into eq.(65), we obtain, after some algebra, an octic equation in τ_2, i.e.,

$$\tau_2^8 + 4\tau_2^7 + 4\tau_2^6 + 4\tau_2^5 + 10\tau_2^4 + 12\tau_2^3 - 4\tau_2^2 - 20\tau_2 + 5 = 0 \tag{66}$$

The eight roots of this equation are

$$(\tau_2)_1 = -2.798907440, \quad (\tau_2)_2 = -1.663251939, \quad (\tau_2)_{3,4} = -1 \pm j1.111785941,$$
$$(\tau_2)_{5,6} = 0.7071067812 \pm j1.383551070, \quad (\tau_2)_7 = 0.798907440, \quad (\tau_2)_8 = 0.2490383764$$

where j is the imaginary unit, i.e., $j \equiv \sqrt{-1}$. The four real solutions correspond to the four intersecting points in Fig. 13, namely,

$$(\theta_2)_1 = -140.678184°, \quad (\theta_2)_2 = -117.968752°, \quad (\theta_2)_7 = 77.2432°, \quad (\theta_2)_8 = 27.968752°$$

Using eqs.(63) and (64), we obtain

$$(u)_1 = -1.578094854, \quad (v)_1 = 1.975316257$$
$$(u)_2 = -1.13224188, \quad (v)_2 = 0.1167964943$$
$$(u)_7 = 1.025308450, \quad (v)_7 = 0.3663245289$$
$$(u)_8 = 2.132241882, \quad (v)_8 = 1.468989944$$

which lead to reach values of

$$(r)_1 = 2.058171027a, \quad (r)_2 = 0.1764350771a, \quad (r)_7 = 2.058171027, \quad (r)_8 = 3.459605564a$$

for a global maximum reach of

$$r_M = 3.459605564a$$

which is very close of the approximate value found above graphically.

We can now determine the length a of the manipulator of Fig. 4.15 to attain the maximum reach of the Puma robot, which was found to be $R = 0.8772$ m. The value sought is thus obtained from

$$3.459605564a = 0.8772096591 \quad \Rightarrow \quad a = 0.2535577085 \text{ m}$$

5 Kinetostatics of Serial Robots

5.1 The Jacobian matrix will be obtained using the algorithm of Section 5.3. First, we note

$$\lambda_i = \cos\alpha_i, \quad \mu_i = \sin\alpha_i, \quad c_i = \cos\theta_i, \quad s_i = \sin\theta_i$$

Then, using the DH parameters of Table 1 of the Exercises, we have

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$$

$$\mu_1 = \mu_3 = \mu_5 = 1, \quad \mu_2 = \mu_4 = \mu_6 = -1$$

$$c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0$$

$$s_1 = s_3 = s_5 = 1, \quad s_2 = s_4 = s_6 = -1$$

Therefore, using eq.(4.1d), we obtain

$$\mathbf{Q}_1 = \mathbf{Q}_3 = \mathbf{Q}_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q}_2 = \mathbf{Q}_4 = \mathbf{Q}_6 = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

and, from eq.(4.3b),

$$[\mathbf{a}_1]_1 = [\mathbf{a}_3]_3 = [\mathbf{a}_5]_5 = \begin{bmatrix} 0 \\ 50 \\ 50 \end{bmatrix}, \quad [\mathbf{a}_2]_2 = [\mathbf{a}_4]_4 = \begin{bmatrix} 0 \\ -50 \\ 50 \end{bmatrix}, \quad [\mathbf{a}_6]_6 = \begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix}$$

i) Evaluation of submatrix $\mathbf{A}$:

$$[\mathbf{e}_1]_1 = [0 \quad 0 \quad 1]^T$$

$$\mathbf{P}_1 = \mathbf{Q}_1, \quad \text{i.e.,}\ [\mathbf{e}_2]_1 = [1 \quad 0 \quad 0]^T$$

$$\mathbf{P}_2 = \mathbf{P}_1\mathbf{Q}_2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}, \quad \text{i.e.,}\ [\mathbf{e}_3]_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{P}_3 = \mathbf{P}_2\mathbf{Q}_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \text{i.e.,}\ [\mathbf{e}_4]_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\mathbf{P}_4 = \mathbf{P}_3\mathbf{Q}_4 = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \text{i.e.,}\ [\mathbf{e}_5]_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{P}_5 = \mathbf{P}_4\mathbf{Q}_5 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \text{i.e.,}\ [\mathbf{e}_6]_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

Therefore,

$$[\mathbf{A}]_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

ii) Evaluation of submatrix $\mathbf{B}$:

We have

$$[\mathbf{r}_6]_6 = [\mathbf{a}_6]_6 = [0 \quad 0 \quad 50]^T$$

and

$$[\mathbf{r}_i]_i = [\mathbf{a}_i]_i + \mathbf{Q}_i[\mathbf{r}_{i+1}]_{i+1}, \quad i = 5, 4, 3, 2, 1$$

Thus,

$$[\mathbf{r}_5]_5 = \begin{bmatrix} 50 \\ 50 \\ 50 \end{bmatrix}, \quad [\mathbf{r}_4]_4 = \begin{bmatrix} 50 \\ -100 \\ 0 \end{bmatrix}, \quad [\mathbf{r}_3]_3 = \begin{bmatrix} 0 \\ 100 \\ -50 \end{bmatrix},$$

$$[\mathbf{r}_2]_2 = \begin{bmatrix} -50 \\ -50 \\ -50 \end{bmatrix}, \quad [\mathbf{r}_1]_1 = \begin{bmatrix} -50 \\ 0 \\ 0 \end{bmatrix}$$

Therefore,

$$[\mathbf{e}_1 \times \mathbf{r}_1]_1 = [0 \quad -50 \quad 0]^T$$

and

$$[\mathbf{e}_i \times \mathbf{r}_i]_1 = \mathbf{P}_{i-1}[\mathbf{e}_i \times \mathbf{r}_i]_i, \quad i = 2, \cdots, 6$$

Thus,

$$[\mathbf{e}_2 \times \mathbf{r}_2]_1 = \begin{bmatrix} 0 \\ 50 \\ -50 \end{bmatrix}, \quad [\mathbf{e}_3 \times \mathbf{r}_3]_1 = \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix}, \quad [\mathbf{e}_4 \times \mathbf{r}_4]_1 = \begin{bmatrix} -100 \\ 50 \\ 0 \end{bmatrix},$$

$$[\,\mathbf{e}_5 \times \mathbf{r}_5\,]_1 = \begin{bmatrix} 0 \\ 50 \\ 50 \end{bmatrix}, \quad [\,\mathbf{e}_6 \times \mathbf{r}_6\,]_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence,

$$[\,\mathbf{B}\,]_1 = \begin{bmatrix} 0 & 0 & 0 & -100 & 0 & 0 \\ -50 & 50 & 0 & 50 & 50 & 0 \\ 0 & -50 & 100 & 0 & 50 & 0 \end{bmatrix}$$

Finally

$$[\,\mathbf{J}\,]_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -100 & 0 & 0 \\ -50 & 50 & 0 & 50 & 50 & 0 \\ 0 & -50 & 100 & 0 & 50 & 0 \end{bmatrix}$$

5.2 We have here a decoupled manipulator, and thus,

$$\mathbf{J}_{12}^T \mathbf{n}_w = \Delta \boldsymbol{\tau}_w \tag{67a}$$

$$\mathbf{J}_{21}^T \mathbf{f} = \Delta \boldsymbol{\tau}_a - \mathbf{J}_{11}^T \mathbf{n}_w \tag{67b}$$

with matrices $\mathbf{J}_{11}$, $\mathbf{J}_{12}$ and $\mathbf{J}_{21}$ given by

$$\mathbf{J}_{11} = [\,\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3\,], \quad \mathbf{J}_{12} = [\,\mathbf{e}_4 \quad \mathbf{e}_5 \quad \mathbf{e}_6\,], \quad \mathbf{J}_{21} = [\,\mathbf{e}_1 \times \mathbf{r}_1 \quad \mathbf{e}_2 \times \mathbf{r}_2 \quad \mathbf{e}_3 \times \mathbf{r}_3\,]$$

where $\mathbf{r}_i$ is defined as the vector directed from O_i to C. Referring to Fig. 14, and expressing all quantities in Frame 1, we have

$$\mathbf{e}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; \mathbf{e}_2 = \mathbf{e}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; \mathbf{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; \mathbf{e}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; \mathbf{e}_6 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{r}_1 = \mathbf{r}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{r}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus,

$$\mathbf{e}_1 \times \mathbf{r}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 \times \mathbf{r}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{e}_3 \times \mathbf{r}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Therefore,

$$\mathbf{J}_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{J}_{12} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{J}_{21} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

Hence,

$$\mathbf{J}_{12}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{J}_{21}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

From eq.(67a), we have

$$\mathbf{n}_w = \left(\mathbf{J}_{12}^T\right)^{-1} \Delta \boldsymbol{\tau}_w = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

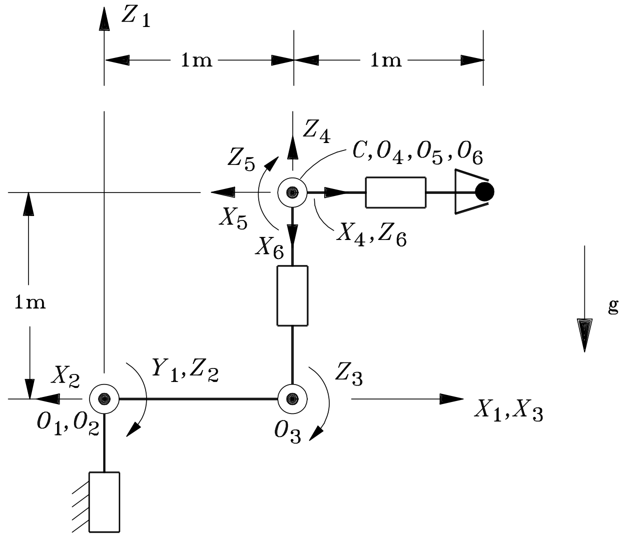

Figure 14:

and, from eq.(67b),

$$\mathbf{f} = \left(\mathbf{J}_{21}^T\right)^{-1}\left[\Delta\boldsymbol{\tau}_a - \mathbf{J}_{11}^T\mathbf{n}_w\right] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}\left(\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

which yields

$$\mathbf{f} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

This last expression represents the weight of the tool, which is directed in the negative Z_1 direction, as expected. Hence,

$$w = 1 \text{ N}$$

Now, $\mathbf{n}_w$ is the resultant moment acting on the EE when the load $\mathbf{f}$ is applied at the center C of the wrist. Therefore, defining

$$\mathbf{r} = [-d \quad 0 \quad 0]^T$$

we have

$$\mathbf{n}_w = \mathbf{f} \times \mathbf{r} = [0 \quad d \quad 0]^T$$

Comparing this expression with eq.(5), we finally obtain

$$d = 1 \text{ m}$$

5.3 We show here two approaches, the first being geometric, and is what we recommend; the second is included for verification purposes, and is based on the formulas developed in the book.

Within the geometric approach, we note that the manipulator is at the posture sketched in Fig. 15, from which

$$\mathbf{r}_1 = \begin{bmatrix} 0 \\ 1+\sqrt{2} \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} -\sqrt{2}/2 \\ 1+\sqrt{2}/2 \end{bmatrix}, \quad \mathbf{r}_3 = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

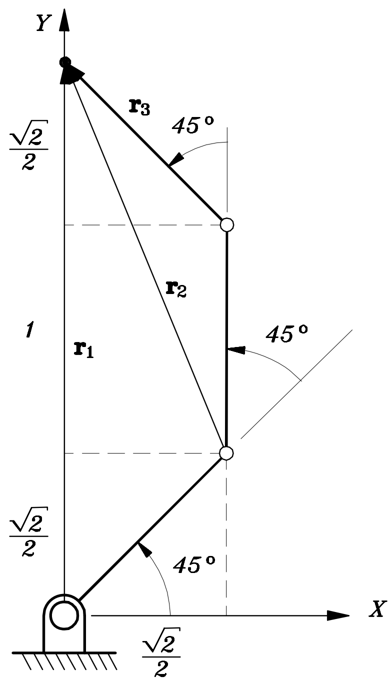

Figure 15: Manipulator at posture $\theta_1 = \theta_2 = \theta_3 = 45°$

Hence,

$$\mathbf{E}\mathbf{r}_1 = \begin{bmatrix} -1-\sqrt{2} \\ 0 \end{bmatrix}, \quad \mathbf{E}\mathbf{r}_2 = \begin{bmatrix} -1-\sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}, \quad \mathbf{E}\mathbf{r}_3 = \begin{bmatrix} -\sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$$

Therefore,

$$\mathbf{J} = \begin{bmatrix} 1 & 1 & 1 \\ -1-\sqrt{2} & -1-\sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}, \quad \boldsymbol{\tau} = \begin{bmatrix} -\sqrt{2} \\ -\sqrt{2} \\ 1-\sqrt{2} \end{bmatrix}$$

If we let the wrench acting at the EE be $\mathbf{w} = [\,n,\ f_x,\ f_y\,]^T$, then the torque-wrench relation $\mathbf{J}^T\mathbf{w} = \boldsymbol{\tau}$ leads to

$$\begin{bmatrix} 1 & -1-\sqrt{2} & 0 \\ 1 & -1-\sqrt{2}/2 & -\sqrt{2}/2 \\ 1 & -\sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix} \begin{bmatrix} n \\ f_x \\ f_y \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ -\sqrt{2} \\ 1-\sqrt{2} \end{bmatrix}$$

We now obtain a reduced system of two equations with two unknowns upon subtracting the first of the above three scalar equations from the second and the third, which thus yields

$$\begin{bmatrix} 1 & -1-\sqrt{2} & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & 1+\sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix} \begin{bmatrix} n \\ f_x \\ f_y \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 0 \\ 1 \end{bmatrix}$$

The 2×2 system is, then,

$$\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ 1+\sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The determinant Δ of the foregoing matrix turns out to be

$$\Delta = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

and hence,

$$\begin{bmatrix} f_x \\ f_y \end{bmatrix} = \sqrt{2}\begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ -1-\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \sqrt{2}\begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ N}$$

whence,

$$n = -\sqrt{2} + (1+\sqrt{2})f_x = -\sqrt{2} + 1 + \sqrt{2} = 1 \text{ Nm}$$

In order to verify the foregoing results, we resort to eq.(5.72). To use this equation, the quantities below must be evaluated:

$$\mathbf{s}_3 = \begin{bmatrix} c_{123} \\ s_{123} \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

$$\mathbf{s}_2 = \begin{bmatrix} c_{12} + c_{123} \\ s_{12} + s_{123} \end{bmatrix} = \begin{bmatrix} 0 - \sqrt{2}/2 \\ 1 + \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 \\ 1+\sqrt{2}/2 \end{bmatrix}$$

$$\mathbf{s}_1 = \begin{bmatrix} c_1 + c_{12} + c_{123} \\ s_1 + s_{12} + s_{123} \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 + 0 - \sqrt{2}/2 \\ \sqrt{2}/2 + 1 + \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1+\sqrt{2} \end{bmatrix}$$

Furthermore, using eq.(5.64) we have

$$\Delta \equiv -\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -\frac{\sqrt{2}}{2} \\ -1-\frac{\sqrt{2}}{2} \end{bmatrix} = \frac{\sqrt{2}}{2}$$

which is rightfully identical to the value obtained above. Therefore, from eq.(5.72),

$$\mathbf{f} = \frac{1}{\Delta}[(\tau_2 - \tau_1)(\mathbf{s}_3 - \mathbf{s}_1) - (\tau_3 - \tau_1)(\mathbf{s}_2 - \mathbf{s}_1)]$$
$$= \frac{2}{\sqrt{2}}\left(0(\mathbf{s}_3 - \mathbf{s}_1) - \begin{bmatrix} -\sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ N}$$

Finally,

$$n = \tau_1 + \mathbf{s}_1^T\mathbf{E}\mathbf{f} = -\sqrt{2} + \begin{bmatrix} 0 & 1+\sqrt{2} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} n = 1 \text{ Nm}$$

thereby confirming the correctness of the results.

5.4 We have

$$\mathbf{J}_{12}\dot{\boldsymbol{\theta}} = \boldsymbol{\omega}$$

Therefore,

$$\|\boldsymbol{\omega}\|^2 = \boldsymbol{\omega}^T\boldsymbol{\omega} = \dot{\boldsymbol{\theta}}^T\mathbf{J}_{12}^T\mathbf{J}_{12}\dot{\boldsymbol{\theta}}$$

Matrix $\mathbf{J}_{12}$ is given by

$$\mathbf{J}_{12} = [\,\mathbf{e}_4 \quad \mathbf{e}_5 \quad \mathbf{e}_6\,]$$

We have

$$[\mathbf{e}_4]_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad [\mathbf{e}_5]_4 = \begin{bmatrix} \sin\theta_4 \\ -\cos\theta_4 \\ 0 \end{bmatrix}, \quad [\mathbf{e}_6]_4 = \begin{bmatrix} \xi \\ \eta \\ 0 \end{bmatrix}$$

Hence

$$\mathbf{J}_{12} = \begin{bmatrix} 0 & \sin\theta_4 & \xi \\ 0 & -\cos\theta_4 & \eta \\ 1 & 0 & 0 \end{bmatrix}$$