8.9 (a)

$$\mathbf{R} = [\,\mathbf{p}_1 - \mathbf{c} \quad \mathbf{p}_2 - \mathbf{c} \quad \mathbf{p}_3 - \mathbf{c}\,] \begin{bmatrix} (\mathbf{p}_1 - \mathbf{c})^T \\ (\mathbf{p}_2 - \mathbf{c})^T \\ (\mathbf{p}_3 - \mathbf{c})^T \end{bmatrix} = \sum_1^3 (\mathbf{p}_i - \mathbf{c})(\mathbf{p}_i - \mathbf{c})^T$$

and hence,

$$\mathbf{J} = \sum_1^3 \left[\|\mathbf{p}_i - \mathbf{c}\|^2 \mathbf{1} - (\mathbf{p}_i - \mathbf{c})(\mathbf{p}_i - \mathbf{c})^T\right] = \operatorname{tr}(\mathbf{R})\mathbf{1} - \mathbf{R} \tag{141}$$

which is indeed, according to eq.(3.136), the moment of inertia of a set of three punctual unit masses placed at the three given points, with respect to the centroid C of the given points.

(b) Upon differentiation of eq.(141), we have

$$\dot{\mathbf{J}} = \frac{d}{dt}\left(\operatorname{tr}(\mathbf{R})\mathbf{1} - \mathbf{R}\right) = \frac{d}{dt}\left(\operatorname{tr}(\mathbf{P}\mathbf{P}^T)\mathbf{1} - \mathbf{P}\mathbf{P}^T\right)$$
$$\dot{\mathbf{J}} = \operatorname{tr}(\dot{\mathbf{P}}\mathbf{P}^T)\mathbf{1} + \operatorname{tr}(\mathbf{P}\dot{\mathbf{P}}^T)\mathbf{1} - \dot{\mathbf{P}}\mathbf{P}^T - \mathbf{P}\dot{\mathbf{P}}^T$$

From eq.(8.16), $\operatorname{tr}(\mathbf{P}\dot{\mathbf{P}}^T) = 0$, which implies that $\operatorname{tr}(\dot{\mathbf{P}}\mathbf{P}^T) = 0$. Moreover, it is apparent that

$$\dot{\mathbf{P}} = \boldsymbol{\Omega}\mathbf{P}, \qquad \dot{\mathbf{P}}^T = \mathbf{P}^T\boldsymbol{\Omega}^T = -\mathbf{P}^T\boldsymbol{\Omega}$$

and hence,

$$\dot{\mathbf{J}} = -\boldsymbol{\Omega}\mathbf{P}\mathbf{P}^T + \mathbf{P}\mathbf{P}^T\boldsymbol{\Omega}$$

Therefore,

$$\dot{\mathbf{J}} = \mathbf{R}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{R} \tag{142}$$

(c) Now, upon differentiation of eq.(142), we have

$$\begin{aligned}
\ddot{\mathbf{J}} &= \frac{d}{dt}\left(\mathbf{P}\mathbf{P}^T\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{P}\mathbf{P}^T\right) \\
&= \dot{\mathbf{P}}\mathbf{P}^T\boldsymbol{\Omega} + \mathbf{P}\dot{\mathbf{P}}^T\boldsymbol{\Omega} + \mathbf{P}\mathbf{P}^T\dot{\boldsymbol{\Omega}} - \dot{\boldsymbol{\Omega}}\mathbf{P}\mathbf{P}^T - \boldsymbol{\Omega}\dot{\mathbf{P}}\mathbf{P}^T - \boldsymbol{\Omega}\mathbf{P}\dot{\mathbf{P}}^T \\
&= \boldsymbol{\Omega}\mathbf{P}\mathbf{P}^T\boldsymbol{\Omega} + \mathbf{P}\mathbf{P}^T\boldsymbol{\Omega}^T\boldsymbol{\Omega} + \mathbf{P}\mathbf{P}^T\dot{\boldsymbol{\Omega}} - \dot{\boldsymbol{\Omega}}\mathbf{P}\mathbf{P}^T - \boldsymbol{\Omega}\boldsymbol{\Omega}\mathbf{P}\mathbf{P}^T - \boldsymbol{\Omega}\mathbf{P}\mathbf{P}^T\boldsymbol{\Omega}^T \\
&= \boldsymbol{\Omega}\mathbf{R}\boldsymbol{\Omega} - \mathbf{R}\boldsymbol{\Omega}^2 + \mathbf{R}\dot{\boldsymbol{\Omega}} - \dot{\boldsymbol{\Omega}}\mathbf{R} - \boldsymbol{\Omega}^2\mathbf{R} + \boldsymbol{\Omega}\mathbf{R}\boldsymbol{\Omega} \\
\ddot{\mathbf{J}} &= \mathbf{R}\dot{\boldsymbol{\Omega}} - \dot{\boldsymbol{\Omega}}\mathbf{R} - \boldsymbol{\Omega}^2\mathbf{R} - \mathbf{R}\boldsymbol{\Omega}^2 + 2\boldsymbol{\Omega}\mathbf{R}\boldsymbol{\Omega}
\end{aligned}$$

8.10 The moment $\mathbf{n}_i$ of the force $\mathbf{f}$ about the three given points $\{P_i\}_1^3$ can be expressed as

$$\mathbf{n}_i = \mathbf{n} + (\mathbf{p}_i - \mathbf{c}) \times \mathbf{f} \Longrightarrow \mathbf{n}_i - \mathbf{n} = (\mathbf{p}_i - \mathbf{c}) \times \mathbf{f} \tag{143}$$

The moment about the centroid C can be determined from eq. (143)

$$\sum_1^3 (\mathbf{n}_i - \mathbf{n}) = \sum_1^3 (\mathbf{p}_i - \mathbf{c}) \times \mathbf{f} \tag{144}$$

Moreover, eq. (144) can be written in the form $\mathbf{M} = -\mathbf{F}\mathbf{p}$ where $\mathbf{F} = \operatorname{CPM}(\mathbf{f})$. Upon application of Theorem A.1,

$$-\operatorname{vect}(\mathbf{M}) = \mathbf{D}\mathbf{f} \Longrightarrow \mathbf{f} = -\mathbf{D}^{-1}\operatorname{vect}(\mathbf{M})$$