.0192 - \mu = 0 \\ -50.0192 + \mu = 0 \end{cases} \Rightarrow \mu = 50.0192 > 0$$

$g(\vec{x}_{opt}) = 0.009800 - 0.509800 + 0.500000 = -0.50000 + 0.500000 = 0$

$\Rightarrow$ Karush-Kuhn-Tucker conditions verified $\Rightarrow$ $\vec{x}_{opt}$ is acceptable.

Ans.

(b)

$$\nabla\nabla f\Big|_{\vec{x}_{opt}} = 2\begin{bmatrix} \overbrace{600\times 0.5098^2 - 200\times 0.0098}^{200(3\times 0.5098^2-0.0098)} + 1 & -200\times 0.5098 \\ -200\times 0.5098 & 100 \end{bmatrix}$$

$$= 2\underbrace{\begin{bmatrix} 154.978 & -101.960 \\ -101.960 & 100 \end{bmatrix}}_{\underset{\sim}{A}}$$

$tr(\underset{\sim}{A}) = 254.978 > 0$; $\det(\underset{\sim}{A}) = 5101.96 > 0$

$\Rightarrow \underset{\sim}{A}$ is positive-definite $\Rightarrow \underset{\sim}{H}_u$ bound to be p.d.

$\Rightarrow$ feasible stationary point $\vec{x}_{opt}$ is a local minimum

Ans.

For the record, $\underset{\sim}{H}_u$ is computed now:

Compute orthogonal complement of $\underset{\sim}{G}(\vec{x}_{opt})$

$$\underset{\sim}{L} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \underset{\sim}{H}_u = \underset{\sim}{L}^T \nabla\nabla f\Big|_{\vec{x}_{opt}} \underset{\sim}{L} =$$

$$= \underbrace{2\times\left(\frac{\sqrt{2}}{2}\right)^2}_{1}\begin{bmatrix} 1 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} 154.978 & -101.960 \\ -101.960 & 100 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\begin{bmatrix} 53.0180 \\ -1.96000 \end{bmatrix}}$$

$= [51.0580] > 0 \Rightarrow$ positive-definite. Check