305-573B Mechanics of Robotic Systems

Solution of Assignment 6

1 Dynamics of Complex Robotic Mechanical Systems

10.1 The condition for a symmetric matrix to be isotropic is that the matrix be proportional to the identity matrix, with a nonzero proportionality factor. Hence, it is apparent that for $\mathbf{I}(\boldsymbol{\theta})$ of Subsection 10.5.2 to be isotropic, all we need is $\beta = 0$, i.e.,[1]

$$H + 3J = 3m_w r^2 + 2m_p r^2 \tag{1}$$

which yields an equality constraint at the design stage among the set of parameters H, J, m_w, m_p, and r. If $\mathbf{I}(\boldsymbol{\theta})$ is isotropic, then we have

$$\mathbf{I}(\boldsymbol{\theta}) = \alpha \mathbf{1} \tag{2}$$

where $\mathbf{1}$ is the 3×3 identity matrix. Now, under condition (1), α becomes

$$\begin{aligned} \alpha &= I + \lambda^2 (3m_w + 2m_p + 15m_w + 4m_p) r^2 \\ &= I + \lambda^2 (18m_w + 6m_p) r^2 > 0 \end{aligned} \tag{3}$$

In order to gain more insight into the dynamics of the isotropic robot, let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

whose cross-product matrices are denoted by $\mathbf{U}_1$, $\mathbf{U}_2$ and $\mathbf{U}_3$, respectively, and need not displayed here. We thus have, now

$$\begin{aligned} \mathbf{I} &= \alpha \mathbf{1} \\ \mathbf{C}(\dot{\boldsymbol{\theta}}_a) &= \underbrace{2\sqrt{3}(\lambda^2(3m_w + m_p)r^2}_{\gamma} \omega \left(\sum_1^3 \mathbf{U}_i \right) \boldsymbol{\theta}_a \end{aligned}$$

where

$$\left(\sum_1^3 \mathbf{U}_i \right) \dot{\boldsymbol{\theta}}_a = \begin{bmatrix} -\dot{\theta}_1 + \dot{\theta}_3 \\ \dot{\theta}_1 - \dot{\theta}_3 \\ -\dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

The mathematical model in component form, thus reduces to

$$\alpha\ddot{\theta}_1 + \gamma\omega(-\dot{\theta}_2 + \dot{\theta}_3) = \tau_1 - \delta_1 \tag{4}$$

$$\alpha\ddot{\theta}_2 + \gamma\omega(\dot{\theta}_1 - \dot{\theta}_3) = \tau_2 - \delta_2 \tag{5}$$

$$\alpha\ddot{\theta}_3 + \gamma\omega(-\dot{\theta}_1 + \dot{\theta}_2) = \tau_3 - \delta_3 \tag{6}$$

[1]Notice typos in α and β in the book

Upon adding all three ODEs, we obtain an equation for ω:

$$3\alpha\dot{\omega} = \sum_1^3 \tau_i - \sum_1^3 \delta_i \Longrightarrow \omega = \frac{1}{3\alpha}\int_0^t [\tau_i(u) - \delta(u)]du$$

where u is a dummy variable of integration. Upon substitution of the above integral into the three odes,a system of three linear, time-varying ODEs are derived for three joint rates. The system, then, in state-variable from becomes

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}\mathbf{v}$$

where

$$\mathbf{x} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} \tau_1 - \delta_1 \\ \tau_2 - \delta_2 \\ \tau_3 - \delta_3\theta_3 \end{bmatrix}$$

And

$$\mathbf{A}(t) = -\frac{\gamma\omega(t)}{\alpha}\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \frac{1}{\alpha}\mathbf{1}$$

10.2 Under pure translation, the kinematic constraint can be written as

$$\omega = \frac{a}{2r}\sum_1^3 \dot{\theta}_i = 0$$

or alternatively,

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = 0$$

Now the robot has 2-dof, and hence $\dot{\boldsymbol{\theta}}_a$ can be expressed as

$$\dot{\boldsymbol{\theta}}_a = \mathbf{L}\mathbf{u}$$

where

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

And so the generalized inertia matrix under pure translation is

$$\begin{aligned} \mathbf{I}' &= \mathbf{L}^T\mathbf{I}\mathbf{L} \\ &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} \alpha & \beta & \beta \\ \beta & \alpha & \beta \\ \alpha & \beta & \alpha \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \\ &= (\alpha - \beta)\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{aligned}$$

10.4 The system mass and angular velocity matrices $\mathbf{M}$ and $\mathbf{W}$ are, in this case, of $6r \times 6r$. Moreover, $\mathbf{T}$ is of $6r \times n$. We can thus express the $6r$-dimensional twist vector $\mathbf{t}$ of the whole system as

$$\mathbf{t} = \mathbf{T}\dot{\boldsymbol{\theta}}_a, \qquad \dot{\mathbf{t}} = \mathbf{T}\ddot{\boldsymbol{\theta}}_a + \dot{\mathbf{T}}\dot{\boldsymbol{\theta}}_a \tag{7}$$

and hence, the $6r$ Newton-Euler equations of the whole set of bodies takes the form

$$\mathbf{M}(\mathbf{T}\ddot{\boldsymbol{\theta}}_a + \dot{\mathbf{T}}\dot{\boldsymbol{\theta}}_a) + \mathbf{WMT}\dot{\boldsymbol{\theta}}_a = \mathbf{w}^A + \mathbf{w}^G + \mathbf{w}^C - \mathbf{w}^D \tag{8}$$

with the usual definitions for $\mathbf{w}^A$, $\mathbf{w}^G$, $\mathbf{w}^C$, and $\mathbf{w}^D$. Now, $\mathbf{t}$ and $\mathbf{w}^C$ are reciprocal, in the sense that

$$\mathbf{t}^T\mathbf{w}^C = 0 \tag{9}$$

and, if we replace $\mathbf{t}$ in the above equation for its expression in eq.(7),

$$\dot{\boldsymbol{\theta}}_a^T\mathbf{T}^T\mathbf{w}^C = 0$$

which holds for any $\dot{\boldsymbol{\theta}}_a$, and hence,

$$\mathbf{T}^T\mathbf{w}^C = 0 \tag{10}$$

i.e., $\mathbf{w}^C$ lies in the nullspace of $\mathbf{T}^T$. Thus, upon multiplying both sides of eq.(8) from the left by $\mathbf{T}^T$, $\mathbf{w}^C$ disappears, and we obtain

$$\mathbf{T}^T\mathbf{M}(\mathbf{T}\ddot{\boldsymbol{\theta}}_a + \dot{\mathbf{T}}\dot{\boldsymbol{\theta}}_a) + \mathbf{T}^T\mathbf{WMT}\dot{\boldsymbol{\theta}}_a = \mathbf{T}^T\mathbf{w}^A + \mathbf{T}^T\mathbf{w}^G - \mathbf{T}^T\mathbf{w}^D$$

or

$$\mathbf{I}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}}_a + \mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a)\dot{\boldsymbol{\theta}}_a = \boldsymbol{\tau}^A + \boldsymbol{\gamma} - \boldsymbol{\delta} \tag{11}$$

with $\mathbf{I}(\boldsymbol{\theta})$ and $\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a)$ defined as in the holonomic case, except that now we distinguish between $\boldsymbol{\theta}$ and $\boldsymbol{\theta}_a$. Thus,

$$\mathbf{I}(\boldsymbol{\theta}) = \mathbf{T}^T\mathbf{MT} \tag{12}$$

$$\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a) = \mathbf{T}^T\mathbf{M}\dot{\mathbf{T}} + \mathbf{T}^T\mathbf{WMT} \tag{13}$$

Moreover, $\mathbf{T} = \mathbf{T}(\boldsymbol{\theta})$, $\dot{\mathbf{T}} = \dot{\mathbf{T}}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a)$, and $\mathbf{W} = \mathbf{W}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a)$. For brevity, we shall omit the arguments in the derivations below. Let us now calculate $\dot{\mathbf{I}}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a)$:

$$\dot{\mathbf{I}}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a) = \dot{\mathbf{T}}^T\mathbf{MT} + \mathbf{T}^T\dot{\mathbf{M}}\mathbf{T} + \mathbf{T}^T\mathbf{M}\dot{\mathbf{T}} \tag{14}$$

Furthermore, we recall the expression for $\dot{\mathbf{M}}$ given in Exercise 3.15 for one single body. For r bodies, $\dot{\mathbf{M}}$ takes on an identical form, except that, now, $\dot{\mathbf{M}}$, $\mathbf{W}$, and $\mathbf{M}$ are $6r \times 6r$ block-diagonal, i.e.,

$$\mathbf{M} = \mathrm{diag}(\mathbf{M}_1, \cdots, \mathbf{M}_r), \quad \mathbf{W} = \mathrm{diag}(\mathbf{W}_1, \cdots, \mathbf{W}_r), \quad \dot{\mathbf{M}} = \mathrm{diag}(\dot{\mathbf{M}}_1, \cdots, \dot{\mathbf{M}}_r)$$

eq.(14) thus yielding

$$\dot{\mathbf{I}}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a) = \dot{\mathbf{T}}^T\mathbf{MT} + \mathbf{T}^T\mathbf{M}\dot{\mathbf{T}} + \mathbf{T}^T(\mathbf{WM} - \mathbf{MW})\mathbf{T} \tag{15}$$

Upon substituting eq.(13) into eq.(15), we obtain

$$\dot{\mathbf{I}}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a) = \mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a) + \dot{\mathbf{T}}^T\mathbf{M}\mathbf{T} - \mathbf{T}^T\mathbf{MWT}$$

Hence,

$$\begin{aligned}\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a) &= \dot{\mathbf{I}}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a) - \mathbf{T}^T\mathbf{M}\dot{\mathbf{T}} + \mathbf{T}^T\mathbf{MWT} \\ &\equiv \frac{1}{2}\dot{\mathbf{I}}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a) + \mathbf{P}\end{aligned}$$

where

$$\begin{aligned}\mathbf{P} &\equiv \frac{1}{2}\dot{\mathbf{I}}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a) - \mathbf{T}^T\mathbf{M}\dot{\mathbf{T}} + \mathbf{T}^T\mathbf{MWT} \\ &= \frac{1}{2}\left[\dot{\mathbf{T}}^T\mathbf{MT} + \mathbf{T}^T\mathbf{M}\dot{\mathbf{T}} + \mathbf{T}^T(\mathbf{WM} - \mathbf{MW})\mathbf{T}\right] - \mathbf{T}^T\mathbf{M}\dot{\mathbf{T}} + \mathbf{T}^T\mathbf{MWT} \\ &= \frac{1}{2}\left[\dot{\mathbf{T}}^T\mathbf{MT} - \mathbf{T}^T\mathbf{M}\dot{\mathbf{T}} + \mathbf{T}^T(\mathbf{WM} + \mathbf{MW})\mathbf{T}\right]\end{aligned}$$

thereby obtaining the derived expression for $\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a)$.

10.6

(a) Because the robot undergoes pure translation, the kinematic constraint can be written in the from

$$\omega = \frac{r}{l}(\dot{\theta}_1 - \dot{\theta}_2) = 0$$

which yields

$$\dot{\boldsymbol{\theta}}_a = \mathbf{L}\dot{\theta}_1$$

where the matrix $\mathbf{L}$ is

$$\mathbf{L} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$$

The inertia matrix with pure translation is, then, a scalar I_a, namely,

$$I_a = \mathbf{L}^T\mathbf{I}\mathbf{L} = I_w^{'} + I_c^{'} + I_b^{'} + I_p^{'}$$

where

$$\begin{aligned}I_w^{'} &= 2(I + m_w r^2) \\ I_c^{'} &= \frac{3m_c r^2 \rho^2}{2} \\ I_b^{'} &= m_b d^2 \rho^2 \\ I_p^{'} &= m_p r^2\end{aligned}$$

(b) Now, for the case in which the midpoint of segment O_1O_2 is stationary, the kinematic constraint can be written as

$$\dot{\theta}_1 + \dot{\theta}_2 = 0 \Longrightarrow \dot{\boldsymbol{\theta}}_a = \mathbf{U}\dot{\theta}_1$$

with

$$\mathbf{U} = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

The expression of the inertia matrix in this case is

$$I_b = \mathbf{U}^T \mathbf{I} \mathbf{U} = I_w^{'} + I_c^{'} + I_b^{'} + I_p^{'}$$

where

$$\begin{aligned} I_w^{'} &= 2(I + m_w r^2 + 4(\rho\delta)^2 H) \\ I_c^{'} &= m_c r^2 \alpha^2 (6\cos^2\psi + \rho^2(1 - \cos^2\psi)) \\ I_b^{'} &= \alpha^2\rho^2(4I_b \sin^2\psi + m_b d^2(1 + 3\cos^2\psi)) \\ I_p^{'} &= 4I_p(\rho\delta)^2 + 4\lambda^2 m_p r^2 \end{aligned}$$

10.8 The Coriolis and centrifugal forces matrix $\mathbf{C}(\sigma, \boldsymbol{\theta}_a, \dot{\boldsymbol{\theta}}_a)\dot{\boldsymbol{\theta}}_a$ can be expressed as

$$\mathbf{C}(\sigma, \boldsymbol{\theta}_a, \dot{\boldsymbol{\theta}}_a)\dot{\boldsymbol{\theta}}_a = \mathbf{T}^T \mathbf{M} \dot{\mathbf{T}} \dot{\boldsymbol{\theta}}_a \equiv \left[\mathrm{vect}(\mathbf{T}^T \mathbf{M} \dot{\mathbf{T}})\right] \times \dot{\boldsymbol{\theta}}_a \tag{16}$$

where from eq.(10.88), we notice that $\mathbf{T}^T \mathbf{M} \dot{\mathbf{T}}$ is skew symmetric, its vector being

$$\mathrm{vect}(\mathbf{T}^T \mathbf{M} \dot{\mathbf{T}}) = \sqrt{3}\lambda^2(3m_w + m_p)r^2\omega \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{17}$$

By recalling eq.(10.74),

$$\mathrm{vect}(\mathbf{T}^T \mathbf{M} \dot{\mathbf{T}}) = 3\lambda^2(3m_w + m_p)r^2(-\lambda) \underbrace{\left(\sum_1^3 \dot{\theta}_i\right)}_{\sqrt{3}\mathbf{e}^T \dot{\boldsymbol{\theta}}_a} \mathbf{e}$$

with

$$\mathbf{e} \equiv \frac{\sqrt{3}}{3}[1 \quad 1 \quad 1]^T$$

Hence,

$$\mathbf{C}(\sigma, \boldsymbol{\theta}_a, \dot{\boldsymbol{\theta}}_a)\dot{\boldsymbol{\theta}}_a = -3\sqrt{3}\lambda^3(3m_w + m_p)r^2(\mathbf{e}^T \dot{\boldsymbol{\theta}}_a)\mathbf{e} \times \dot{\boldsymbol{\theta}}_a$$

Now, $\mathbf{C}(\sigma, \boldsymbol{\theta}_a, \dot{\boldsymbol{\theta}}_a)\dot{\boldsymbol{\theta}}_a$ vanishes under the following nontrivial conditions :

i) The mean value of $\{\dot{\theta}_i\}_1^3$ vanishes, i.e., $\dot{\boldsymbol{\theta}}_a$ is normal to **e**. This implies that the platform undergoes pure translation;

ii) All three wheel rates are identical, i.e., $\dot{\boldsymbol{\theta}}_a$ is parallel to **e**. This implies that the platform undergoes pure rotation.