Finally, the minimum-magnitude moment is obtained as

$$\mathbf{n}_0 = \frac{\mathbf{n}_i \cdot \mathbf{f}}{\|\mathbf{f}\|^2}\mathbf{f}, \quad \text{for } i = 1, 2 \text{ or } 3$$

However, $\mathbf{n}_i \cdot \mathbf{f} = 0$, for $i = 1, 2, 3$, and hence,

$$\mathbf{n}_0 = \mathbf{0}$$

8.12 We have

$$\dot{\mathbf{P}} = \mathbf{\Omega P}, \quad \mathbf{R} = \mathbf{PP}^T$$

Multiplying both sides of eq. (8.6) by $\mathbf{P}^T$, we obtain

$$\dot{\mathbf{P}}\mathbf{P}^T = \mathbf{\Omega P P}^T = \mathbf{\Omega R}$$

Recalling Theorem A.1,

$$\text{vect}(\dot{\mathbf{P}}\mathbf{P}^T) = \frac{1}{2}\mathbf{J}\boldsymbol{\omega}$$

Now, to show that matrix $\mathbf{J}$ is frame-invariant, we introduce two different frames, labeled $\mathcal{A}$ and $\mathcal{B}$, in which the representation of $\mathbf{J}$ is $[\mathbf{J}]_{\mathcal{A}}$ and $[\mathbf{J}]_{\mathcal{B}}$, respectively. From the definition of $\mathbf{p}$, moreover, $[\mathbf{p}]_{\mathcal{A}} = [\mathbf{Q}]_{\mathcal{A}}[\mathbf{p}]_{\mathcal{A}}$. Additionally, let $[\mathbf{Q}]_{\mathcal{A}}$ denote the rotation of frame $\mathcal{A}$ into frame $\mathcal{B}$. Thus,

$$\begin{aligned}
[\mathbf{J}]_{\mathcal{A}} &= \text{tr}([\mathbf{p}]_{\mathcal{A}}[\mathbf{p}^T]_{\mathcal{A}})\mathbf{1} - [\mathbf{p}]_{\mathcal{A}}[\mathbf{p}^T]_{\mathcal{A}} \\
&\equiv \text{tr}([\mathbf{Q}]_{\mathcal{A}}[\mathbf{p}]_{\mathcal{B}}[\mathbf{p}^T]_{\mathcal{B}}[\mathbf{Q}^T]_{\mathcal{A}})[\mathbf{Q}]_{\mathcal{A}}[\mathbf{Q}^T]_{\mathcal{A}} - [\mathbf{Q}]_{\mathcal{A}}[\mathbf{p}]_{\mathcal{B}}[\mathbf{p}^T]_{\mathcal{B}}[\mathbf{Q}^T]_{\mathcal{A}} \\
&= [\mathbf{Q}]_{\mathcal{A}}\left(\text{tr}([\mathbf{Q}^T]_{\mathcal{A}}[\mathbf{Q}]_{\mathcal{A}}[\mathbf{p}]_{\mathcal{B}}[\mathbf{p}^T]_{\mathcal{B}})\mathbf{1} - [\mathbf{p}]_{\mathcal{B}}[\mathbf{p}^T]_{\mathcal{B}}\right)[\mathbf{Q}^T]_{\mathcal{A}} \\
&= [\mathbf{Q}]_{\mathcal{A}}\left(\text{tr}([\mathbf{p}]_{\mathcal{B}}[\mathbf{p}^T]_{\mathcal{B}})\mathbf{1} - [\mathbf{p}]_{\mathcal{B}}[\mathbf{p}^T]_{\mathcal{B}}\right)[\mathbf{Q}^T]_{\mathcal{A}} \\
&\equiv [\mathbf{Q}]_{\mathcal{A}}[\mathbf{J}]_{\mathcal{B}}[\mathbf{Q}^T]_{\mathcal{A}}
\end{aligned}$$

which shows that $\mathbf{J}$ is indeed frame-invariant.

8.13

$$\mathbf{p}_1 = \begin{bmatrix} 0 \\ -30 \\ 30 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 0 \\ 30 \\ 30 \end{bmatrix}, \quad \mathbf{p}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Rightarrow \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix}$$

$$\dot{\mathbf{p}}_1 = \begin{bmatrix} 120 \\ -60 \\ -60 \end{bmatrix}, \quad \dot{\mathbf{p}}_2 = \begin{bmatrix} 0 \\ -60 \\ 60 \end{bmatrix}, \quad \dot{\mathbf{p}}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Rightarrow \dot{\mathbf{c}} = \begin{bmatrix} 40 \\ -40 \\ 0 \end{bmatrix}$$

$$\Rightarrow \mathbf{P} = \begin{bmatrix} 0 & 0 & 0 \\ -30 & 30 & 0 \\ 10 & 10 & -20 \end{bmatrix}, \quad \dot{\mathbf{P}} = \begin{bmatrix} 80 & -40 & -40 \\ -20 & -20 & 40 \\ -60 & 60 & 0 \end{bmatrix}$$

(a)

$$\begin{aligned}
\mathbf{P}^T\dot{\mathbf{P}} &= \begin{bmatrix} 0 & -30 & 10 \\ 0 & 30 & 10 \\ 0 & 0 & -20 \end{bmatrix} \begin{bmatrix} 80 & -40 & -40 \\ -20 & -20 & 40 \\ -60 & 60 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 12 & -12 \\ -12 & 0 & 12 \\ 12 & -12 & 0 \end{bmatrix} \times 100 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \times 1200,
\end{aligned}$$

which is apparently skew-symmetric, and hence, estimates are compatible.

(*b*) $\mathbf{D}\boldsymbol{\omega} = \mathrm{vect}(\dot{\mathbf{p}})$ where

$$\mathbf{D} = \frac{1}{2}[\mathrm{tr}(\mathbf{P})\mathbf{1} - \mathbf{P}] = \frac{1}{2}\begin{bmatrix} 10 & 0 & 0 \\ 30 & -20 & 0 \\ -10 & -10 & 30 \end{bmatrix}, \quad \mathrm{vect}(\dot{\mathbf{P}}) = \frac{1}{2}\begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}$$

Hence,

$$\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & 0 \\ -1 & -1 & 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Matrix is lower-triangular. Use forward substitution: $\omega_1 = 2$, $3\omega_1 - 2\omega_2 = 2 \Rightarrow -2\omega_2 = -4 \Rightarrow \omega_2 = 2$. Then, $-\omega_1 - \omega_2 + 3\omega_3 = 2 \Rightarrow 3\omega_3 = 6 \Rightarrow \omega_3 = 2$

$$\Rightarrow \quad \boldsymbol{\omega} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \text{ rad/s}$$

9 Geometry of General Serial Robots

9.1 A reflection $\mathbf{H}$ onto a plane Π of a unit normal $\mathbf{n}$ can be expressed as

$$\mathbf{H} = \mathbf{1} - 2\mathbf{n}\mathbf{n}^T \tag{149}$$

Now, if the reflection plane is normal to $\mathbf{f}$, which is not necessarily of unit magnitude, then vector $\mathbf{h}$ is mapped by $\mathbf{H}$ into $\mathbf{h}'$ given by

$$\mathbf{h}' = (\mathbf{1} - 2\,\mathbf{f}\,\mathbf{f}^T/\|\mathbf{f}\|^2)\mathbf{h} = \frac{(\mathbf{f}\cdot\mathbf{f})\mathbf{h} - 2(\mathbf{f}\cdot\mathbf{h})\mathbf{f}}{\|\mathbf{f}\|^2}$$

Notice that $\|\mathbf{f}\|^2\mathbf{h}'$ is the left-hand side of eq.(8.22f). By the same token, we can write, for $\mathbf{i}'$

$$\mathbf{i}' = (\mathbf{1} - 2\,\mathbf{g}\,\mathbf{g}^T/\|\mathbf{g}\|^2)\mathbf{i} = \frac{(\mathbf{g}\cdot\mathbf{g})\mathbf{i} - 2\,(\mathbf{g}\cdot\mathbf{i})\mathbf{g}}{\|\mathbf{g}\|^2}$$

Obviously, the right-hand side of eq.(8.22f) is equal to $\|\mathbf{g}\|^2\mathbf{i}'$

9.3 See Maple worksheet in Appendix 1.

9.5 Assume that we have an orthogonal matrix $\mathbf{H}$ chosen as a product of 12 Householder reflections[5] that will render the matrix $\mathbf{S}$ in an upper triangular form $\mathbf{U}$. That is,

$$\mathbf{H}\mathbf{S} = \mathbf{U}$$

Equation (8.51d) can be written as

$$\mathbf{S}^T\mathbf{H}^T\mathbf{H}\tilde{\mathbf{x}}_{45} = \mathbf{0}_{12} \Longrightarrow (\mathbf{H}\mathbf{S})^T\mathbf{H}\tilde{\mathbf{x}}_{45} = \mathbf{0}_{12} \tag{150}$$

where $\mathbf{0}_{12}$ is the 12-dimensional zero vector. Then, we can write eq.(150) as

$$\mathbf{U}^T\mathbf{v} = \mathbf{0}_{12}$$

with

$$\mathbf{v} = \mathbf{H}\,\tilde{\mathbf{x}}_{45} \tag{151}$$

[5]See Appendix B.

Since $\mathbf{S}$ is singular, the 12th row of $\mathbf{U}$ is full of zeros, and so is the 12th column of $\mathbf{U}^T$. As a consequence, the nullspace of $\mathbf{U}^T$ is spanned by the unit vector $\mathbf{n} = [\mathbf{0}_{11}^T \; 1]^T$. Consequently, vector $\mathbf{v}$ is a multiple of $\mathbf{n}$, i.e., $\mathbf{v} = \alpha\mathbf{n}$, where α is a scalar, as yet to be determined, which is done below. From eq.(151),

$$\tilde{\mathbf{x}}_{45} = \mathbf{H}^T\mathbf{v} = \alpha\mathbf{H}^T\mathbf{n} = \alpha \left[\; h_{12,1} \ldots h_{12,12} \; \right]^T$$

where $h_{12,i}$ denotes, as usual, the ith component of the 12th row of $\mathbf{H}$. Upon comparison of the above expression of $\mathbf{x}_{45}$ with its definition in eq.(8.27a), we find α as

$$\alpha h_{12,12} = 1 \Rightarrow \alpha = \frac{1}{h_{12,12}}$$

9.7 We have four equations according to eq.(9.70a). By selecting any two of them, we end up with a system of two nonlinear equations $\mathbf{f}$ in two unknowns $\mathbf{x}$. The Jacobian matrix $\mathbf{F}$ is given by

$$\mathbf{F} \equiv \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$$

According to the Newton-Raphson method, we have

$$\mathbf{x}_{i+1} = \mathbf{x}_i - \mathbf{F}_i^{-1}\mathbf{f}_i, \quad i = 0, 1, 2, \ldots$$

where $\mathbf{F}_i$ and $\mathbf{f}_i$ represent $\mathbf{F}$ and $\mathbf{f}$ evaluated at $\mathbf{x}_i$, respectively.

With an initial guess $\mathbf{x}_0$ close enough to a root, the Newton-Raphson method may converge to that root rapidly. Given a tolerance ϵ, the criterion to stop the iteration is

$$\|\mathbf{x}_{i+1} - \mathbf{x}_i\|_\infty < \epsilon$$

where $\|\cdot\|_\infty$ denotes the Chebyshev norm. The Maple code implementing this calculation is given in Appendix 2.

By monitoring the condition number of $\mathbf{F}$ based on the Frobenius norm, we observe that the Newton-Raphson method converges faster when the condition number is smaller. The condition number introduced in Section 5.8, with the Frobenius-norm condition number discussed in eqs.(5.79)–(5.82).

10 Kinematics of Complex Robotic Mechanical Systems

10.1 For the parallel manipulator of Fig. 9.7, the matrix mapping joint forces into wrenches acting on the moving platform can be obtained by relating the power generated by the actuators and the power consumed by the load. From eq.(9.102a),

$$\dot{\mathbf{b}} = \mathbf{K}\mathbf{t}$$

where $\mathbf{K}$ is the Jacobian of the manipulator given in eq.(9.102b). Under static, conservative conditions, the power delivered by the actuators equals that developed by the load, i.e., $\Pi_a = \Pi_L$, where

$$\Pi_a = \dot{\mathbf{b}}^T\boldsymbol{\tau}$$
$$\Pi_L = \mathbf{t}^T\mathbf{w}$$

with $\dot{\mathbf{b}}$ being the vector of actuated joint rates, $\boldsymbol{\tau}$ the vector of actuated joint torques, $\mathbf{t}$ the twist of the moving platform, and $\mathbf{w}$ the wrench acting on the moving platform. Then,

$$\dot{\mathbf{b}}^T\boldsymbol{\tau} = \mathbf{t}^T\mathbf{w}$$
$$\mathbf{t}^T\mathbf{K}^T\boldsymbol{\tau} = \mathbf{t}^T\mathbf{w}$$

which is valid for every possible motion, i.e., for every possible twist $\mathbf{t}$, and hence, the above equation leads to

$$\mathbf{w} = \mathbf{K}^T\boldsymbol{\tau}$$