The innermost do-loop, as pertaining to revolute manipulators, involves one coordinate transformation between *two consecutive coordinate frames*, from $\mathcal{F}_i$- to $\mathcal{F}_{i+1}$-coordinates, plus two vector sums, which consumes $4(n-i)$ multiplications and $6(n-i)$ additions; this loop also consumes one matrix-times-vector multiplication, with $\mathbf{E}$ being the said matrix, which involves zero floating-point operations, as mentioned above, and one scalar-times-vector multiplication, which requires $2(n-i)$ multiplications and zero additions. Thus, the total numbers of operations required by this algorithm, for a n-revolute manipulator, are M_{ia} multiplications and A_{ia} additions, as given below:

$$M_{ia} = 2n + \sum_{i=1}^{n} 6(n-i) = n(3n-1)$$

$$A_{ia} = \sum_{i=1}^{n} 6(n-i) = 3n(n-1)$$

the presence of prismatic pairs reducing the above figures.

8 Special Topics on Rigid-Body Kinematics

8.1 (a) We have, from Fig. 7,

$$\mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \\ 0 \end{bmatrix}, \quad \mathbf{p}_3 = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \\ 0 \end{bmatrix}, \quad \mathbf{p}_4 = \begin{bmatrix} 0 \\ \sqrt{3}/3 \\ \sqrt{6}/3 \end{bmatrix}$$

and from the data,

$$\dot{\mathbf{p}}_1 = [\,0 \quad \sqrt{3}/3 \quad \sqrt{6}/3\,]^T$$

Now, let us express $\dot{\mathbf{p}}_2$ in the form

$$\dot{\mathbf{p}}_2 = [\,\dot{x}_2 \quad 0 \quad 0\,]^T$$

Then, from the relation

$$(\dot{\mathbf{p}}_2 - \dot{\mathbf{p}}_1) \cdot (\mathbf{p}_2 - \mathbf{p}_1) = 0$$

we obtain $\dot{x}_2 = 1$. Hence,

$$\dot{\mathbf{p}}_2 = [\,1 \quad 0 \quad 0\,]^T$$

(b) Now, we must have, for compatibility, with $\dot{\mathbf{p}}_3 = [\dot{x}_3,\ \dot{y}_3,\ \dot{z}_3]^T$,

$$(\dot{\mathbf{p}}_3 - \dot{\mathbf{p}}_1) \cdot (\mathbf{p}_3 - \mathbf{p}_1) = -\frac{1}{2}\dot{x}_3 + \frac{\sqrt{3}}{2}\left(\dot{y}_3 - \frac{\sqrt{3}}{3}\right) = 0$$

which leads to

$$\dot{x}_3 = \sqrt{3}\,\dot{y}_3 - 1 \tag{130}$$

If $\dot{\mathbf{p}}_3 = \mathbf{0}$, we will have $\dot{x}_3 = \dot{y}_3 = \dot{z}_3 = 0$, which does not agree with eq.(130). Thus, $\dot{\mathbf{p}}_3$ cannot be zero.

(c) If $\dot{\mathbf{p}}_3$ lies in the $P_1P_2P_3$ plane, then $\dot{z}_3 = 0$. Again, we must have

$$(\dot{\mathbf{p}}_3 - \dot{\mathbf{p}}_2) \cdot (\mathbf{p}_3 - \mathbf{p}_2) = -(\dot{x}_3 - 1) = 0 \tag{131}$$

From eqs.(130) and (131), we have $\dot{x}_3 = 1$ and $\dot{y}_3 = 2\sqrt{3}/3$, i.e.,

$$\dot{\mathbf{p}}_3 = [\,1 \quad 2\sqrt{3}/3 \quad 0\,]^T$$

(d) Let the angular velocity be

$$\boldsymbol{\omega} = [\,\omega_1 \quad \omega_2 \quad \omega_3\,]^T$$

Then,

$$\dot{\mathbf{p}}_2 - \dot{\mathbf{p}}_1 = \boldsymbol{\omega} \times (\mathbf{p}_2 - \mathbf{p}_1)$$

That is

$$\begin{bmatrix} 1 \\ -\sqrt{3}/3 \\ -\sqrt{6}/3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -\sqrt{3}\,\omega_3 \\ \omega_3 \\ \sqrt{3}\,\omega_1 - \omega_2 \end{bmatrix}$$

from which we obtain

$$\omega_3 = -2\sqrt{3}/3 \tag{132}$$

$$\sqrt{3}\,\omega_1/2 - \omega_2/2 = -\sqrt{6}/3 \tag{133}$$

We also have

$$\dot{\mathbf{p}}_3 - \dot{\mathbf{p}}_1 = \boldsymbol{\omega} \times (\mathbf{p}_3 - \mathbf{p}_1)$$

which yields

$$\sqrt{3}\,\omega_1/2 + \omega_2/2 = -\sqrt{6}/3 \tag{134}$$

The solution of eqs.(132)–(134) is thus

$$\boldsymbol{\omega} = [\,-2\sqrt{2}/3 \quad 0 \quad -2\sqrt{3}/3\,]^T$$

Alternatively, we can apply eq.(8.9) to compute $\boldsymbol{\omega}$.

(e) Let $\mathbf{p}$ and $\dot{\mathbf{p}}$ be the position and velocity vectors of a point P on the instantaneous screw axis. We have

$$\dot{\mathbf{p}} = \dot{\mathbf{p}}_1 + \boldsymbol{\omega} \times (\mathbf{p} - \mathbf{p}_1) = \frac{1}{3}\begin{bmatrix} 2\sqrt{3} \\ \sqrt{3} - 2\sqrt{3}\,x + 2\sqrt{2}\,z \\ \sqrt{6} - 2\sqrt{2}\,y \end{bmatrix}$$

From $\dot{\mathbf{p}} \times \boldsymbol{\omega} = \mathbf{0}$, we have

$$\frac{1}{3}\begin{bmatrix} 3 - 6x + 2\sqrt{6}\,z \\ 2\sqrt{3} - 10y \\ -\sqrt{6} + 2\sqrt{6}\,x - 4z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{135}$$

Solving eq.(135) for y and z, we obtain $y = \sqrt{3}/5$ and $z = \sqrt{6}\,(2x-1)/4$. Thus, the distance between P and the origin is

$$d^2 = x^2 + y^2 + z^2 = \frac{2}{5}x^2 - \frac{3}{2}x + \frac{3}{8} + \frac{3}{25}$$

Now, d^2 is minimized by setting its derivative with respect to x equal to zero, thereby obtaining $x = 3/10$. Hence,

$$\mathbf{p} = [3/10 \quad \sqrt{3}/5 \quad -\sqrt{6}/10]^T \tag{136}$$

Moreover, we can readily obtain the unit vector $\mathbf{e}$ in the direction of the instant screw axis as

$$\mathbf{e} = \frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|} = \frac{3\sqrt{5}}{10}[\,-2\sqrt{2}/3 \quad 0 \quad -2\sqrt{3}/3\,]^T \tag{137}$$

From eqs.(136) and (137), the screw axis is totally determined. Alternatively, we can apply formulas (3.72) to find the above value of $\mathbf{p}$.

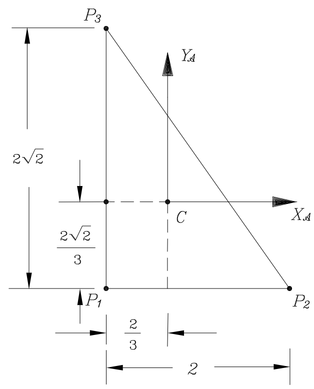

Figure 29: Auxiliary coordinate frame.

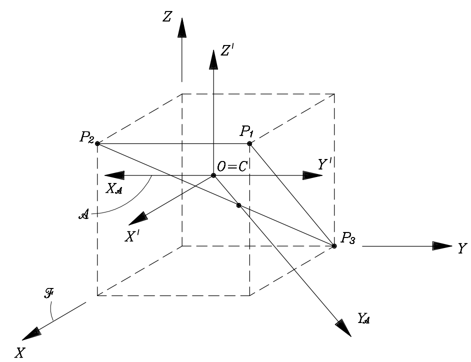

Figure 30: Relative layout of the $\mathcal{A}$ and $\mathcal{F}$ frames.

8.2 (a) We first verify compatibility, which requires computing matrices $\mathbf{P}$ and $\dot{\mathbf{P}}$, and these require, in turn,

$$\mathbf{c} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \dot{\mathbf{c}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Therefore,

$$\mathbf{P} = \frac{2}{3}\begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \quad \dot{\mathbf{P}} = 2\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

whence,

$$\dot{\mathbf{P}}^T\mathbf{P} = \mathbf{O}$$

which is apparently skew-symmetric, the motion thus being possible.

(b) To compute $\boldsymbol{\omega}$, we must verify first whether the points are collinear. A quick calculation shows that

$$(\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1) = 4\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \neq \mathbf{0}$$

the points thus being noncollinear. Moreover, $\mathrm{tr}(\mathbf{P}) = -2$, while

$$\mathbf{P}^2 = \frac{4}{3}\begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

and hence, $\mathrm{tr}^2(\mathbf{P}) = \mathrm{tr}(\mathbf{P}^2)$, even though the points are noncollinear. Thus, we cannot find $\boldsymbol{\omega}$ from eq.(8.9), for matrix $\mathbf{D}$ is singular. However, since $\mathbf{P}$ is not frame-invariant, it is possible to render $\mathbf{D}$ invertible upon a change of frame. Thus, let $\mathcal{A}$ be an auxiliary frame $\{X_{\mathcal{A}}, Y_{\mathcal{A}}, Z_{\mathcal{A}}\}$, with origin at C and axes $X_{\mathcal{A}}$ and $Y_{\mathcal{A}}$ defined as in Fig. 29. Hence,

$$[\mathbf{p}_1]_{\mathcal{A}} = \begin{bmatrix} -2/3 \\ -2\sqrt{2}/3 \\ 0 \end{bmatrix}, \quad [\mathbf{p}_2]_{\mathcal{A}} = \begin{bmatrix} 4/3 \\ -2\sqrt{2}/3 \\ 0 \end{bmatrix}, \quad [\mathbf{p}_3]_{\mathcal{A}} = \begin{bmatrix} -2/3 \\ 4\sqrt{2}/3 \\ 0 \end{bmatrix},$$

and

$$[\mathbf{P}]_{\mathcal{A}} = \begin{bmatrix} -2/3 & 4/3 & -2/3 \\ -2\sqrt{2}/3 & -2\sqrt{2}/3 & 4\sqrt{2}/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \mathrm{tr}([\mathbf{P}]_{\mathcal{A}}) = \frac{-2-2\sqrt{2}}{3} \Rightarrow \mathrm{tr}^2([\mathbf{P}]_{\mathcal{A}}) = \frac{12+8\sqrt{2}}{9}$$

Moreover,

$$[\mathbf{P}^2]_{\mathcal{A}} = \frac{1}{9}\begin{bmatrix} 4-8\sqrt{2} & -8-8\sqrt{2} & 4+16\sqrt{2} \\ 8+4\sqrt{2} & 8-8\sqrt{2} & -16+4\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathrm{tr}([\mathbf{P}^2]_{\mathcal{A}}) = \frac{12-16\sqrt{2}}{9} \neq \mathrm{tr}^2([\mathbf{P}]_{\mathcal{A}})$$

which means that matrix $[\mathbf{D}]_{\mathcal{A}}$ is invertible. In fact,

$$[\mathbf{D}]_{\mathcal{A}} = \frac{1}{3}\begin{bmatrix} -\sqrt{2} & -2 & 1 \\ \sqrt{2} & -1 & -2\sqrt{2} \\ 0 & 0 & -1-\sqrt{2} \end{bmatrix}$$

whose determinant is readily calculated as

$$\det([\mathbf{D}]_{\mathcal{A}}) = \frac{-2-\sqrt{(2)}}{9} \neq 0$$

In order to find $\boldsymbol{\omega}$, then, all we need is the right-hand side of eq.(8.9), which requires calculating the given velocities in $\mathcal{A}$. We show in Fig. 30 the relative layout between the original frame $\mathcal{F}$ and the auxiliary frame $\mathcal{A}$.

Let $\mathbf{i}_{\mathcal{A}}$, $\mathbf{j}_{\mathcal{A}}$, $\mathbf{k}_{\mathcal{A}}$ be the unit vectors parallel to $X_{\mathcal{A}}, Y_{\mathcal{A}}, Z_{\mathcal{A}}$, respectively, while $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are their counterparts associated with $\mathcal{F}$. From Fig. 30, it is apparent that

$$[\mathbf{i}_{\mathcal{A}}]_{\mathcal{F}} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad [\mathbf{j}_{\mathcal{A}}]_{\mathcal{F}} = \frac{[\overrightarrow{P_1P_3}]_{\mathcal{F}}}{\|[\overrightarrow{P_1P_3}]_{\mathcal{F}}\|}$$

where

$$[\overrightarrow{P_1P_3}]_{\mathcal{F}} = [\mathbf{p}_3 - \mathbf{p}_1] = \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}$$

Hence,

$$[\mathbf{j}_{\mathcal{A}}]_{\mathcal{F}} = \frac{\sqrt{2}}{2}\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \Rightarrow [\mathbf{k}_{\mathcal{A}}]_{\mathcal{F}} = [\mathbf{i}_{\mathcal{A}} \times \mathbf{j}_{\mathcal{A}}]_{\mathcal{F}} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Now, from Definition 2.2.1, the matrix $\mathbf{Q}$ rotating $\mathcal{F}$ into $\mathcal{A}$ is given by

$$[\mathbf{Q}]_{\mathcal{F}} = [\,\mathbf{i}_{\mathcal{A}} \quad \mathbf{j}_{\mathcal{A}} \quad \mathbf{k}_{\mathcal{A}}\,]_{\mathcal{F}} = \begin{bmatrix} 0 & -\sqrt{2}/2 & \sqrt{2}/2 \\ -1 & 0 & 0 \\ 0 & -\sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}$$

Therefore,

$$[\dot{\mathbf{p}}_i]_{\mathcal{A}} = [\mathbf{Q}^T]_{\mathcal{F}}[\dot{\mathbf{p}}_i]_{\mathcal{F}} \Rightarrow [\dot{\mathbf{P}}]_{\mathcal{A}} = [\mathbf{Q}^T]_{\mathcal{F}}[\dot{\mathbf{P}}]_{\mathcal{F}}$$

where

$$[\dot{\mathbf{P}}]_{\mathcal{F}} = 2\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow [\dot{\mathbf{P}}]_{\mathcal{A}} = 2\sqrt{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Thus,

$$\operatorname{vect}([\dot{\mathbf{P}}]_{\mathcal{A}}) = \sqrt{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and hence, eq.(8.9) leads to

$$\boldsymbol{\omega} = \mathbf{D}^{-1}\operatorname{vect}([\dot{\mathbf{P}}]_{\mathcal{A}})$$

with $\mathbf{D}^{-1}$ calculated, using computer algebra, as

$$\mathbf{D}^{-1} = \frac{2-\sqrt{2}}{2}\begin{bmatrix} -1-\sqrt{2} & 2(1+\sqrt{2}) & -1-4\sqrt{2} \\ -2-\sqrt{2} & -2-\sqrt{2} & 4-\sqrt{2} \\ 0 & 0 & -3\sqrt{2} \end{bmatrix}$$

whence,

$$[\boldsymbol{\omega}]_{\mathcal{A}} = \frac{2-\sqrt{2}}{2}\begin{bmatrix} -2-\sqrt{2} \\ -2-2\sqrt{2} \\ 0 \end{bmatrix}$$

Therefore,

$$[\boldsymbol{\omega}]_{\mathcal{F}} = [\mathbf{Q}]_{\mathcal{F}}[\boldsymbol{\omega}]_{\mathcal{A}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

8.3 If the three points are collinear, all three vectors $\mathbf{p}_i - \mathbf{c}$, for $i = 1, 2, 3$, are linearly dependent, and matrix $\mathbf{P}$ is of rank 1, its nullspace being a plane passing through the origin and normal to line $P_1P_2P_3$. Matrix $\mathbf{P}$ being of rank 1, its three eigenvalues are $\{\pi_1, 0, 0\}$, with $\pi_1 \neq 0$. Hence, $\operatorname{tr}^2(\mathbf{P}) = \pi_1^2 = \operatorname{tr}(\mathbf{P}^2)$. Notice that the foregoing relation holds regardless of whether the origin is collinear with the three points or not.

8.4 From the compatibility condition,

$$\dot{\mathbf{P}}^T\mathbf{P} + \mathbf{P}\dot{\mathbf{P}}^T = \mathbf{O}$$

Taking the trace of both sides, we obtain

$$\operatorname{tr}(\dot{\mathbf{P}}^T\mathbf{P} + \mathbf{P}\dot{\mathbf{P}}^T) = 0 \Rightarrow \operatorname{tr}(\dot{\mathbf{P}}^T\mathbf{P}) + \operatorname{tr}(\mathbf{P}\dot{\mathbf{P}}^T) = 0$$

Now, because $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{AB})$ for all square matrices $\mathbf{A}$ and $\mathbf{B}$, then

$$2\operatorname{tr}(\mathbf{P}\dot{\mathbf{P}}^T) = 0 \Longrightarrow \operatorname{tr}(\mathbf{P}\dot{\mathbf{P}}^T) = 0$$

Alternatively, one can write

$$\mathbf{P}\mathbf{P}^T = [\,\mathbf{p}_1 - \mathbf{c} \quad \mathbf{p}_2 - \mathbf{c} \quad \mathbf{p}_3 - \mathbf{c}\,] \begin{bmatrix} (\mathbf{p}_1 - \mathbf{c})^T \\ (\mathbf{p}_2 - \mathbf{c})^T \\ (\mathbf{p}_3 - \mathbf{c})^T \end{bmatrix} = \sum_1^3 (\mathbf{p}_i - \mathbf{c})(\mathbf{p}_i - \mathbf{c})^T$$

Upon differentiation of both sides with respect to time, we have

$$\dot{\mathbf{P}}^T\mathbf{P} + \mathbf{P}\dot{\mathbf{P}}^T = \sum_1^3 \left[(\dot{\mathbf{p}}_i - \dot{\mathbf{c}})(\mathbf{p}_i - \mathbf{c})^T + (\mathbf{p}_i - \mathbf{c})(\dot{\mathbf{p}}_i - \dot{\mathbf{c}})^T\right]$$

Taking the trace of both sides, while considering $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$,

$$2\,\mathrm{tr}(\mathbf{P}\dot{\mathbf{P}}^T) = 2\sum_1^3 (\dot{\mathbf{p}}_i - \dot{\mathbf{c}})^T(\mathbf{p}_i - \mathbf{c}) = 0$$

and hence, $\mathrm{tr}(\mathbf{P}\dot{\mathbf{P}}^T) = 0$.

8.7 (a) The data satisfy the relation $\mathrm{tr}^2(\mathbf{P}) = \mathrm{tr}(\mathbf{P}^2)$ with $\mathbf{P}$ as defined in eq.(8.4), and hence, a change of frame is required. We will consider a frame $\mathcal{F}_1$ with its origin at P_1 and its Z_1-axis perpendicular to the plane defined by P_1, P_2 and P_3. Moreover, its X_1-axis is aligned with a vector going from P_2 to P_1. In the new frame $\mathcal{F}_1$ we have

$$[\mathbf{p}_1]_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad [\mathbf{p}_2]_1 = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \quad [\mathbf{p}_3]_1 = \begin{bmatrix} 0 \\ -2\sqrt{2} \\ 0 \end{bmatrix}$$

Moreover,

$$[\mathbf{c}]_1 = \frac{1}{3}\sum_1^3 [\mathbf{p}_i]_1 = \frac{2}{3}\begin{bmatrix} -1 \\ -\sqrt{2} \\ 0 \end{bmatrix}$$

The rotation matrix that brings $\mathcal{F}_1$ into an orientation coincident with the original frame $\mathcal{F}$ is given by

$$\mathbf{Q} = \begin{bmatrix} 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{bmatrix}$$

The accelerations of the three given points in $\mathcal{F}_1$ are thus

$$[\ddot{\mathbf{p}}_i]_1 = \mathbf{Q}[\ddot{\mathbf{p}}_i]_{\mathcal{F}}, \qquad \text{for } i = 1, 2, 3$$

i.e.,

$$[\ddot{\mathbf{p}}_1]_1 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 0 \end{bmatrix}, \quad [\ddot{\mathbf{p}}_2]_1 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 2\sqrt{2} \end{bmatrix}, \quad [\ddot{\mathbf{p}}_3]_1 = \begin{bmatrix} 1 \\ \sqrt{2} \\ -2\sqrt{2} \end{bmatrix}$$

Moreover,

$$[\mathbf{P}]_1 = \frac{2}{3}\begin{bmatrix} 1 & -2 & 0 \\ \sqrt{2} & \sqrt{2} & -2\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}, \qquad [\mathbf{P}^2]_1 = \frac{4}{9}\begin{bmatrix} 1-2\sqrt{2} & -2-2\sqrt{2} & 4\sqrt{2} \\ 2+\sqrt{2} & 2-2\sqrt{2} & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Furthermore,

$$\mathrm{tr}[\mathbf{P}]_1 = \frac{2}{3}(1+\sqrt{2}), \quad \mathrm{tr}^2[\mathbf{P}]_1 = \frac{4}{9}(3+2\sqrt{2}), \quad \mathrm{tr}([\mathbf{P}^2]_1) = \frac{4}{9}(3-4\sqrt{2}) \neq \mathrm{tr}^2[\mathbf{P}]_1$$