(b) With $\mathbf{f}$ available, $\mathbf{n}_w$ is readily computed as

$$\mathbf{n}_w = (\mathbf{P} - \mathbf{C})\mathbf{f} = \begin{bmatrix} 0 & -0.5 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 50 \\ 125/b_3 \end{bmatrix} = \begin{bmatrix} -25 \\ 0 \\ 0 \end{bmatrix} \text{ Nm}$$

Now we calculate $\boldsymbol{\tau}_w$ from eq.(71), where we need $\mathbf{J}_{12}$. From Problem 4.19,

$$\mathbf{J}_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus,

$$\boldsymbol{\tau}_w = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -25 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -25 \\ 0 \end{bmatrix} \text{ Nm}$$

Therefore, the readings of the wrist joints are $\tau_4 = 0$, $\tau_5 = -25$ Nm, and $\tau_6 = 0$.

5.9 (a) Since we have here a decoupled manipulator, its Jacobian matrix at point C can be written as

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} \\ \mathbf{J}_{21} & \mathbf{O} \end{bmatrix}$$

with

$$\mathbf{J}_{11} = [\,\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{0}\,], \quad \mathbf{J}_{12} = [\,\mathbf{e}_4 \quad \mathbf{e}_5 \quad \mathbf{e}_6\,], \quad \mathbf{J}_{21} = [\,\mathbf{e}_1 \times \mathbf{r}_1 \quad \mathbf{e}_2 \times \mathbf{r}_2 \quad \mathbf{e}_3\,]$$

where $\mathbf{r}_i$ is the vector directed from O_i to the center of the wrist, C. With all quantities expressed in $\mathcal{F}_1$, we have

$$\mathbf{e}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_4 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_5 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{e}_6 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

and

$$\mathbf{r}_1 = \mathbf{r}_2 = \begin{bmatrix} -(b_3 + b_4) \\ 0 \\ 0 \end{bmatrix}$$

Thus,

$$\mathbf{e}_1 \times \mathbf{r}_1 = \begin{bmatrix} 0 \\ -(b_3 + b_4) \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 \times \mathbf{r}_2 = \begin{bmatrix} 0 \\ 0 \\ -(b_3 + b_4) \end{bmatrix}$$

Therefore,

$$\mathbf{J}_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{J}_{12} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{J}_{21} = \begin{bmatrix} 0 & 0 & -1 \\ -b_{34} & 0 & 0 \\ 0 & -b_{34} & 0 \end{bmatrix}$$

with b_{34} defined as $b_3 + b_4$, and

$$\mathbf{J} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -(b_3 + b_4) & 0 & 0 & 0 & 0 & 0 \\ 0 & -(b_3 + b_4) & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) The twist $\mathbf{t}_P$ of the EE at point P is given. However, we found in (a) the Jacobian for the decoupled manipulator with its EE twist defined at the center C of the wrist. Thus, the twist at the center of the wrist, $\mathbf{t}_C$, is required. This is obtained using the twist-transfer formula given by eqs.(3.84a & b), as

$$\mathbf{t}_C = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{P} - \mathbf{C} & \mathbf{1} \end{bmatrix} \mathbf{t}_P$$

where $\mathbf{C}$ and $\mathbf{P}$ are the cross-product matrices of the position vectors $\mathbf{c}$ and $\mathbf{p}$, respectively. Here,

$$\mathbf{c} = [\,-(b_3 + b_4) \quad 0 \quad 0\,]^T$$
$$\mathbf{p} = [\,-(b_3 + b_4) \quad -b_6 \quad 0\,]^T$$

Thus

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & (b_3 + b_4) \\ 0 & -(b_3 + b_4) & 0 \end{bmatrix}$$
$$\mathbf{P} = \begin{bmatrix} 0 & 0 & -b_6 \\ 0 & 0 & (b_3 + b_4) \\ b_6 & -(b_3 + b_4) & 0 \end{bmatrix}$$

and

$$\mathbf{P} - \mathbf{C} = \begin{bmatrix} 0 & 0 & -b_6 \\ 0 & 0 & 0 \\ b_6 & 0 & 0 \end{bmatrix}$$

Therefore,

$$\dot{\mathbf{c}} = (\mathbf{P} - \mathbf{C})\boldsymbol{\omega} + \dot{\mathbf{p}} = \omega \begin{bmatrix} -b_6 \\ 0 \\ b_6 \end{bmatrix} + v \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} v - b_6\omega \\ v \\ v + b_6\omega \end{bmatrix}$$

For a decoupled manipulator, we have

$$\mathbf{J}_{11}\dot{\boldsymbol{\theta}}_a + \mathbf{J}_{12}\dot{\boldsymbol{\theta}}_w = \boldsymbol{\omega}$$
$$\mathbf{J}_{21}\dot{\boldsymbol{\theta}}_a = \dot{\mathbf{c}}$$

Thus,

$$\dot{\boldsymbol{\theta}}_a = \mathbf{J}_{21}^{-1}\dot{\mathbf{c}} = \begin{bmatrix} 0 & -1/(b_3 + b_4) & 0 \\ 0 & 0 & -1/(b_3 + b_4) \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v - b_6\omega \\ v \\ v + b_6\omega \end{bmatrix}$$
$$= -\frac{1}{b_3 + b_4} \begin{bmatrix} v \\ v + b_6\omega \\ (b_3 + b_4)(v - b_6\omega) \end{bmatrix}$$

and

$$\dot{\boldsymbol{\theta}}_w = \mathbf{J}_{12}^{-1}(\boldsymbol{\omega} - \mathbf{J}_{11}\dot{\boldsymbol{\theta}}_a) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \left(\begin{bmatrix} \omega \\ \omega \\ \omega \end{bmatrix} - \frac{1}{b_3 + b_4} \begin{bmatrix} 0 \\ v + b_6\omega \\ -v \end{bmatrix} \right)$$
$$= \frac{-1}{b_3 + b_4} \begin{bmatrix} \omega(b_3 + b_4) \\ v + (b_3 + b_4)\omega \\ -v + (b_3 + b_4 - b_6)\omega \end{bmatrix}$$

(c) We have, for a decoupled manipulator,

$$\mathbf{J}_{11}^T\mathbf{n}_w + \mathbf{J}_{21}^T\mathbf{f} = \boldsymbol{\tau}_a$$
$$\mathbf{J}_{12}^T\mathbf{n}_w = \boldsymbol{\tau}_w$$

where $\mathbf{n}_w$ is the resultant moment acting on the EE when $\mathbf{f}$ is applied at the center C of the wrist. Therefore,

$$\mathbf{n}_w = \mathbf{n} + (\mathbf{p}-\mathbf{c})\times\mathbf{f} = T\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}0\\-b_6\\0\end{bmatrix}\times F\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}T-b_6F\\T\\T+b_6F\end{bmatrix}$$

Thus,

$$\boldsymbol{\tau}_w = \mathbf{J}_{12}^T\mathbf{n}_w = \begin{bmatrix}-1 & 0 & 0\\0 & 0 & -1\\0 & -1 & 0\end{bmatrix}\begin{bmatrix}T-b_6F\\T\\T+b_6F\end{bmatrix} = -\begin{bmatrix}T-b_6F\\T+b_6F\\T\end{bmatrix}$$

and

$$\begin{aligned}\boldsymbol{\tau}_a &= \mathbf{J}_{21}^T\mathbf{f} + \mathbf{J}_{11}^T\mathbf{n}_w\\ &= \begin{bmatrix}0 & -(b_3+b_4) & 0\\0 & 0 & -(b_3+b_4)\\-1 & 0 & 0\end{bmatrix}\begin{bmatrix}F\\F\\F\end{bmatrix} + \begin{bmatrix}0 & 0 & 1\\0 & -1 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}T-b_6F\\T\\T+b_6F\end{bmatrix}\\ &= \begin{bmatrix}T-(b_3+b_4-b_6)F\\-T-(b_3+b_4)F\\-F\end{bmatrix}\end{aligned}$$

whose first two components are torques, its third component being, consistently, a force.

5.10 The procedure outlined in the problem statement was followed using computer algebra. The characteristic equation, resulting from $\det(\mathbf{M}) = 0$, is thus obtained as

$$\begin{aligned}&\big(-C^2F^2 - 2\,C^2G^2 - 2\,A^2H^2 - 2\,H^2B^2 + 2\,C^2GF + 2\,CAFH + 4\,CGBH\\ &-2\,CFBH\big)\cos^2\theta_3 + \big(2\,A^2 + 2\,B^2 - 2\,D^2G^2 - D^2F^2 + 4\,IDGB - 2\,IDFB\\ &+ 2\,IADF + 2\,D^2GF\big)\sin^2\theta_3 + \big(2\,ICAF + 4\,ICGB + 4\,CGFD + 4\,DGBH\\ &-2\,DFBH + 2\,ADFH - 2\,ICFB - 4\,CG^2D - 4\,IHB^2 - 2\,CDF^2 - 4\,IA^2H\big)\cos\theta_3\sin\theta_3\\ &+ \big(-4\,CG^2E - CBF^2 - 2\,A^2FH - 4\,A^2HJ - 2\,HB^2F + ACF^2 - 4\,HB^2J - 2\,CEF^2\\ &+2\,CGBF - 2\,CFBJ + 2\,CAFJ + 4\,CGBJ + 4\,CGFE + 2\,AEFH + 4\,EGBH\\ &-2\,EFBH\big)\cos\theta_3 + \big(-4\,DG^2E - 2\,DEF^2 - DBF^2 + ADF^2 - 4\,IA^2J - 4\,IB^2J\\ &-2\,IEFB - 2\,IB^2F + 2\,IAEF + 2\,DGBF + 4\,DGBJ - 2\,DFBJ + 4\,DGFE\\ &+2\,ADFJ + 4\,IEGB - 2\,IA^2F\big)\sin\theta_3 - 2\,A^2FJ - 2\,GFA^2 + 2\,GFE^2 + AEF^2\\ &+ABF^2 + 2\,AEFJ - 2\,AGBF - 2\,EFBJ + 2\,EGBF + 4\,EGBJ - 2\,A^2J^2\\ &+2\,G^2A^2 - E^2F^2 - 2\,G^2E^2 - 2\,B^2J^2 - EBF^2 - 2\,JB^2F = 0\end{aligned}$$

This equation is obviously quadratic in $\cos\theta_3$ and $\sin\theta_3$, as expected. Moreover, using the identities

$$\cos\theta_3 = \frac{1-t_3^2}{1+t_3^2}, \quad \sin\theta_3 = \frac{2t_3}{1+t_3^2}, \quad t_3 = \tan\left(\frac{\theta_3}{2}\right)$$

a quartic equation in t_3 is obtained, namely,

$$a_4t_3^4 + a_3t_3^3 + a_2t_3^2 + a_1t_1 + a_0 = 0$$

where

$$
\begin{aligned}
a_0 = & -2A^2FJ - 2GFA^2 + 2GFE^2 + AEF^2 + ABF^2 + 2AEFJ - 2AGBF \\
& -2EFBJ + 2EGBF + 4EGBJ + 2C^2GF - 2A^2J^2 + 2G^2A^2 - E^2F^2 - 4CG^2E \\
& -CBF^2 - 2C^2G^2 - 2A^2FH - 4A^2HJ - 2HB^2F + ACF^2 - 4HB^2J - 2CEF^2 \\
& -C^2F^2 - 2A^2H^2 - 2H^2B^2 + 2CAFH + 4CGBH - 2CFBH - 2G^2E^2 - 2B^2J^2 \\
& -EBF^2 - 2JB^2F + 2CGBF - 2CFBJ + 2CAFJ + 4CGBJ + 4CGFE \\
& +2AEFH + 4EGBH - 2EFBH \\
a_1 = & -8IA^2J - 4IA^2F - 4IEFB + 8IEGB - 4IB^2F - 8DG^2E - 4DEF^2 - 2DBF^2 \\
& +2ADF^2 + 4DGBF + 8DGBJ - 4DFBJ + 8DGFE + 4ADFJ - 8IB^2J \\
& -4ICFB + 4IACF + 8ICGB + 4IAEF + 8CGFD + 8DGBH - 4DFBH \\
& +4ADFH - 8IHB^2 - 8IA^2H - 8CG^2D - 4CDF^2 \\
a_2 = & -4A^2FJ - 4GFA^2 + 4GFE^2 + 2AEF^2 + 2ABF^2 + 4AEFJ - 4AGBF - 4EFBJ \\
& +4EGBF + 8EGBJ + 8A^2 + 8IADF + 16IDGB - 4C^2GF - 8IDFB - 4A^2J^2 \\
& +4G^2A^2 - 2E^2F^2 + 8B^2 + 8D^2GF - 8D^2G^2 - 4D^2F^2 + 2C^2F^2 + 4C^2G^2 + 4A^2H^2 \\
& +4H^2B^2 - 4CAFH - 8CGBH + 4CFBH - 4G^2E^2 - 4B^2J^2 - 2EBF^2 - 4JB^2F \\
a_3 = & -8IA^2J - 4IA^2F + 8IA^2H + 8IHB^2 - 4IEFB + 8IEGB - 4IACF \\
& -8ICGB - 4IB^2F - 8DG^2E - 4DEF^2 - 2DBF^2 + 2ADF^2 + 4DGBF \\
& +8DGBJ - 4DFBJ + 8DGFE + 4ADFJ - 8IB^2J + 4ICFB + 4IAEF \\
& -8CGFD - 8DGBH + 4DFBH - 4ADFH + 8CG^2D + 4CDF^2 \\
a_4 = & -2A^2FJ - 2GFA^2 + 2GFE^2 + AEF^2 + ABF^2 + 2AEFJ - 2AGBF - 2EFBJ \\
& +2EGBF + 4EGBJ + 2C^2GF - 2A^2J^2 + 2G^2A^2 - E^2F^2 + 4CG^2E + CBF^2 \\
& +2A^2FH + 4A^2HJ + 2HB^2F - ACF^2 + 4HB^2J + 2CEF^2 - C^2F^2 - 2C^2G^2 \\
& -2A^2H^2 - 2H^2B^2 + 2CAFH + 4CGBH - 2CFBH - 2G^2E^2 - 2B^2J^2 - EBF^2 \\
& -2B^2J^2 - EBF^2 - 2JB^2F - 2CGBF + 2CFBJ - 2CAFJ - 4CGBJ - 4CGFE \\
& -2AEFH - 4EGBH + 2EFBH
\end{aligned}
$$

5.12 Apparently, the workspace is generated by the sector of a circle of radius R and angle $2\alpha = 2\sin^{-1}(r/R)$ from which a triangle of height b_3 and base $2r$ has been removed, upon rotating it about its diameter, as shown schematically in Fig. 16. Now we use the additivity relation of first moments, i.e., *the moment of a composed figure about an axis equals the algebraic sum of the moments of the individual figures about the same axis.* Hence, for α in radian,

$$
q = R^2\alpha\frac{2R\sin\alpha}{3\alpha} - rb_3\frac{2b_3}{3} = \frac{2}{3}rR^2 - \frac{2}{3}rb_3^2 = \frac{2}{3}r(R^2 - b_3^2) = \frac{2}{3}r^3
$$

Therefore,

$$
V = 2\pi q = \frac{4\pi r^3}{3}
$$

and hence, the value of the workspace volume of the Puma robot is that of a sphere of radius $r = \sqrt{R^2 - b_3^2}$.

5.13 We have

$$
\mathbf{J} = [\,\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3\,]
$$

which is a square matrix, and hence,

$$
\mu \equiv \sqrt{\det(\mathbf{J}\mathbf{J}^T)} = |\det(\mathbf{J})|
$$

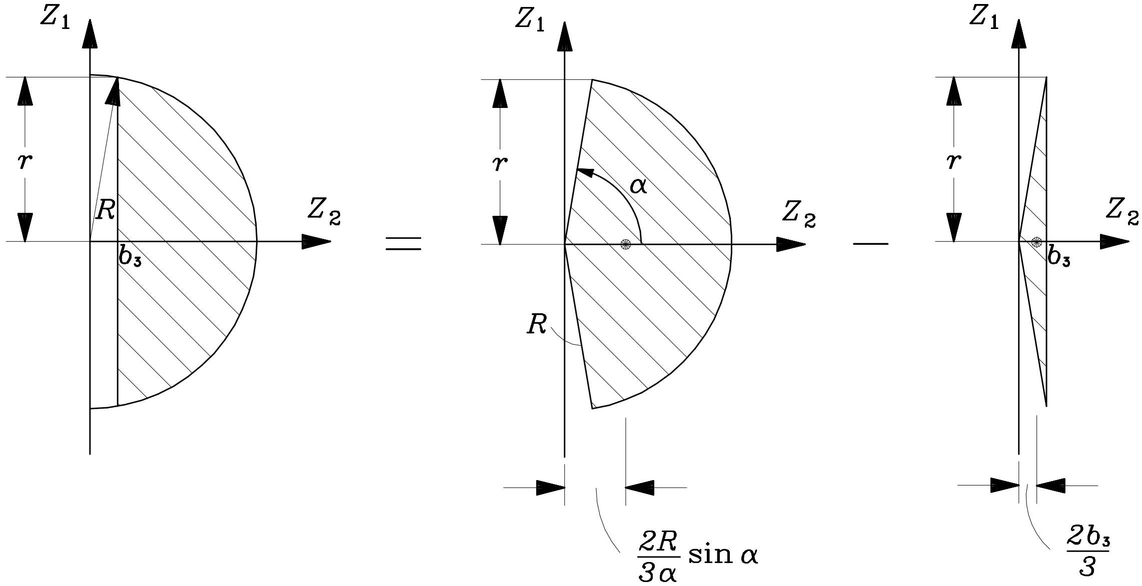

Figure 16: Planar region generating the workspace of the Puma manipulator, decomposed into two parts

i.e.,

$$\mu = |\mathbf{e}_2 \times \mathbf{e}_3 \cdot \mathbf{e}_1| = |\mathbf{e}_3 \times \mathbf{e}_1 \cdot \mathbf{e}_2| = |\mathbf{e}_1 \times \mathbf{e}_2 \cdot \mathbf{e}_3|$$

Since $\{\mathbf{e}_i\}_1^3$ are unit vectors, we have

$$|\mathbf{e}_2 \times \mathbf{e}_3 \cdot \mathbf{e}_1| = \|\mathbf{e}_1\| \|\mathbf{e}_2 \times \mathbf{e}_3\| |\cos\phi| = \|\mathbf{e}_2 \times \mathbf{e}_3\| |\cos\phi|$$

where ϕ is the angle between $\mathbf{e}_1$ and $\mathbf{e}_2 \times \mathbf{e}_3$. Furthermore, if α_2 denotes the angle between $\mathbf{e}_2$ and $\mathbf{e}_3$,

$$\|\mathbf{e}_2 \times \mathbf{e}_3\| = \|\mathbf{e}_2\| \|\mathbf{e}_3\| |\sin(\mathbf{e}_2, \mathbf{e}_3)| = |\sin\alpha_2|$$

Then,

$$\mu = |\sin\alpha_2 \cos\phi|$$

which attains a maximum value of 1 when both $|\sin\alpha_2| = 1$ and $|\cos\phi| = 1$. The foregoing values of $|\sin\alpha_2|$ and $|\cos\phi|$ correspond to $\alpha_2 = \pm 90°$ and $\phi = 0°$. Thus, μ attains its maximum when $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are mutually orthogonal, the maximum value of μ being 1.

5.14 We have

$$\mathbf{J} = [\,\mathbf{e}_4 \quad \mathbf{e}_5 \quad \mathbf{e}_6\,]$$

where, expressing all vectors in $\mathcal{F}_5$,

$$[\mathbf{e}_4]_5 = \mathbf{Q}_4^T [\mathbf{e}_4]_4\,, \quad [\mathbf{e}_5]_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad [\mathbf{e}_6]_5 = \mathbf{Q}_5 [\mathbf{e}_6]_6$$

Thus,

$$[\mathbf{e}_4]_5 = \begin{bmatrix} 0 \\ \mu_4 \\ \lambda_4 \end{bmatrix}, \quad [\mathbf{e}_5]_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad [\mathbf{e}_6]_5 = \begin{bmatrix} \mu_5 s_5 \\ -\mu_5 c_5 \\ \lambda_5 \end{bmatrix}$$

the Jacobian thus becoming

$$\mathbf{J} \leftarrow [\mathbf{J}]_5 = \begin{bmatrix} 0 & 0 & \mu_5 s_5 \\ \mu_4 & 0 & -\mu_5 c_5 \\ \lambda_4 & 1 & \lambda_5 \end{bmatrix}$$

In the sequel, we will need an expression for $\mathbf{J}^{-1}$:

$$\mathbf{J}^{-1} = \frac{1}{\mu_4 \mu_5 s_5} \begin{bmatrix} \mu_5 c_5 & \mu_5 s_5 & 0 \\ -(\mu_4 \lambda_5 + \mu_5 \lambda_4 c_5) & -\mu_5 \lambda_4 s_5 & \mu_4 \mu_5 s_5 \\ \mu_4 & 0 & 0 \end{bmatrix}$$

Now, we compute $\|\mathbf{J}\|$ and $\|\mathbf{J}^{-1}\|$, using the weighted Frobenius norm

$$\|\mathbf{J}\|_F = \sqrt{\mathrm{tr}(\mathbf{J}\mathbf{W}\mathbf{J}^T)}, \quad \mathbf{W} \equiv \left(\frac{1}{3}\right) \mathbf{1}$$

Hence,

$$\|\mathbf{J}\|_F = \sqrt{\frac{1}{3}} \sqrt{\mathrm{tr}(\mathbf{J}\mathbf{J}^T)}, \qquad \|\mathbf{J}^{-1}\|_F = \sqrt{\frac{1}{3}} \sqrt{\mathrm{tr}(\mathbf{J}^{-1}\mathbf{J}^{-T})}$$

and $-T$ denotes the transpose of the inverse or, equivalently, the inverse of the transpose. We thus have

$$\kappa_F^2 = \frac{\mu_4^2 + \mu_5^2 + (\mu_4 \lambda_5 c_5 + \mu_5 \lambda_4)^2 + \mu_4^2 s_5^2}{3 \mu_4^2 \mu_5^2 s_5^2}$$

which is, apparently, a function of α_4, α_5 and θ_5 only.

The above expression can be minimized over α_4, α_5 and θ_5 upon solving a system of three equations in three unknowns, derived from

$$f_4(\alpha_4, \alpha_5, \theta_5) \equiv \frac{\partial \kappa^2}{\partial \alpha_4} = 0, \quad f_5(\alpha_4, \alpha_5, \theta_5) \equiv \frac{\partial \kappa^2}{\partial \alpha_5} = 0, \quad f_6(\alpha_4, \alpha_5, \theta_5) \equiv \frac{\partial \kappa^2}{\partial \theta_5} = 0$$

A simpler approach follows, based on the hint, which suggests that $\mathbf{J}$ can be rendered isotropic. For isotropy, the three columns of $\mathbf{J}$ must be of identical Euclidean norm and mutually orthogonal. Orthogonality of the second with the first and the third columns readily leads to

$$\lambda_4 = \lambda_5 = 0 \quad \Rightarrow \quad \alpha_4 = \pm\frac{\pi}{2}, \quad \alpha_5 = \pm\frac{\pi}{2}$$

i.e., the wrist must be orthogonal. Furthermore, orthogonality of the first and third rows leads to

$$c_5 = 0 \quad \Rightarrow \quad \theta_5 = \pm\frac{\pi}{2}$$

and hence, isotropy is reached when three unit vectors $\mathbf{e}_4$, $\mathbf{e}_5$ and $\mathbf{e}_6$ form an orthogonal triad, a result that should have been expected.

5.15 First and foremost, we recall the formulas for the determinant of a block matrix (CRC Standard Mathematical Tables, 1987) $\mathbf{A}$, given as

$$\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{bmatrix}$$

The formulas are

$$\det(\mathbf{A}) = \det(\mathbf{E}) \det(\mathbf{B} - \mathbf{C}\mathbf{E}^{-1}\mathbf{D}) = \det(\mathbf{B}) \det(\mathbf{E} - \mathbf{D}\mathbf{B}^{-1}\mathbf{C})$$

On the other hand, the Jacobian can be written as

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} \\ \mathbf{J}_{21} & \mathbf{O}_{3\times 3} \end{bmatrix}$$

where $\mathbf{O}_{3\times 3}$ is the 3×3 zero matrix. Hence, applying the second of the above formulas,

$$\det\left(\begin{bmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} \\ \mathbf{J}_{21} & \mathbf{O}_{3\times 3} \end{bmatrix}\right) = \det(\mathbf{J}_{11})\ \det(\mathbf{O}_{3\times 3} - \mathbf{J}_{21}\mathbf{J}_{11}^{-1}\mathbf{J}_{12})$$

and hence,

$$\det(\mathbf{J}) = -\det(\mathbf{J}_{21}\mathbf{J}_{12})$$

Now, since $\mathbf{J}_{21}$ and $\mathbf{J}_{12}$ are of 3×3, we have

$$\mu_a = |\det(\mathbf{J}_{21})|, \quad \mu_w = |\det(\mathbf{J}_{12})|$$

and

$$\mu_a\mu_w = |\det(\mathbf{J}_{21})| \cdot |\det(\mathbf{J}_{12})| = |\det(\mathbf{J}_{21}\mathbf{J}_{12})| = |-\det(\mathbf{J}_{21}\mathbf{J}_{12})|$$

Therefore,

$$\mu = |\det(\mathbf{J}_{21})| \cdot |\det(\mathbf{J}_{12})|$$

and hence, $\mu = \mu_a\mu_w$, q.e.d.

5.16 The Jacobian for a 2R planar manipulator is

$$\mathbf{J} = a_1 \begin{bmatrix} -s_1 - rs_{12} & -rs_{12} \\ c_1 + rc_{12} & rc_{12} \end{bmatrix}$$

where

$$s_1 = \sin\theta_1, \quad s_{12} = \sin(\theta_1 + \theta_2)$$
$$c_1 = \cos\theta_1, \quad c_{12} = \cos(\theta_1 + \theta_2)$$
$$r = \frac{a_2}{a_1}$$

The inverse of $\mathbf{J}$ is, hence,

$$\mathbf{J}^{-1} = \frac{1}{a_1 r s_2}\begin{bmatrix} rc_2 & rs_2 \\ -(1 + rc_2) & -rs_2 \end{bmatrix}$$

and the condition number is computed using the Frobenius norm:

$$\|\mathbf{J}\|_F = \sqrt{\mathrm{tr}(\mathbf{J}\mathbf{W}\mathbf{J}^T)} = \frac{a_1\sqrt{1 + 2r^2 + 2rc_2}}{\sqrt{2}}, \quad \mathbf{W} = \left(\frac{1}{2}\right)\mathbf{1}$$

Therefore,

$$\|\mathbf{J}^{-1}\|_F = \sqrt{\mathrm{tr}(\mathbf{J}^{-1}\mathbf{W}\mathbf{J}^{-T})} = \frac{\sqrt{1 + 2r^2 + 2rc_2}}{\sqrt{2}\,a_1 r s_2}$$

Then,

$$\kappa_F = \frac{1 + 2r^2 + 2rc_2}{2rs_2}$$

We now solve

$$\kappa_F \to \min_{r,\theta_2}$$

which we do upon zeroing the partial derivatives of κ with respect to r and θ_2, namely,

$$\frac{\partial \kappa_F}{\partial r} = 0 \quad \Rightarrow \quad 2r^2 - 1 = 0 \tag{74}$$

$$\frac{\partial \kappa_F}{\partial \theta_2} = 0 \quad \Rightarrow \quad 2r + (2r^2 + 1)c_2 = 0 \tag{75}$$

From eq.(74), $r = \pm\sqrt{2}/2$, where we reject the negative sign, which has no geometrical meaning. With this value of r, eq.(75) leads to $c_2 = -\sqrt{2}/2$ and $s_2 = \pm\sqrt{2}/2$. Hence,

$$(\kappa_F)_{\min} = 1$$

which is attained with $r = \sqrt{2}/2$ and $\theta_2 = \pm 3\pi/4$.

5.17 Shown in Fig.17 is a cross-section of the workspace of the manipulator at hand, which is a hollow sphere, its volume V_A and reach R_A being

$$V_A = \frac{4}{3}\pi l^3[(\sqrt{2}+1)^3 - (\sqrt{2}-1)^3] = \frac{56}{3}\pi l^3 \tag{76}$$

$$R_A = (\sqrt{2}+1)l \tag{77}$$

For a similar orthogonal manipulator, with identical link lengths λ, the corresponding volume V_B and reach R_B are

$$V_B = \frac{4}{3}\pi(2\lambda^3) = \frac{32}{3}\pi\lambda^3$$

$$R_B = 2\lambda$$

Now, since the two manipulators have the same reach,

$$2\lambda = (\sqrt{2}+1)l \quad \Rightarrow \quad \lambda = \frac{\sqrt{2}+1}{2}l \quad \Rightarrow \quad \frac{V_A}{V_B} = 0.9949$$

The Jacobian matrix of the second manipulator takes the form

$$\mathbf{J} = [\ \mathbf{e}_1 \times \mathbf{r}_1 \quad \mathbf{e}_2 \times \mathbf{r}_2 \quad \mathbf{e}_3 \times \mathbf{r}_3\ ]$$

where

$$\mathbf{e}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{e}_2 = \mathbf{e}_3 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}$$

and

$$\mathbf{r}_1 = \lambda \begin{bmatrix} (c_2 + c_{23})c_1 \\ (c_2 + c_{23})s_1 \\ s_2 + s_{23} \end{bmatrix}, \quad \mathbf{r}_2 = \mathbf{r}_1, \quad \mathbf{r}_3 = \lambda \begin{bmatrix} c_{23}c_1 \\ c_{23}s_1 \\ s_{23} \end{bmatrix}.$$

Since θ_1 does not affect the condition number of the Jacobian, we can fix θ_1 to any value, and so, we set $\theta_1 = 0$, the Jacobian thus becoming

$$\mathbf{J} = \lambda \begin{bmatrix} 0 & -(s_2 + s_{23}) & -s_{23} \\ c_2 + c_{23} & 0 & 0 \\ 0 & c_2 + c_{23} & c_{23} \end{bmatrix}$$

In order to determine θ_2 and θ_3 that minimize the condition number $\kappa_F(\mathbf{J})$of the Jacobian, based on the Frobenius norm, we can use the Matlab built-in function `fminsearch`. The results reported by

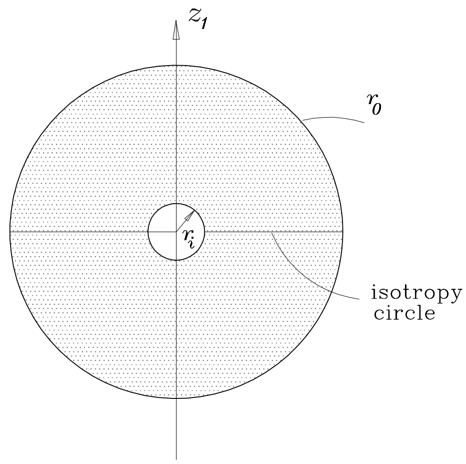

Figure 17: A cross-section of the workspace of the isotropic manipulator, with $r_i = (\sqrt{2}-1)l,\ r_0 = (\sqrt{2}+1)l$

Matlab are

$$\theta_2 = -65.3393^\circ, \quad \theta_3 = 130.6760^\circ$$

Furthermore, the minimum condition number attained at the optimum posture is

$$\kappa_m = 1.0798$$

Therefore, the KCI of the manipulator at hand is

$$\text{KCI} = 92.6097\%$$

Apparently, this robot is not far from being kinematically isotropic. However, its workspace is only slightly bigger than that of its isotropic counterpart, for the same reach. The smaller volume of the isotropic robot is due to the void in its workspace.

6 Trajectory Planning: Pick-and-Place Operations

6.1 (a) The area A of the trapezoidal profile is

$$A = \frac{1}{2}\tau_1 s'_{\max} + (\tau_2 - \tau_1)s'_{\max} + \frac{1}{2}(1-\tau_2)s'_{\max} = \frac{1}{2}(1-\tau_1+\tau_2)s'_{\max}$$

We need then $A = 1$, and thus

$$\frac{1}{2}(1-\tau_1+\tau_2)s'_{\max} = 1$$

from which we obtain

$$s'_{\max} = \frac{2}{1-\tau_1+\tau_2} \tag{78}$$

(b) Using eq.(78), we have

$$s(\tau) = \begin{cases} \dfrac{\tau^2}{\tau_1(1-\tau_1+\tau_2)}, & 0 \le \tau \le \tau_1 \\ \dfrac{2\tau - \tau_1}{1-\tau_1+\tau_2}, & \tau_1 \le \tau \le \tau_2 \\ \dfrac{\tau^2+\tau_2^2-2\tau-\tau_1(\tau_2-1)}{(\tau_2-1)(1-\tau_1+\tau_2)}, & \tau_2 \le \tau \le 1 \end{cases}$$

The plot of $s(\tau)$ vs. τ appears in Fig. 18(a). The decomposition of $s(\tau)$ into a linear part and a

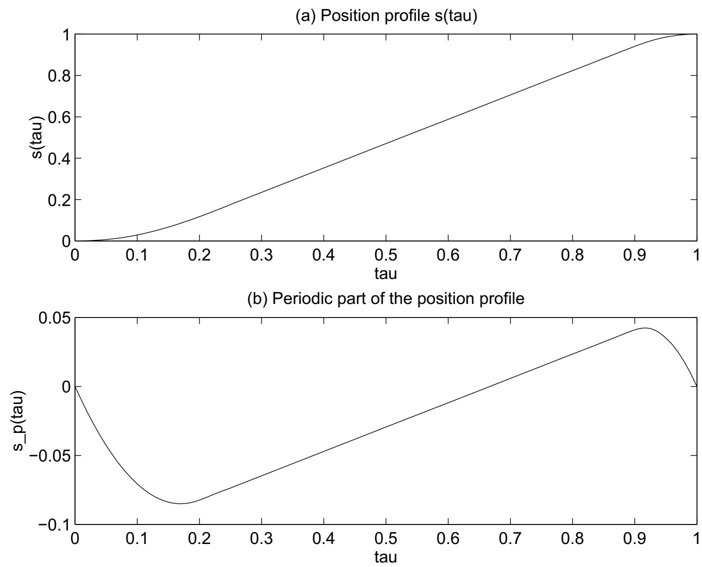

Figure 18:

periodic part, is, then,

$$s_l(\tau) = \tau, \quad s_p(\tau) = s(\tau) - \tau$$

with $s_p(\tau)$ displayed in Fig. 18(b).

(c) For a periodic cubic spline, we have the conditions

$$s_1 = s_N \tag{79a}$$

$$s_1' = s_N' \tag{79b}$$

$$s_1'' = s_N'' \tag{79c}$$

As explained in Section 6.6, condition (79c) can be used to eliminate one unknown, namely s_N'', while condition (79b) leads to an additional equation given by eq.(6.63). Thus, recalling the definitions of eqs.(6.58d–f), we have now the system

$$\mathbf{A}\mathbf{s}'' = 6\mathbf{C}\mathbf{s} \tag{80}$$

where $\mathbf{A}$ and $\mathbf{C}$ are $(N-1) \times (N-1)$ matrices defined as:

$$\mathbf{A} = \begin{bmatrix} 2\alpha_{1,N'} & \alpha_1 & 0 & 0 & \cdots & \alpha_{N'} \\ \alpha_1 & 2\alpha_{1,2} & \alpha_2 & 0 & \cdots & 0 \\ 0 & \alpha_2 & 2\alpha_{2,3} & \alpha_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \alpha_{N'''} & 2\alpha_{N','' N''} & \alpha_{N''} \\ \alpha_{N'} & 0 & 0 & \cdots & \alpha_{N''} & 2\alpha_{N,''N'} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} -\beta_{1,N'} & \beta_1 & 0 & 0 & \cdots & \beta_{N'} \\ \beta_1 & -\beta_{1,2} & \beta_2 & 0 & \cdots & 0 \\ 0 & \beta_2 & -\beta_{2,3} & \beta_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_{N'''} & -\beta_{N','' N''} & \beta_{N''} \\ \beta_{N'} & 0 & 0 & \cdots & \beta_{N''} & -\beta_{N,''N'} \end{bmatrix}$$

and

$$\mathbf{s} = [\, s_1, \cdots, s_{N-1} \,]^T, \qquad \mathbf{s}'' = [\, s_1'', \cdots, s_{N-1}'' \,]^T$$

Moreover,

$$\Delta x_k = \frac{1}{N-1}, \qquad k = 1, \cdots, N-1$$

Thus, for $i, j, k = 1, \ldots, N-1$,

$$\alpha_k = \frac{1}{N-1}, \qquad \alpha_{i,j} = \frac{2}{N-1} \tag{81}$$

$$\beta_k = N-1, \qquad \beta_{i,j} = 2(N-1) \tag{82}$$

and matrices **A** and **C** reduce to

$$\mathbf{A} = \frac{1}{N-1} \begin{bmatrix} 4 & 1 & 0 & 0 & \cdots & 1 \\ 1 & 4 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 4 & 1 \\ 1 & 0 & 0 & \cdots & 1 & 4 \end{bmatrix}$$

$$\mathbf{C} = (N-1) \begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -2 & 1 \\ 1 & 0 & 0 & \cdots & 1 & -2 \end{bmatrix}$$

Now, vector $\mathbf{s} = [\, s_1, \cdots, s_{N-1} \,]^T$ is readily known from the N equally spaced points, while the vector $\mathbf{s}''$ of eq.(80) is obtained as

$$\mathbf{s}'' = 6\mathbf{A}^{-1}\mathbf{C}\mathbf{s}$$

Then, the coefficient A_k, B_k, C_k and D_k of each cubic spline, for $k = 1, \ldots, N-1$, are obtained directly using eqs.(6.55a–d). The *Matlab* code implementing the foregoing calculations is displayed below:

```
clear
N=10;
tau1=0.2;
tau2=0.9;

delta=1/(N-1);
for i=1:N-1,
   if i==1
      A(i,:)=[4,1,zeros(1,N-4),1]/(N-1);
      C(i,:)=[-2,1,zeros(1,N-4),1]*(N-1);
   elseif i==N-1
      A(i,:)=[1,zeros(1,N-4),1,4]/(N-1);
      C(i,:)=[1,zeros(1,N-4),1,-2]*(N-1);
   else
      A(i,:)=[zeros(1,i-2),1,4,1,zeros(1,N-2-i)]/(N-1);
      C(i,:)=[zeros(1,i-2),1,-2,1,zeros(1,N-2-i)]*(N-1);
   end
end
```

```
t=(0:delta:1-delta);

for i=1:length(t),
   if t(i)<=tau1
      s(i)=t(i)^2/(tau1*(1-tau1+tau2));
   elseif t(i)>=tau2
      s(i)=(t(i)^2+tau2^2-2*t(i)-tau1*(tau2-1))/((tau2-1)*(1-tau1+tau2));
   else
      s(i)=(2*t(i)-tau1)/(1-tau1+tau2);
   end
end

sl=t;
sp=s-t;

spp=6*inv(A)*C*sp';

t(N)=1;
sp(N)=sp(1);
spp(N)=spp(1);

step=delta/10;

for i=1:N-1,
   Ak(i)=(spp(i+1)-spp(i))/(6*delta);
   Bk(i)=spp(i)/2;
   Ck(i)=(sp(i+1)-sp(i))/delta-delta*(spp(i+1)+2*spp(i))/6;
   Dk(i)=sp(i);

   if i==N-1
      tk=((i-1)*delta:step:i*delta);
   else
      tk=((i-1)*delta:step:i*delta-step);
   end
   sk=Ak(i)*(tk-t(i)).^3+Bk(i)*(tk-t(i)).^2+Ck(i)*(tk-t(i))+Dk(i);
   sppk=6*Ak(i)*(tk-t(i))+2*Bk(i);
   tt=[tt,tk];
   spline=[spline,sk];
   splinepp=[splinepp,sppk];
end
```

The resulting periodic cubic spline and its acceleration profile are displayed in Figs. 19(a) and (b), respectively, for eight supporting points. This number of supporting points gives a very good approximation of the original profile $s(\tau)$, while smoothing its acceleration profile. Moreover, the maximum acceleration value, about eight, is only slightly higher than the original acceleration level, which is about six, and thus, seems quite reasonable. Also note that, as a matter of comparison, the maximum acceleration of the cycloidal motion is slightly over six, namely, 2π.

6.2 The acceleration program is obtained by differentiating the trapezoidal joint-rate profile of Fig. 13, as plotted in Fig. 20. As in Problem 6.1, we consider the system of equations of eqs.(6.58a–c). What we

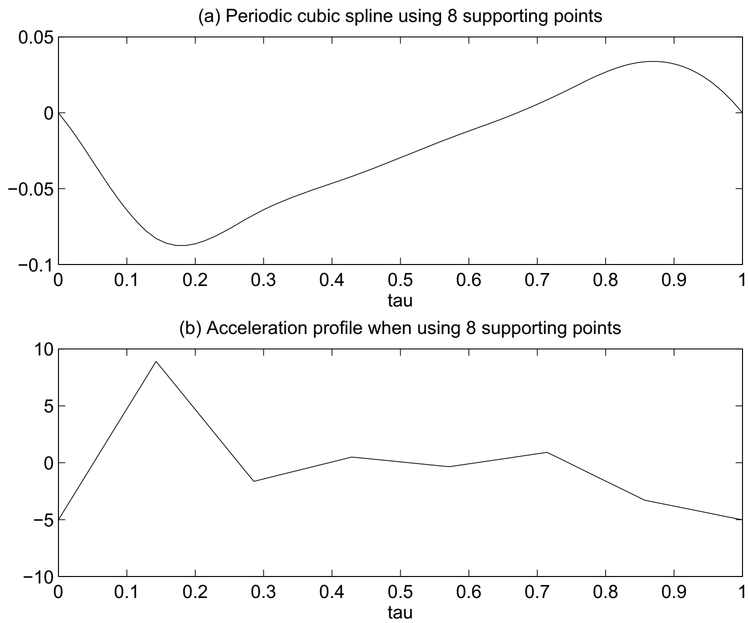

Figure 19:

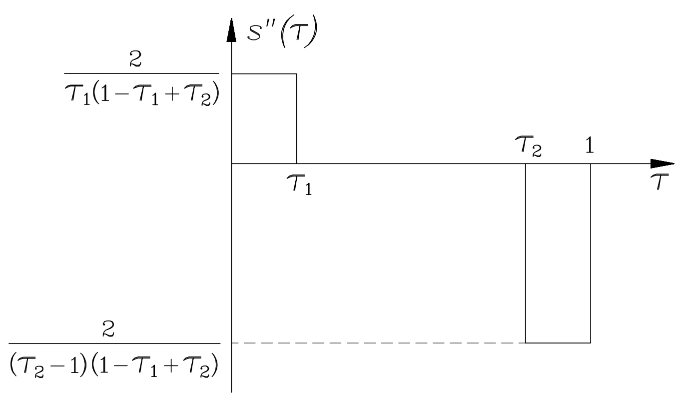

Figure 20:

want is to obtain vector $\mathbf{s}$ using eq.(6.58a) in the form

$$\mathbf{s} = \frac{1}{6}\mathbf{C}^{-1}\mathbf{A}\mathbf{s}'' \tag{83}$$

where vector $\mathbf{s}''$ contains the sampled values of the equally spaced supporting points of the acceleration profile of Fig. 20. Note that for this profile to be periodic, we must have $s''(0^-) = s''(1^+) = 0$. Here, we denote by superscripts $(-)$ and $(+)$ the instant "just before" and "just after," in order to accommodate the jump discontinuities. Moreover, in order to determine uniquely the displacement program, we must give the initial displacement of $s(\tau)$ since the acceleration profile contains no information about it. We thus set $s(0) = 0$, and since the acceleration profile s'' does not contain information on the linear part of $s(\tau)$, what we obtain using eq.(83) is just the periodic part $s_p(\tau)$ of $s(\tau)$, and thus, $s(1) = 0$. Using these two conditions, namely $s(0) = s(1) = 0$, vector $\mathbf{s}$ can be chosen as

$$\mathbf{s} = [\, s_2, \cdots, s_{N-1}\,]^T$$

and matrix $\mathbf{C}$ of eq.(6.58c) reduces to

$$\mathbf{C} = \begin{bmatrix} -\beta_{1,2} & \beta_2 & 0 & \cdots & 0 \\ \beta_2 & -\beta_{2,3} & \beta_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \beta_{N'''} & -\beta_{N',N''} & \beta_{N''} \\ 0 & 0 & \cdots & \beta_{N''} & -\beta_{N'',N'} \end{bmatrix}$$

which is invertible. Moreover, for N equally spaced supporting points, we have

$$\Delta x_k = \frac{1}{N-1}, \qquad k = 1, \cdots, N-1$$

Thus, for $i, j, k = 1, \ldots, N-1$,

$$\alpha_k = \frac{1}{N-1}, \qquad \alpha_{i,j} = \frac{2}{N-1} \tag{84}$$

$$\beta_k = N-1, \qquad \beta_{i,j} = 2(N-1) \tag{85}$$

matrices $\mathbf{A}$ and $\mathbf{C}$ reducing to

$$\mathbf{A} = \frac{1}{N-1}\begin{bmatrix} 1 & 4 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 4 & 1 & 0 \\ 0 & 0 & \cdots & 1 & 4 & 1 \end{bmatrix}, \quad \mathbf{C} = (N-1)\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & \cdots & 1 & -2 \end{bmatrix}$$

Using eq.(83), $\mathbf{s}$ is readily obtained and the coefficients A_k, B_k, C_k and D_k of the cubic spline, for $k = 1, \ldots, N-1$, are computed directly using eqs.(6.55a–d). The *Matlab* code implementing the foregoing calculations is shown below:

```
clear
N=25;
tau1=0.2;
tau2=0.9;

delta=1/(N-1);
for i=1:N-2,
```

```
   if i==1
      A(i,:)=[zeros(1,i-1),1,4,1,zeros(1,N-2-i)]/(N-1);
      C(i,:)=[-2,1,zeros(1,N-4)]*(N-1);
   elseif i==N-2
      A(i,:)=[zeros(1,i-1),1,4,1,zeros(1,N-2-i)]/(N-1);
      C(i,:)=[zeros(1,N-4),1,-2]*(N-1);
   else
      A(i,:)=[zeros(1,i-1),1,4,1,zeros(1,N-2-i)]/(N-1);
      C(i,:)=[zeros(1,i-2),1,-2,1,zeros(1,N-3-i)]*(N-1);
   end
end

t=(delta:delta:1-delta);

for i=1:length(t),
   if t(i)<=tau1
      spp(i)=2/(tau1*(1-tau1+tau2));
   elseif t(i)>=tau2
      spp(i)=2/((tau2-1)*(1-tau1+tau2));
   else
      spp(i)=0;
   end
end
t=[0,t,0];
spp=[0,spp,0];

sp=inv(C)*A*spp'/6;
sp=[0;sp;0];

step=delta/10;

for i=1:N-1,
   Ak(i)=(spp(i+1)-spp(i))/(6*delta);
   Bk(i)=spp(i)/2;
   Ck(i)=(sp(i+1)-sp(i))/delta-delta*(spp(i+1)+2*spp(i))/6;
   Dk(i)=sp(i);

   if i==N-1
      tk=((i-1)*delta:step:i*delta);
   else
      tk=((i-1)*delta:step:i*delta-step);
   end
   sk=Ak(i)*(tk-t(i)).^3+Bk(i)*(tk-t(i)).^2+Ck(i)*(tk-t(i))+Dk(i);
   sppk=6*Ak(i)*(tk-t(i))+2*Bk(i);
   tt=[tt,tk];
   spline=[spline,sk];
   splinepp=[splinepp,sppk];
end
```

The resulting periodic cubic spline and its acceleration profile are plotted in Figs. 21(a) and (b), respectively, for 25 supporting points. Below this number of points, the deceleration part of the

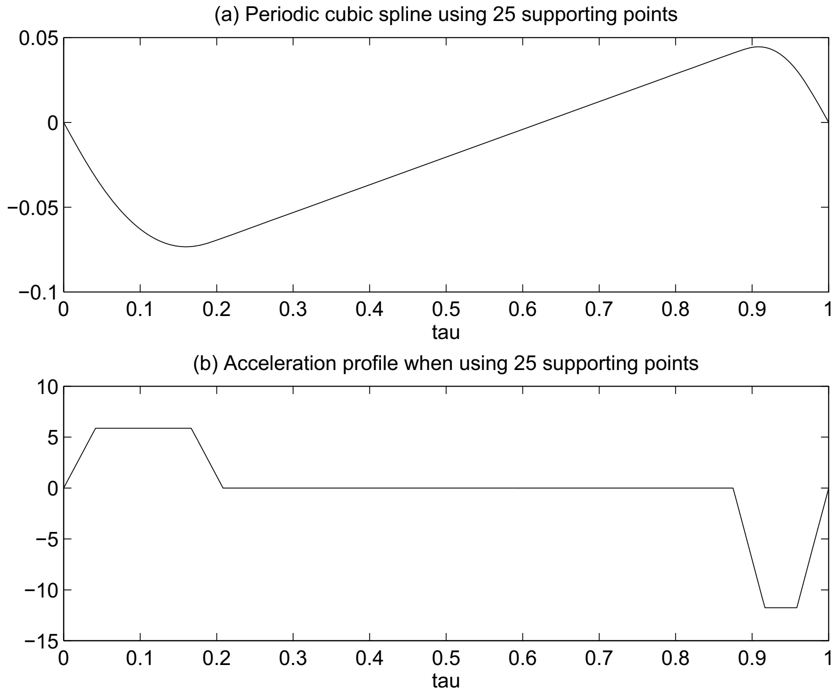

Figure 21:

acceleration profile ($\tau_2 \leq \tau \leq 1$) is not well approximated. Moreover, this number of supporting points gives a good approximation of the original profile $s(\tau)$, while smoothing its acceleration profile, as expected.

6.3 Here we want to use cycloidal motions to smooth the joint-rate profile of Fig. 6.7 of the text. To this end, we define a segment of a cycloidal-motion function between $u = 0$ and $u = u_1$ such that $s_1'(u_1) = s'_{\max}$. We have, from eqs.(6.38a & b) of the text,

$$s(u) = A(u - \frac{1}{2\pi} \sin 2\pi u)$$
$$s'(u) = A(1 - \cos 2\pi u), \text{ for } 0 \leq u \leq 1$$

as depicted in Fig. 22(a). From this figure, it is clear that the slope is horizontal, as required, when $u = 1/2$. Since we want to obtain this point at $\tau = \tau_1$, a change of variable is needed to shrink the plot in the horizontal direction, as shown in Fig. 22(b). This is done by defining u as $u = \tau/(2\tau_1)$, and hence

$$s(\tau) = \frac{A}{2}\left[\frac{\tau}{\tau_1} - \frac{1}{\pi}\sin\left(\pi\frac{\tau}{\tau_1}\right)\right] \tag{86}$$

$$s'(\tau) = \frac{A}{2\tau_1}\left[1 - \cos\left(\pi\frac{\tau}{\tau_1}\right)\right] \tag{87}$$

where A is a constant to be determined. We have

$$s'(\tau_1) = \frac{A}{2\tau_1}(1 - \cos\pi) = \frac{A}{\tau_1} = s'_{\max}$$