305-573B Mechanics of Robotic Systems

Solution of Assignment 5

5.1 (a) The area A of the trapezoidal profile is

$$\begin{aligned} A &= \frac{1}{2}\tau_1 s'_{\max} + (\tau_2 - \tau_1)s'_{\max} + \frac{1}{2}(1-\tau_2)s'_{\max} \\ &= \frac{1}{2}(1-\tau_1+\tau_2)s'_{\max} \end{aligned}$$

We need then $A = 1$, and thus

$$\frac{1}{2}(1-\tau_1+\tau_2)s'_{\max} = 1$$

from which we obtain

$$s'_{\max} = \frac{2}{1-\tau_1+\tau_2} \tag{1}$$

(b) Using eq.(1), we have

$$s(\tau) = \begin{cases} \frac{\tau^2}{\tau_1(1-\tau_1+\tau_2)} & 0 \leq \tau \leq \tau_1 \\ \frac{2\tau-\tau_1}{1-\tau_1+\tau_2} & \tau_1 \leq \tau \leq \tau_2 \\ \frac{\tau^2+\tau_2^2-2\tau-\tau_1(\tau_2-1)}{(\tau_2-1)(1-\tau_1+\tau_2)} & \tau_2 \leq \tau \leq 1 \end{cases}$$

The plot of $s(\tau)$ vs. τ appears in Fig. 1(a). The decomposition of $s(\tau)$ into a linear part and a periodic part, is, then,

$$\begin{aligned} s_l(\tau) &= \tau \\ s_p(\tau) &= s(\tau) - \tau \end{aligned}$$

with $s_p(\tau)$ displayed in Fig. 1(b).

(c) For a periodic cubic spline, we have the conditions

$$s_1 = s_N \tag{2}$$

$$s'_1 = s'_N \tag{3}$$

$$s''_1 = s''_N \tag{4}$$

As explained in Section 5.6, condition (4) can be used to eliminate one unknown, namely s''_N, while condition (3) leads to an additional equation given by eq.(5.63). Thus, recalling the definitions of eqs.(5.58d–f), we have now the system

$$\mathbf{A}\mathbf{s}'' = 6\mathbf{C}\mathbf{s} \tag{5}$$

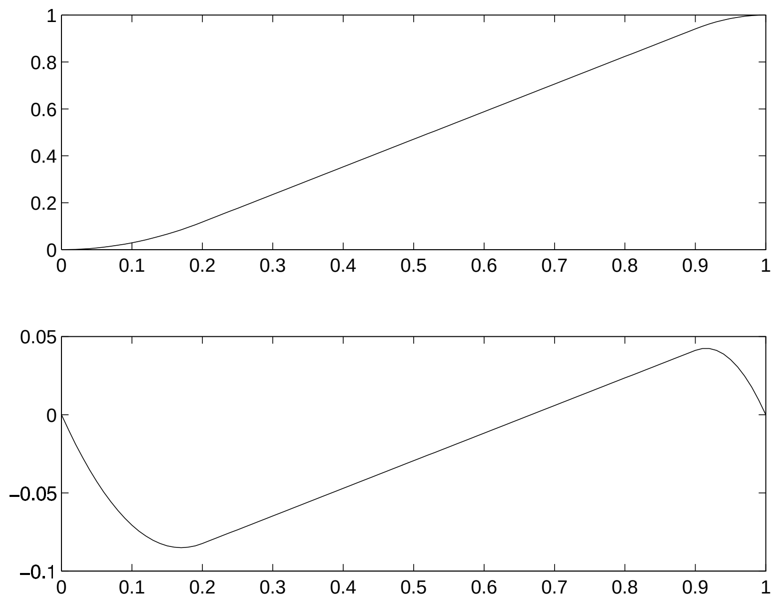

Figure 1:

where $\mathbf{A}$ and $\mathbf{C}$ are $(N-1)\times(N-1)$ matrices defined as:

$$\mathbf{A} = \begin{bmatrix} 2\alpha_{1,N'} & \alpha_1 & 0 & 0 & \cdots & \alpha_{N'} \\ \alpha_1 & 2\alpha_{1,2} & \alpha_2 & 0 & \cdots & 0 \\ 0 & \alpha_2 & 2\alpha_{2,3} & \alpha_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \alpha_{N'''} & 2\alpha_{N''',N''} & \alpha_{N''} \\ \alpha_{N'} & 0 & 0 & \cdots & \alpha_{N''} & 2\alpha_{N'',N'} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} -\beta_{1,N'} & \beta_1 & 0 & 0 & \cdots & \beta_{N'} \\ \beta_1 & -\beta_{1,2} & \beta_2 & 0 & \cdots & 0 \\ 0 & \beta_2 & -\beta_{2,3} & \beta_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_{N'''} & -\beta_{N''',N''} & \beta_{N''} \\ \beta_{N'} & 0 & 0 & \cdots & \beta_{N''} & -\beta_{N'',N'} \end{bmatrix}$$

and

$$\mathbf{s} = [\, s_1, \cdots, s_{N-1} \,]^T, \qquad \mathbf{s}'' = [\, s_1'', \cdots, s_{N-1}'' \,]^T$$

Moreover,

$$\Delta x_k = \frac{1}{N-1}, \qquad k = 1, \cdots, N-1$$

Thus, for $i, j, k = 1, \ldots, N-1$,

$$\alpha_k = \frac{1}{N-1}, \qquad \alpha_{i,j} = \frac{2}{N-1} \tag{6}$$

$$\beta_k = N-1, \qquad \beta_{i,j} = 2(N-1) \tag{7}$$

and matrices **A** and **C** reduce to

$$\mathbf{A} = \frac{1}{N-1}\begin{bmatrix} 4 & 1 & 0 & 0 & \cdots & 1 \\ 1 & 4 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 4 & 1 \\ 1 & 0 & 0 & \cdots & 1 & 4 \end{bmatrix}$$

$$\mathbf{C} = (N-1)\begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -2 & 1 \\ 1 & 0 & 0 & \cdots & 1 & -2 \end{bmatrix}$$

Now, vector $\mathbf{s} = [\, s_1, \cdots, s_{N-1}]^T$ is readily known from the N equally spaced points, while vector $\mathbf{s}''$ of eq.(5) is obtained as

$$\mathbf{s}'' = 6\mathbf{A}^{-1}\mathbf{C}\mathbf{s}$$

Then, the coefficient A_k, B_k, C_k and D_k of each cubic spline, for $k = 1, \dots, N-1$, are obtained directly using eqs.(5.55a–d). The *Matlab* code implementing the foregoing calculations is displayed below:

```
clear
N=10;
tau1=0.2;
tau2=0.9;

delta=1/(N-1);
for i=1:N-1,
   if i==1
      A(i,:)=[4,1,zeros(1,N-4),1]/(N-1);
      C(i,:)=[-2,1,zeros(1,N-4),1]*(N-1);
   elseif i==N-1
      A(i,:)=[1,zeros(1,N-4),1,4]/(N-1);
      C(i,:)=[1,zeros(1,N-4),1,-2]*(N-1);
   else
      A(i,:)=[zeros(1,i-2),1,4,1,zeros(1,N-2-i)]/(N-1);
      C(i,:)=[zeros(1,i-2),1,-2,1,zeros(1,N-2-i)]*(N-1);
   end
end

t=(0:delta:1-delta);

for i=1:length(t),
```

```
    if t(i)<=tau1
        s(i)=t(i)^2/(tau1*(1-tau1+tau2));
    elseif t(i)>=tau2
        s(i)=(t(i)^2+tau2^2-2*t(i)-tau1*(tau2-1))/((tau2-1)*(1-tau1+tau2));
    else
        s(i)=(2*t(i)-tau1)/(1-tau1+tau2);
    end
end

sl=t;
sp=s-t;

spp=6*inv(A)*C*sp';

t(N)=1;
sp(N)=sp(1);
spp(N)=spp(1);

step=delta/10;

for i=1:N-1,
    Ak(i)=(spp(i+1)-spp(i))/(6*delta);
    Bk(i)=spp(i)/2;
    Ck(i)=(sp(i+1)-sp(i))/delta-delta*(spp(i+1)+2*spp(i))/6;
    Dk(i)=sp(i);

    if i==N-1
        tk=((i-1)*delta:step:i*delta);
    else
        tk=((i-1)*delta:step:i*delta-step);
    end
    sk=Ak(i)*(tk-t(i)).^3+Bk(i)*(tk-t(i)).^2+Ck(i)*(tk-t(i))+Dk(i);
    sppk=6*Ak(i)*(tk-t(i))+2*Bk(i);
    tt=[tt,tk];
    spline=[spline,sk];
    splinepp=[splinepp,sppk];
end
```

The resulting periodic cubic spline and its acceleration profile are displayed in Figs. 2, for 8 supporting points. This number of supporting points gives a very good approximation of the original profile $s(\tau)$, while smoothing its acceleration profile. Moreover, the maximum acceleration value, about 8, is only slightly higher the the original acceleration level, which is about 6, and thus, seems quite reasonable. Also note that the maximum acceleration of the cycloidal motion is slightly over 6, namely, 2π.

5.3 Here we want to use cycloidal motions to smooth the joint-rate profile of Fig. 13 of the

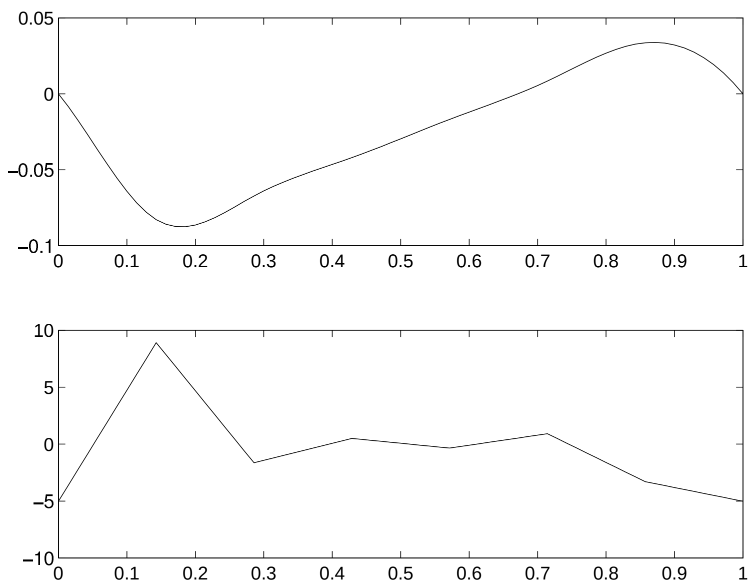

Figure 2:

Exercises. To this end, we define a segment of a cycloidal-motion function between $\tau = 0$ and $\tau = \tau_1$ such that $s'(\tau_1) = s'_{\rm max}$. We have, from eqs.(5.38a & b) of the text,

$$\begin{aligned} s(\tau) &= \tau - \frac{1}{2\pi}\sin 2\pi\tau \\ s'(\tau) &= 1 - \cos 2\pi\tau \end{aligned}$$

the last function being depicted in Fig. 3(a). From that figure, it is clear that the slope is horizontal, as required, when $\tau = 1/2$. However, we want to obtain this point at $\tau = \tau_1$, thereby requiring a change of variable to shrink the plot in the horizontal direction, as shown

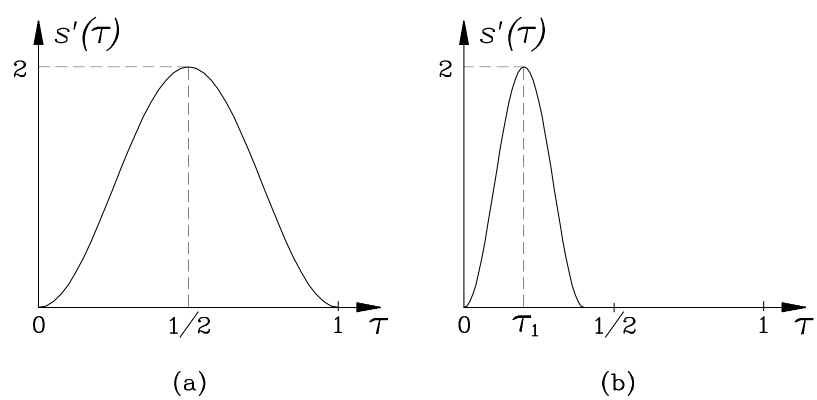

Figure 3:

in Fig. 3(b). This is done by defining a new variable u as

$$u \equiv 2\tau_1\tau$$

such that

$$\tau = \frac{u}{2\tau_1}$$

and hence

$$s'(u) = A[1 - \cos(\pi\frac{u}{\tau_1})]$$

where A is a constant to be determined. We have

$$s'(\tau_1) = A(1 - \cos\pi) = 2A = s'_{\text{max}}$$

and thus

$$A = \frac{s'_{\text{max}}}{2}$$

Finally, we have

$$s'(u) = \frac{s'_{\text{max}}}{2}[1 - \cos(\pi\frac{u}{\tau_1})] \tag{8}$$

Now, integrating this equation, under the condition that $s(0) = 0$, we obtain

$$s(u) = \frac{s'_{\text{max}}}{2}\left[u - \frac{\tau_1}{\pi}\sin(\pi\frac{u}{\tau_1})\right] \tag{9}$$

Differentiating eq.(8), we have

$$s''(u) = \frac{\pi}{\tau_1}\frac{s'_{\text{max}}}{2}\sin(\pi\frac{u}{\tau_1})$$

From τ_1 to τ_2, the velocity is kept constant and equal to s'_{max}. Therefore, we have

$$s'(\tau) = s'_{\text{max}}, \quad \text{for} \quad \tau_1 \leq \tau \leq \tau_2 \tag{10}$$

and thus

$$s(\tau) = s'_{\text{max}}\tau + C_1 \tag{11}$$

where C_1 is a constant of integration which can be determined by substituting $u = \tau_1$ into eq.(9) and using this value in eq.(11) as an initial condition, namely,

$$C_1 = -\frac{1}{2}s'_{\text{max}}\tau_1$$

Hence,

$$s(\tau) = s'_{\text{max}}(\tau - \frac{1}{2}\tau_1) \tag{12}$$

Moreover, differentiating eq.(10), we have

$$s''(\tau) = 0$$

Finally, for $\tau_2 \leq \tau \leq 1$, we define another cycloidal segment where $s'(\tau_2) = s'_{\text{max}}$ and $s'(1) = 0$. Here, we consider

$$s'(\tau) = 1 - \cos 2\pi\tau$$

In this case, we must shrink the plot of this equation, which is displayed in Fig. 3(a), such that the slope is horizontal at $\tau = \tau_2$. This is done by defining the variable u as

$$u \equiv 2(1 - \tau_2)\tau$$

such that

$$\tau = \frac{u}{2(1 - \tau_2)}$$

Finally, in order to have $s'(1) = 0$, we must shift this plot to the right by an amount $2\tau_2 - 1$, which is done by introducing the variable v defined as

$$v \equiv u + (2\tau_2 - 1)$$

such that

$$u = v - (2\tau_2 - 1)$$

Therefore, we have

$$s'(v) = B\left[1 - \cos\left(\pi\frac{v + 1 - 2\tau_2}{1 - \tau_2}\right)\right]$$

where B is a constant that can be determined using

$$s'(\tau_2) = B(1 - \cos\pi) = 2B = s'_{\text{max}}$$

and thus

$$B = \frac{s'_{\text{max}}}{2}$$

Hence

$$s'(v) = \frac{s'_{\text{max}}}{2}\left[1 - \cos\left(\pi\frac{v + 1 - 2\tau_2}{1 - \tau_2}\right)\right] \tag{13}$$

Now, if we integrate this equation, using eq.(12) evaluated at $\tau = \tau_2$, to obtain the initial condition, we have

$$s(v) = \frac{s'_{\text{max}}}{2}\left[v - \tau_1 + \tau_2 - \frac{1 - \tau_2}{\pi}\sin\left(\pi\frac{v + 1 - 2\tau_2}{1 - \tau_2}\right)\right]$$

Moreover, differentiating eq.(13), we have

$$s''(v) = \frac{\pi s'_{\text{max}}}{2(1 - \tau_2)}\sin\left(\pi\frac{v + 1 - 2\tau_2}{1 - \tau_2}\right)$$

In summary, we have

i) for $0 \le \tau \le \tau_1$

$$\begin{aligned}
s(\tau) &= \frac{s'_{\max}}{2}\left[\tau - \frac{\tau_1}{\pi}\sin(\pi\frac{\tau}{\tau_1})\right] \\
s'(\tau) &= \frac{s'_{\max}}{2}[1-\cos(\pi\frac{\tau}{\tau_1})] \\
s''(\tau) &= \frac{\pi}{\tau_1}\frac{s'_{\max}}{2}\sin(\pi\frac{\tau}{\tau_1})
\end{aligned}$$

ii) for $\tau_1 \le \tau \le \tau_2$

$$\begin{aligned}
s(\tau) &= s'_{\max}(\tau - \frac{1}{2}\tau_1) \\
s'(\tau) &= s'_{\max} \\
s''(\tau) &= 0
\end{aligned}$$

iii) for $\tau_2 \le \tau \le 1$

$$\begin{aligned}
s(\tau) &= \frac{s'_{\max}}{2}\left[\tau - \tau_1 + \tau_2 - \frac{1-\tau_2}{\pi}\sin\left(\pi\frac{\tau+1-2\tau_2}{1-\tau_2}\right)\right] \\
s'(\tau) &= \frac{s'_{\max}}{2}\left[1-\cos\left(\pi\frac{\tau+1-2\tau_2}{1-\tau_2}\right)\right] \\
s''(\tau) &= \frac{\pi s'_{\max}}{2(1-\tau_2)}\sin\left(\pi\frac{\tau+1-2\tau_2}{1-\tau_2}\right)
\end{aligned}$$

The plots of the displacement, velocity and acceleration profiles are displayed in Fig. 4(a), for $\tau_1 = 0.2$, $\tau_2 = 0.9$ and $s'_{\max} = 2/(1-\tau_1+\tau_2)$, as defined in Problem 5.1.

5.4 From the problem, the set of conditions for the initial and final poses are

$$\begin{aligned}
\mathbf{p}(0) &= \mathbf{p}_I, & \dot{\mathbf{p}}(0) &= \mathbf{0}, & \ddot{\mathbf{p}}(0) &= \mathbf{0} \\
\mathbf{Q}(0) &= \mathbf{Q}_I, & \boldsymbol{\omega}(0) &= \mathbf{0}, & \dot{\boldsymbol{\omega}}(0) &= \mathbf{0} \\
\mathbf{p}(T) &= \mathbf{p}_F, & \dot{\mathbf{p}}(T) &= \dot{\mathbf{p}}_F, & \ddot{\mathbf{p}}(T) &= \mathbf{0} \\
\mathbf{Q}(T) &= \mathbf{Q}_F, & \boldsymbol{\omega}(T) &= \mathbf{0}, & \dot{\boldsymbol{\omega}}(T) &= \mathbf{0}
\end{aligned}$$

In the absence of singularities, these conditions correspond to

$$\boldsymbol{\theta}(0) = \boldsymbol{\theta}_I, \quad \dot{\boldsymbol{\theta}}(0) = \mathbf{0}, \quad \ddot{\boldsymbol{\theta}}(0) = \mathbf{0} \tag{14}$$

$$\boldsymbol{\theta}(T) = \boldsymbol{\theta}_F, \quad \dot{\boldsymbol{\theta}}(T) = \dot{\boldsymbol{\theta}}_F, \quad \ddot{\boldsymbol{\theta}}(T) = \mathbf{0} \tag{15}$$

First, we consider a fifth-degree polynomial, namely,

$$\theta(t) = at^5 + bt^4 + ct^3 + dt^2 + et + f$$

its two first time-derivative being given by

$$\begin{aligned}
\dot{\theta}(t) &= 5at^4 + 4bt^3 + 3ct^2 + 2dt + e \\
\ddot{\theta}(t) &= 20at^3 + 12bt^2 + 6ct + 2d
\end{aligned}$$

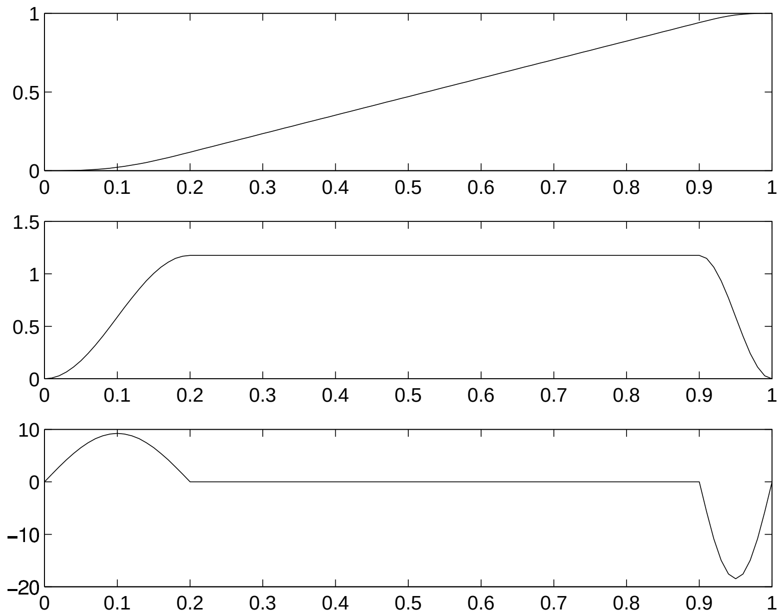

Figure 4:

If θ_i^I is the i-th component of $\boldsymbol{\theta}_I$, with the same notation for θ_i^F and $\dot{\theta}_i^F$, we have, from the conditions of eq.(14) for joint i,

$$d = e = 0, \qquad f = \theta_i^I$$

while the conditions of eq.(15) yield three linear equations in a, b, and c, namely,

$$\begin{aligned} aT^2 + bT + c &= \frac{\theta_i^F - \theta_i^I}{T^3} \\ 5aT^2 + 4bT + 3c &= \frac{\dot{\theta}_i^F}{T^2} \\ 20aT^2 + 12bT + 6c &= 0 \end{aligned}$$

Upon solving the three foregoing equations for the three aforementioned unknowns, we obtain

$$\begin{aligned} a &= \frac{6}{T^5}(\theta_i^F - \theta_i^I) - \frac{3}{T^4}\dot{\theta}_i^F \\ b &= -\frac{15}{T^4}(\theta_i^F - \theta_i^I) + \frac{7}{T^3}\dot{\theta}_i^F \\ c &= \frac{10}{T^3}(\theta_i^F - \theta_i^I) - \frac{4}{T^2}\dot{\theta}_i^F \end{aligned}$$

and hence, the fifth-degree polynomial sought is

$$\theta_i(t) = \theta_i^I + \left[\frac{6}{T^5}(\theta_i^F - \theta_i^I) - \frac{3}{T^4}\dot{\theta}_i^F\right] t^5 + \left[-\frac{15}{T^4}(\theta_i^F - \theta_i^I) + \frac{7}{T^3}\dot{\theta}_i^F\right] t^4 + \left[\frac{10}{T^3}(\theta_i^F - \theta_i^I) - \frac{4}{T^2}\dot{\theta}_i^F\right] t^3$$

To obtain the seventh-degree polynomial, we impose the condition that the joint jerk vanishes at the two endpoints, namely,

$$\boldsymbol{\theta}^{(3)}(0) = \mathbf{0}, \qquad \boldsymbol{\theta}^{(3)}(T) = \mathbf{0} \tag{16}$$

The polynomial has the form

$$\theta(t) = at^7 + bt^6 + ct^5 + dt^4 + et^3 + ft^2 + gt + h$$

and its three first time-derivatives are given by

$$\begin{aligned} \dot{\theta}(t) &= 7at^6 + 6bt^5 + 5ct^4 + 4dt^3 + 3et^2 + 2ft + g \\ \ddot{\theta}(t) &= 42at^5 + 30bt^4 + 20ct^3 + 12dt^2 + 6et + 2f \\ \theta^{(3)}(t) &= 210at^4 + 120bt^3 + 60ct^2 + 24dt + 6e \end{aligned}$$

From the conditions of eq.(14) and the first condition of eq.(16), we have

$$e = f = g = 0, \qquad h = \theta_i^I$$

while the conditions of eq.(15) and the second condition of eq.(16) yield four linear equations in a, b, c, and d, namely,

$$\begin{aligned} aT^3 + bT^2 + cT + d &= \frac{\theta_i^F - \theta_i^I}{T^4} \\ 7aT^3 + 6bT^2 + 5cT + 4d &= \frac{\dot{\theta}_i^F}{T^3} \\ 42aT^3 + 30bT^2 + 20cT + 12d &= 0 \\ 210aT^3 + 120bT^2 + 60cT + 24d &= 0 \end{aligned}$$

Upon solving the four foregoing equations for the four aforementionned unknowns, we obtain

$$\begin{aligned} a &= -\frac{20}{T^7}(\theta_i^F - \theta_i^I) + \frac{10}{T^6}\dot{\theta}_i^F \\ b &= \frac{70}{T^6}(\theta_i^F - \theta_i^I) - \frac{34}{T^5}\dot{\theta}_i^F \\ c &= -\frac{84}{T^5}(\theta_i^F - \theta_i^I) + \frac{39}{T^4}\dot{\theta}_i^F \\ d &= \frac{35}{T^4}(\theta_i^F - \theta_i^I) - \frac{15}{T^3}\dot{\theta}_i^F \end{aligned}$$

and hence, the seventh-degree polynomial sought is

$$\begin{aligned} \theta_i(t) = \; & \theta_i^I + \left[-\frac{20}{T^7}(\theta_i^F - \theta_i^I) + \frac{10}{T^6}\dot{\theta}_i^F\right] t^7 + \left[\frac{70}{T^6}(\theta_i^F - \theta_i^I) - \frac{34}{T^5}\dot{\theta}_i^F\right] t^6 \\ & + \left[-\frac{84}{T^5}(\theta_i^F - \theta_i^I) + \frac{39}{T^4}\dot{\theta}_i^F\right] t^5 + \left[\frac{35}{T^4}(\theta_i^F - \theta_i^I) - \frac{15}{T^3}\dot{\theta}_i^F\right] t^4 \end{aligned}$$

9.1 (a) Many solutions are possible for the location of the robot base. Obviously, the path must lie within the workspace of the robot. Additionally, two alternatives are possible: either the welding is done from outside or from within the helix. The workspace of the PUMA 560 is displayed in Fig. 5, along with a projection of the path. For an interior weld, we could have a path Γ^0:

$$\begin{aligned} x &= 0.3\cos\vartheta \\ y &= 0.3\sin\vartheta \\ z &= \frac{0.8\vartheta}{\pi} \end{aligned}$$

Here, the center[1] O_H of the helicoidal path has $\mathcal{F}_2$ coordinates $(0,0,0)$. For an exterior weld, we have a path Γ^1:

$$\begin{aligned} x &= 0.3\cos\vartheta - 0.5 \\ y &= 0.3\sin\vartheta - 0.5 \\ z &= \frac{0.8\vartheta}{\pi} + 0.33 \end{aligned}$$

with the center of the helicoidal path located at a point of $\mathcal{F}_2$ coordinates $(-0.5, -0.5, 0.33)$ m. Now, let us determine the Frenet-Serret vectors, independent of the center O_H. First, we determine the velocity along the helix:

$$\begin{aligned} \dot{x} &= -0.3\dot{\vartheta}\sin\vartheta \\ \dot{y} &= 0.3\dot{\vartheta}\cos\vartheta \\ \dot{z} &= \frac{0.8\dot{\vartheta}}{\pi} \end{aligned}$$

and the corresponding acceleration:

$$\begin{aligned} \ddot{x} &= -0.3\dot{\vartheta}^2\cos\vartheta - 0.3\ddot{\vartheta}\sin\vartheta \\ \ddot{y} &= 0.3\ddot{\vartheta}\cos\vartheta - 0.3\dot{\vartheta}^2\sin\vartheta \\ \ddot{z} &= \frac{0.8\ddot{\vartheta}}{\pi} \end{aligned}$$

The constant-speed condition (as in Example 9.3.1) leads to:

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v_0^2$$

where v_0 is the constant speed along the helix. Upon substitution of numerical values, the foregoing condition becomes

$$\begin{aligned} 0.3^2\dot{\vartheta}^2 + \left(\frac{0.8}{\pi}\right)^2\dot{\vartheta}^2 &= (0.050)^2 \\ \dot{\vartheta} &= 0.1271 \end{aligned}$$

[1]The center of the helix is defined as the intersection of its axis with the X_2-Y_2 plane.

and hence, with $c = 0.1271\ \mathrm{s}^{-1}$, we obtain

$$\begin{aligned}\dot{x} &= -0.3c\sin(ct)\\ \dot{y} &= 0.3c\cos(ct)\\ \dot{z} &= \frac{0.8c}{\pi}\end{aligned}$$

Following Example 9.3.1, since this part is independent of the robot,

$$\mathbf{e}_t \equiv \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \equiv \frac{\dot{\mathbf{r}}}{\dot{s}} = \frac{c}{v_o}\begin{bmatrix} -a\sin ct \\ a\cos ct \\ b \end{bmatrix} \quad \text{and} \quad \mathbf{e}_n = -\begin{bmatrix} \cos ct \\ \sin ct \\ 0 \end{bmatrix}$$

Thus, the binormal vector $\mathbf{e}_b$ is calculated simply as the cross product of the first two vectors of the Frenet-Serret triad:

$$\mathbf{e}_b \equiv \mathbf{e}_t \times \mathbf{e}_n = -\frac{c}{v_o}\begin{bmatrix} -b\sin ct \\ b\cos ct \\ -a \end{bmatrix}$$

The orientation matrix $\mathbf{Q}$ of the electrode tip is given by

$$\mathbf{Q} \equiv [\mathbf{e}_t \quad \mathbf{e}_n \quad \mathbf{e}_b]$$

Hence,

$$\mathbf{Q} = \frac{c}{v_o}\begin{bmatrix} -a\sin ct & -(v_o/c)\cos ct & b\sin ct \\ a\cos ct & -(v_o/c)\sin ct & -b\cos ct \\ b & 0 & a \end{bmatrix}$$

Now, the center of the wrist is located, with respect to the base, by the following vector:

$$\mathbf{c} = \mathbf{o}_H + \mathbf{p} + \mathbf{Q}\mathbf{c}_P$$

with $\mathbf{o}_H$ denoting the position vector of O_H, $\mathbf{p}$ that of an arbitrary point P on the helix, both given in $\mathcal{F}_2$, and $\mathbf{c}_P$, the position vector of P in the Frenet-Serret frame.

$$\mathbf{o}_H = \begin{bmatrix} -0.500 \\ -0.500 \\ 0.330 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} a\cos\vartheta \\ a\sin\vartheta \\ b\vartheta \end{bmatrix}, \quad \mathbf{c}_P = \begin{bmatrix} 0 \\ -0.050 \\ 0.0867 \end{bmatrix}$$

(b) Now, once $\mathbf{c}$ is available, we proceed with the inverse kinematics of the PUMA 560 as done in Section 4.4. Once we obtain $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$ can be found from Section 4.5 as

$$\dot{\boldsymbol{\theta}} = \mathbf{J}^{-1}\mathbf{t}$$

(c) The joint accelerations can be found from Section 4.6 as

$$\ddot{\boldsymbol{\theta}} = \mathbf{J}^{-1}(\dot{\mathbf{t}} - \dot{\mathbf{J}}\dot{\boldsymbol{\theta}})$$

The joint angle, velocity and acceleration trajectories appear in Figs. 6, 7 and 8, respectively, for the center of the helix at $(0, 0, 0)$, and in Figs. 9, 10 and 11, for the center of the helix at $(-0.5, -0.5, 0.33)$ m.

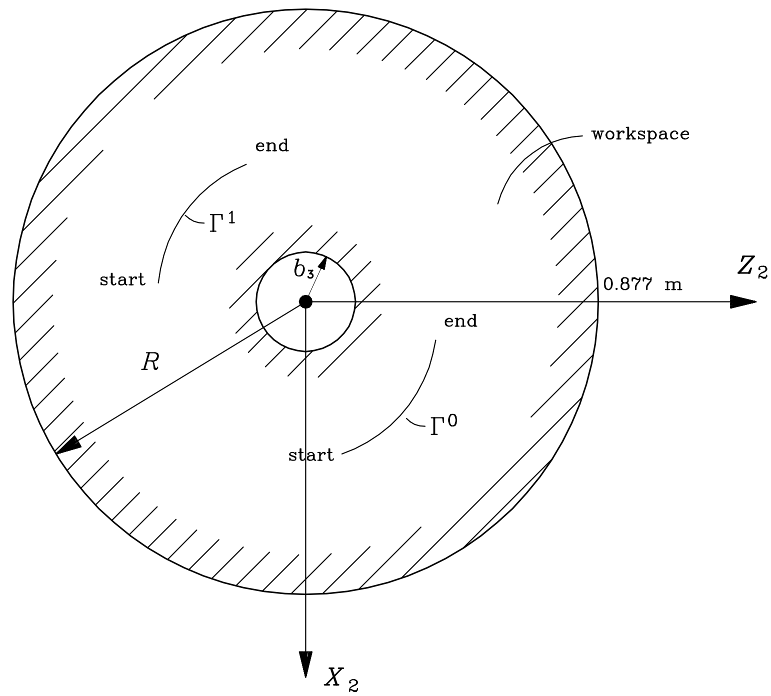

Figure 5: Top view of the workspace of the PUMA 560

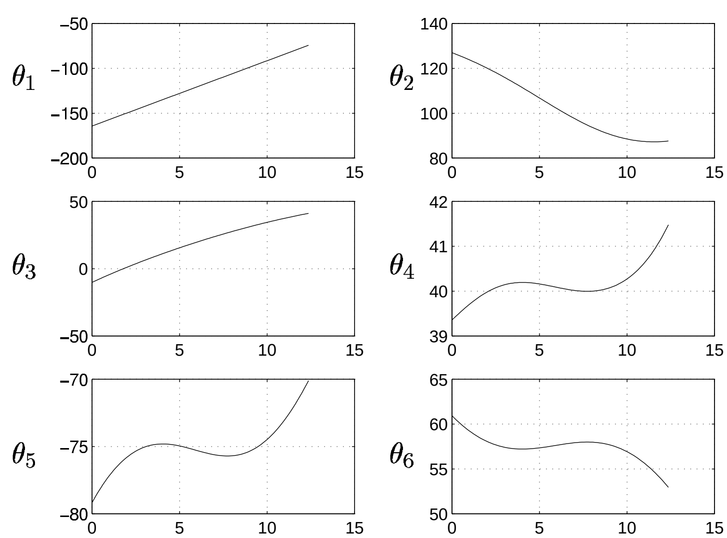

Figure 6: Joint trajectories vs. time (s) for the origin of the helix at $(0, 0, 0)$, in degrees.

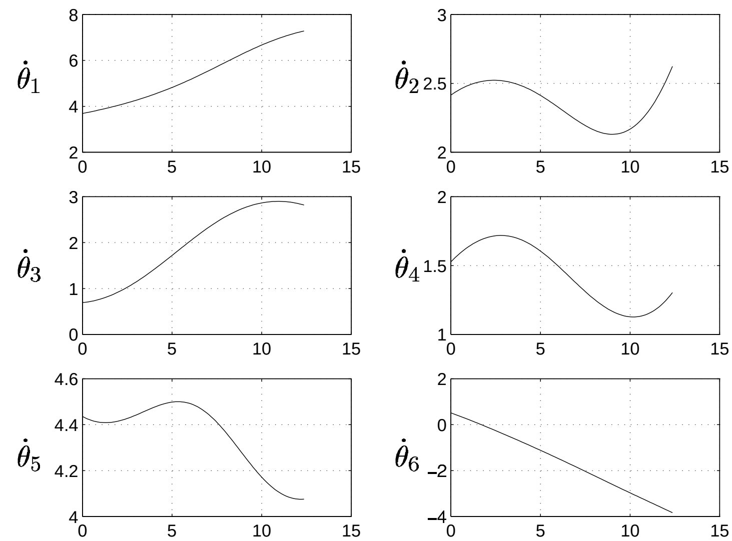

Figure 7: Joint velocities vs. time (s) for the origin of the helix at $(0, 0, 0)$, in rad/s.

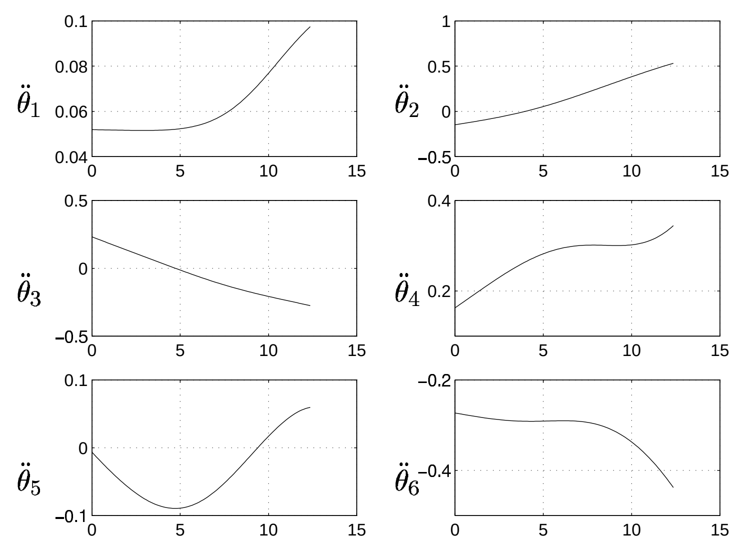

Figure 8: Joint accelerations vs. time (s) for the origin of the helix at $(0, 0, 0)$, in rad/s^2.

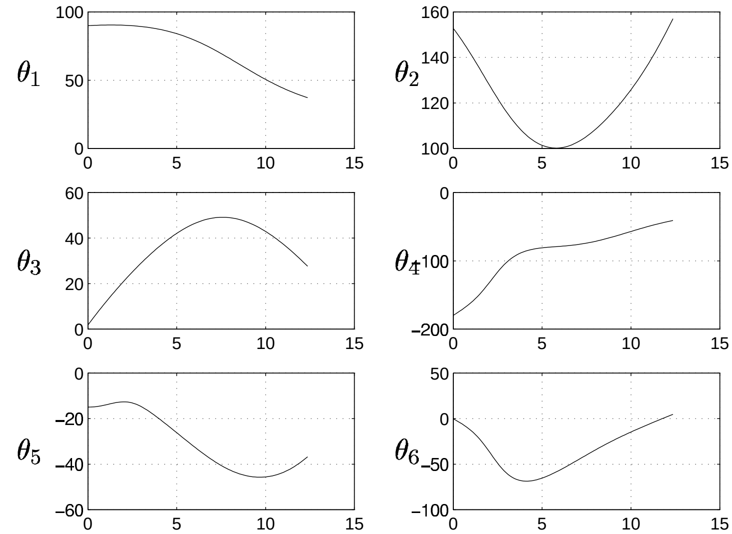

Figure 9: Joint trajectories vs. time (s) for the origin of the helix at $(-0.5, -0.5, 0.33)$ m, in degrees.

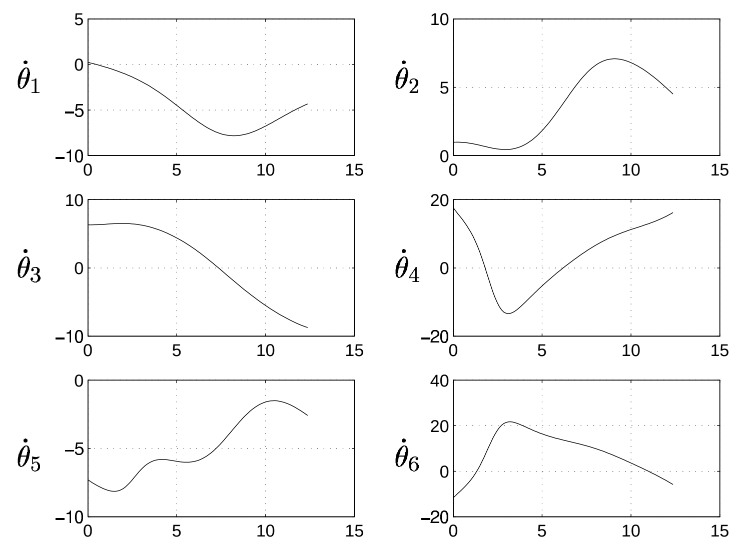

Figure 10: Joint velocities vs. time (s) for the origin of the helix at $(-0.5, -0.5, 0.33)$ m, in rad/s.

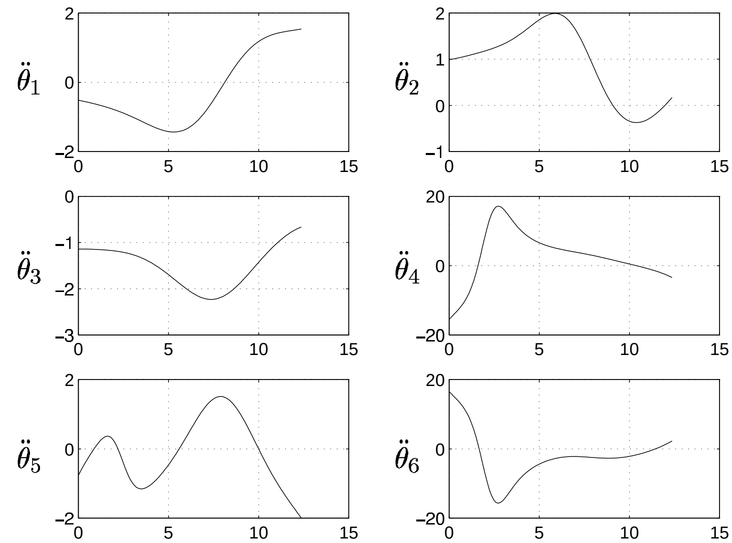

Figure 11: Joint accelerations vs. time (s) for the origin of the helix at $(-0.5, -0.5, 0.33)$ m, in rad/s^2.

9.3 We have

$$x = 2t, \quad y = t^2, \quad z = \frac{t^3}{3}$$

Then the position vector $\mathbf{r}$ of any point on the curve with its first and second-derivatives are

$$\mathbf{r} = \begin{bmatrix} 2t \\ t^2 \\ \frac{t^3}{3} \end{bmatrix}, \quad \dot{\mathbf{r}} \begin{bmatrix} 2 \\ 2t \\ t^2 \end{bmatrix}, \quad \ddot{\mathbf{r}} \begin{bmatrix} 0 \\ 2 \\ 2t \end{bmatrix}$$

Now the Frenet-Serret triad is readily evaluated as

$$\mathbf{e}_t = \frac{\dot{\mathbf{r}}}{\|\dot{\mathbf{r}}\|} = \frac{1}{t^2+2} \begin{bmatrix} 2 \\ 2t \\ t^2 \end{bmatrix}, \quad \mathbf{e}_b = \frac{\dot{\mathbf{r}} \times \ddot{\mathbf{r}}}{\|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}\|} = \frac{1}{t^2+2} \begin{bmatrix} t^2 \\ -2t \\ 2 \end{bmatrix}, \quad \mathbf{e}_n = \mathbf{e}_b \times \mathbf{e}_t = \frac{1}{t^2+2^2} \begin{bmatrix} -2t^3+4t \\ -t^4+4 \\ 2t^3+4t \end{bmatrix}$$

and hence, the orientation matrix $\mathbf{Q}$ is given by

$$\mathbf{Q} = \begin{bmatrix} \mathbf{e}_t & \mathbf{e}_n & \mathbf{e}_b \end{bmatrix} = \frac{1}{t^2+2} \begin{bmatrix} 2 & -2t & t^2 \\ 2t & 2-t^2 & -2t \\ t^2 & 2t & 2 \end{bmatrix}$$

Now the vector of matrix $\mathbf{Q}$ can be computed as

$$\text{vec}(\mathbf{Q}) = \frac{1}{2} \begin{bmatrix} \mathbf{Q}(3,2) - \mathbf{Q}(2,3) \\ \mathbf{Q}(1,3) - \mathbf{Q}(3,1) \\ \mathbf{Q}(2,1) - \mathbf{Q}(1,2) \end{bmatrix} = \frac{1}{t^2+2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \cos\phi = \frac{\text{tr}(\mathbf{Q}) - 1}{2} = \frac{2-t^2}{t^2+2}$$

The expressions for the curvature and torsion in terms of time are readily evaluated

$$\kappa = \frac{||\dot{\mathbf{r}} \times \ddot{\mathbf{r}}||}{||\dot{\mathbf{r}}||^3} = \frac{2}{t^2+2}, \quad \tau = \frac{\dot{\mathbf{r}} \times \ddot{\mathbf{r}} \cdot \mathbf{r}}{||\dot{\mathbf{r}} \times \ddot{\mathbf{r}}||^2} = \frac{2}{t^2+2}$$

The angular velocity $\boldsymbol{\omega}$ and angular acceleration $\dot{\boldsymbol{\omega}}$ take the forms

$$\boldsymbol{\omega} = \dot{s}\boldsymbol{\delta} = \frac{2}{t^2+2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \dot{\boldsymbol{\omega}} = \frac{-4t}{(t^2+2)^2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

where

$$\dot{s} = ||\dot{\mathbf{r}}|| = t^2 + 2, \quad \boldsymbol{\delta} = \frac{2}{(t^2+2)^2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$