A COMPREHENSIVE SOLUTION OF THE CLASSIC BURMESTER PROBLEM

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ABSTRACT

The classic Burmester problem aims at finding the geometric parameters of a planar four-bar linkage whose coupler link attains a prescribed set of finitely separated poses. The solution proposed is claimed to be comprehensive because it (a) includes all four types of dyads—RR (revolute-revolute), PR (prismatic-revolute), RP and PP—and (b) gives due consideration to the numerics behind the solution. A PR dyad is treated as a RR dyad with its fixed joint centre at infinity, similar interpretations applying to RP and PP dyads. The paper includes the synthesis of planar four-bar linkages in its full generality, that of dyads with P joints being given the utmost attention. Finally, the underlying numerics receives the attention seldom found in the literature on the subject, our main concern being numerical *robustness*.

UNE SOLUTION EXHAUSTIVE AU PROBLÈME CLASSIQUE DE BURMESTER

RÉSUMÉ

Le problème classique de Burmester porte sur le calcul des paramètres geométriques d'un mécanisme à quatre barres articulées dont la bièle atteint un ensemble de situations à séparation finie. La solution proposée est dite exhaustive parce qu'elle (a) inclut les quatre types de dyades—RR (rotoïde-rotoïde), PR (prismatique-rotoïde), RP et PP—et (b) traite de façon approfondie les aspects numériques sous-jacents. Une dyade PR est traitée comme une dyade RR dont le centre de l'articulation fixe se trouve à l'infini, les dyades RP et PP admettant des interprétations similaires. Dans cette communication, les auteurs traitent la synthèse des mécanismes à quatre barres articulées dans toute sa généralité, tout particulièrement celle menant à des articulations P. Enfin, les aspects numériques sous-jacents de ce problème, très rarement traités dans la littérature, reçoivent ici une attention particulière, visant surtout la *robustesse* des procédures de calcul.

1 Introduction

The Burmester problem aims at finding the geometric parameters of a four-bar linkage for a prescribed set of finitely separated $poses^1$. It is well known that a RR dyad² can be synthesized exactly for up to five prescribed poses. The synthesis problem discussed here pertains to both four and five poses. The case of one PR dyad was outlined in [3].

The four-pose problem is known to admit infinitely many solutions, each solution dyad being given by a pair of corresponding cubics, the *centrepoint* and the *circlepoint* curves³. The five-pose problem, on the other hand, is known to lead to the solution of a quartic equation, and hence, admits none, two or four real dyads [1]. Extensive research has been reported on the solution of the Burmester problem with different approaches. Bottema and Roth [4], Hunt [5] and McCarthy [6] solved the five-pose problem by intersecting two centrepoint curves of two four-pose problems for two subsets of four poses out of the given five-pose set, to obtain the centrepoints. Beyer [7] and Lichtenheldt [8] reported a method based on projective geometry, while Modler [9, 10, 11, 12] investigated various special cases. Sandor and Erdman applied complex numbers [13]; Ravani and Roth [14] and Haves and Zsombor-Murray [15], in turn, solved the problem via the kinematic mapping. Schröcker et al. [16] applied the kinematic mapping to detect the branch defect in the synthesis of four-bar linkages. Recently, Brunnthaler et al. [17] provided solutions to PR and RR dyad-type determination and dimensional synthesis for the five-pose problem, by means of the kinematic mapping. Furthermore, the Burmester problem is also studied for spatial mechanisms [18]. Except for the kinematic-mapping approach, all foregoing works rely on the location of the poles of the various displacements, which lie at infinity in the presence of a pure translation, and hence, is bound to introduce singularities in this approach. More recently, the authors reported on a formulation that accounts for dyads with one P joint, and illustrated their procedure with the synthesis of one PR dyad [3]. Furthermore, to the authors' knowledge, the four-bar linkage synthesis problem with at least one P dyad is not included in commercial design software, such as LINCAGES [19], which provides for at most four-pose synthesis [20].

In its full generality, the Burmester problem can be stated, with reference to Fig. 1, as: A rigid body, attached to the coupler link of a four-bar linkage, is to be guided through a discrete set of m poses, given by $\{\mathbf{r}_j, \theta_j\}_0^m$, starting with a reference pose labeled 0, where \mathbf{r}_j is the position vector of a landmark point R of the body at the jth pose and θ_j is the corresponding angle of a line of the body, as depicted in Fig. 1. The problem consists in finding the joint centres A_0 and B that define the BA_0R dyad of the guiding four-bar linkage, dyad $B^*A_0^*R$ being determined likewise. Given that A_0 and A_0^* describe circles centred at B and B^* , respectively, the former are termed the circlepoints, the latter the centrepoints of the dyads.

In the balance of this paper we will develop a general synthesis procedure for four and five poses, applicable to problems admitting either a RR dyad or one dyad with at least one P joint. Such dyads can always be found for the four-pose problem, the conditions for the occurrence of the same dyads in terms of the prescribed set of poses being derived for the five-pose case. A synthesis method, for the same case, is developed by resorting to the geometric conditions for the existence of these dyads. Furthermore, the approach followed here relies on the notion of poles⁴ for data-conditioning purposes only, which should make the derivations more geometrically significant. Data-conditioning is a key issue in the numerics underlying the solution proposed here. Other related items are redundancy and least-square filtering. These features aim at producing *robust*, reliable solutions.

2 Determination of RR Dyads

We start with the synthesis of the four-bar linkage shown in Fig. 1, using X_0 - Y_0 as the reference coordinate frame throughout the paper. Under the usual rigid-body assumption, the synthesis equation is readily derived:

$$\|\underbrace{(\mathbf{r}_{j} - \mathbf{b}) + \mathbf{Q}_{j} \mathbf{a}_{0}}_{\mathbf{a}_{j} - \mathbf{b}}\|^{2} = \|\mathbf{a}_{0} - \mathbf{b}\|^{2}, \quad \text{for } j = 1, \dots, m$$
(1a)

¹Burmester: "Are there any points in a rigid body whose corresponding positions lies on a circle of the fixed plane for the four arbitrarily prescribed positions?" [1].

 $^{^{2}}$ A dyad is a well-known concept in the realm of kinematic synthesis, namely, the coupling of two links by means of a lower kinematic pair [2].

³These curves are well known in the literature on the Burmester problem [4]; for completeness they are recalled below.

 $^{^{4}}$ The *pole* of a finite displacement of a rigid body under *planar motion* is standard knowledge in this context: This is the point of the body that remains immovable under the above displacement.



Figure 1: Two finitely separated poses of a rigid body carried by the coupler link of a four-bar linkage

where \mathbf{a}_0 and \mathbf{b} are the position vectors of points A_0 and B, the design parameters of the RR dyad, while \mathbf{Q}_j denotes the rotation matrix carrying the guided body from pose 0 to pose j, i.e.,

$$\mathbf{Q}_{j} = \begin{bmatrix} \cos \phi_{j} & -\sin \phi_{j} \\ \sin \phi_{j} & \cos \phi_{j} \end{bmatrix}, \quad \text{with} \quad \phi_{j} \equiv \theta_{j} - \theta_{0}$$
(1b)

Upon expansion of Eq. (1a) and simplifying the expression thus resulting, we obtain

$$\mathbf{b}^{T}(\mathbf{1}-\mathbf{Q}_{j})\mathbf{a}_{0}+\mathbf{r}_{j}^{T}\mathbf{Q}_{j}\mathbf{a}_{0}-\mathbf{r}_{j}^{T}\mathbf{b}+\frac{\mathbf{r}_{j}^{T}\mathbf{r}_{j}}{2}=0, \quad j=1,\ldots,m$$
(2)

where 1 is the 2×2 identity matrix, thereby obtaining the synthesis equations allowing us to compute the design parameters.

In order to find the RR dyad, we shall first eliminate \mathbf{b} from Eq. (2), which will be achieved by rewriting this equation as:

$$\mathbf{Gz} = \mathbf{0}_m \tag{3a}$$

where $\mathbf{0}_m$ is the *m*-dimensional zero vector, **G** is a $m \times 3$ matrix linear function of \mathbf{a}_0 , and **z** is the three-dimensional array of homogeneous coordinates of B, i.e.,

$$\mathbf{G} \equiv \begin{bmatrix} \mathbf{g}_1^T \\ \vdots \\ \mathbf{g}_m^T \end{bmatrix}, \quad \mathbf{z} \equiv \begin{bmatrix} \mathbf{b} \\ 1 \end{bmatrix}, \quad \mathbf{g}_j = \begin{bmatrix} (\mathbf{1} - \mathbf{Q}_j)\mathbf{a}_0 - \mathbf{r}_j \\ \mathbf{r}_j^T \mathbf{Q}_j \mathbf{a}_0 + \mathbf{r}_j^T \mathbf{r}_j/2 \end{bmatrix}, \qquad j = 1, \dots, m$$
(3b)

By the same token, Eq. (2) is rewritten in the form

$$\mathbf{H}\mathbf{w} = \mathbf{0}_m \tag{4a}$$

with **H** defined as a $m \times 3$ matrix linear function of **b**, while **w** is the *three-dimensional array of homo*geneous coordinates of A_0 , i.e.,

$$\mathbf{H} \equiv \begin{bmatrix} \mathbf{h}_{1}^{T} \\ \vdots \\ \mathbf{h}_{m}^{T} \end{bmatrix}, \quad \mathbf{w} \equiv \begin{bmatrix} \mathbf{a}_{0} \\ 1 \end{bmatrix}, \quad \mathbf{h}_{j} = \begin{bmatrix} (\mathbf{1} - \mathbf{Q}_{j}^{T})\mathbf{b} + \mathbf{Q}_{j}^{T}\mathbf{r}_{j} \\ -\mathbf{r}_{j}^{T}\mathbf{b} + \mathbf{r}_{j}^{T}\mathbf{r}_{j}/2 \end{bmatrix}, \qquad j = 1, \dots, m$$
(4b)

Actually, once \mathbf{a}_0 is available, its numerical value can be substituted into Eq. (2), thereby obtaining a system of m > 2 equations in **b**. Therefore, **b** can be computed *directly*—as opposed to *iteratively*—from Eqs. (2) by linear-equation solving. We suggest to compute \mathbf{a}_0 and **b** independent from each other in order to avoid roundoff-error propagation, as a cascaded computation would carry the roundoff error in the computation of one of these vectors into the computation of the other. This approach is essential to the *robustness* of the algorithm described herein.

2.1 Four-Pose Case

When m = 3, both **G** and **H** are 3×3 matrices. Since **z** and **w** cannot vanish, we must have

$$\mathcal{K}$$
: det(G) = 0, \mathcal{M} : det(H) = 0 (5)

the two equations (5) yielding the *circlepoint curve* \mathcal{K} and the *centrepoint curve* \mathcal{M} of \mathbf{a}_0 and \mathbf{b} , respectively; there are thus infinitely many solutions for \mathbf{a}_0 and \mathbf{b} of the RR dyads in this case. It is noteworthy that each row of \mathbf{G} is linear in \mathbf{a}_0 , each row of \mathbf{H} is linear in \mathbf{b} , the two determinants in Eq. (5), and hence, the two curves in question thus being cubic, a well-known result [4].

2.2 Five-Pose Case

When m = 4, **G** becomes a 4×3 matrix. The condition that **z** be different from zero leads to the rank-deficiency of **G**, and hence, to the singularity of every 3×3 submatrix of **G**, i.e.,

$$\mathcal{K}_j: \ \Delta_j(\mathbf{a}_0) = \Delta_j(x, y) = \det(\mathbf{G}_j) = 0, \text{ for } j = 1, \dots, 4$$
(6)

where \mathbf{G}_j is formed by deleting the *j*th row from \mathbf{G} . Each of the four equations (6) defines one curve in the X_0 - Y_0 plane, thereby leading to the four circlepoint curves \mathcal{K}_j corresponding to the three synthesis equations obtained when deleting the *j*th equation from the given four in Eq. (2). Their common intersections yield the real circlepoints of the RR dyad. If no such common intersection occurs, the problem admits no real solution.

Likewise, the centrepoints of a RR dyad can be found from:

$$\mathcal{M}_j: \ \bar{\Delta}_j(\mathbf{b}) = \bar{\Delta}_j(u, v) = \det(\mathbf{H}_j) = 0, \ \text{for} \ j = 1, \dots, 4$$
(7)

where \mathbf{H}_j is formed by deleting the *j*th row from \mathbf{H} . The four equations thus obtained are functions of $\mathbf{b} = [u, v]^T$, and provide four centrepoint curves \mathcal{M}_j , for $j = 1, \ldots, 4$. The centrepoints are determined as the intersections of all four curves \mathcal{M}_j . Again, if no common intersections occur, then the problem admits no real solution.

Therefore, every triplet of synthesis equations (2), out of the given $m \ge 4$, defines both one centrepoint and one circlepoint curve.

While algebraically any pair of eqs.(6) suffices to determine \mathbf{a}_0 , for numerical robustness we use all four equations (6). Upon regarding these four equations in the two unknown components of \mathbf{a}_0 —the Cartesian coordinates of A_0 —as an overdetermined system of four equations in two unknowns, we compute the two unknowns as the least-square approximation of eqs.(6). A similar rationale applies to Eq.(7) in connection with the centrepoint B. This way of handling the synthesis equations is at the core of the robustness of our approach.

3 Synthesis of Linkages with One PR Dyad

From the second of Eqs. (5), we obtain a PR dyad for the four-pose case when the centrepoint of \mathcal{M} goes to infinity. Hence, we may simply compute the asymptote of the cubic curve \mathcal{M} , and determine the P joint from the slope of the asymptote. Computing the asymptotes of a planar curve is equivalent to finding the points at infinity of the given curve. A novel approach to finding a PR dyad is introduced here,

that obviates the computation of the asymptotes. To simplify the derivation of the synthesis equations for PR dyads, we divide both sides of Eq. (2) by the Euclidean norm of **b**, thus obtaining

$$[\underbrace{(\mathbf{1} - \mathbf{Q}_j)\mathbf{a}_0 - \mathbf{r}_j}_{-\mathbf{u}_j}]^T \frac{\mathbf{b}}{\|\mathbf{b}\|} + \left(\mathbf{r}_j^T \mathbf{Q}_j \mathbf{a}_0 + \frac{\mathbf{r}_j^T \mathbf{r}_j}{2}\right) \frac{1}{\|\mathbf{b}\|} = 0 , \quad j = 1, \dots, m$$
(8)

Furthermore, we define a unit vector $\boldsymbol{\beta}$ as

$$\beta = \frac{\mathbf{b}}{\|\mathbf{b}\|} \tag{9}$$

When $\|\mathbf{b}\| \to \infty$, the centrepoint *B* goes to infinity, which leads to a PR dyad, the unit vector $\boldsymbol{\beta}$ giving the direction of the asymptote of every centrepoint curve that arises from every triplet of Eqs. (8). Moreover, $\boldsymbol{\beta}$ also gives the *line of sight* of *B* at infinity, the normal direction to $\boldsymbol{\beta}$ indicating the direction of the translations allowed by the corresponding P joint.

Under the above condition, the second term of Eq. (8) vanishes. Moreover, upon substitution of Eq. (9) into Eq. (8), we have $\mathbf{u}_j^T \boldsymbol{\beta} = 0$, where $\mathbf{u}_j \equiv \mathbf{a}_j - \mathbf{a}_0$ is the displacement of the circlepoint A_0 at the *j*th pose, i.e.,

$$\mathbf{u}_j = \mathbf{r}_j - (\mathbf{1} - \mathbf{Q}_j)\mathbf{a}_0, \quad j = 1, \dots, m \tag{10}$$



Figure 2: Relation between the *i*th and *j*th poses and the circlepoints

With reference to Fig. 2, \mathbf{u}_i (i = 1, ..., m) is the *i*th displacement vector of the circlepoint. For a PR dyad, all *m* vectors \mathbf{u}_i must be parallel. In other words, the cross product of any two displacement vectors must vanish. However, rather than working with cross products, we simplify the analysis by resorting to the well-known two-dimensional representation of the cross product. This is based on matrix **E** rotating vectors in the plane through 90° ccw. Hence, the parallelism condition between \mathbf{u}_i and \mathbf{u}_j can be expressed as

$$\Delta_{ij} = \mathbf{u}_j^T \mathbf{E} \mathbf{u}_i = 0, \quad i, j = 1, \dots, m, \quad i \neq j$$
(11)

which expands to

$$\Delta_{ij} = \mathbf{a}_0^T (-\mathbf{E}\mathbf{Q}_i - \mathbf{Q}_j^T \mathbf{E} + \mathbf{Q}_j^T \mathbf{E}\mathbf{Q}_i) \mathbf{a}_0 - (\mathbf{E}\mathbf{r}_i - \mathbf{Q}_j^T \mathbf{E}\mathbf{r}_i - \mathbf{E}\mathbf{r}_j + \mathbf{Q}_i^T \mathbf{E}\mathbf{r}_j)^T \mathbf{a}_0 + \mathbf{r}_j^T \mathbf{E}\mathbf{r}_i = 0$$
(12)

We develop below all quadratic terms of Eq. (12), those in the first line of this equation, by writing \mathbf{Q}_i in the form $\mathbf{Q}_i = c_i \mathbf{1} + s_i \mathbf{E}$, in which $s_i \equiv \sin \phi_i$ and $c_i \equiv \cos \phi_i$. Hence,

$$-\mathbf{a}_{0}^{T}\mathbf{E}\mathbf{Q}_{i}\mathbf{a}_{0} = -\mathbf{a}_{0}^{T}\mathbf{E}(c_{i}\mathbf{1} + s_{i}\mathbf{E})\mathbf{a}_{0} = -c_{i}\mathbf{a}_{0}^{T}\mathbf{E}\mathbf{a}_{0} - s_{i}\mathbf{a}_{0}^{T}\mathbf{E}^{2}\mathbf{a}_{0} = s_{i}||\mathbf{a}_{0}||^{2}$$
(13a)

$$-\mathbf{a}_{0}^{T}\mathbf{Q}_{j}^{T}\mathbf{E}\mathbf{a}_{0} = -\mathbf{a}_{0}^{T}\mathbf{E}^{T}\mathbf{Q}_{j}\mathbf{a}_{0} = \mathbf{a}_{0}^{T}\mathbf{E}\mathbf{Q}_{j}\mathbf{a}_{0} = -s_{j}||\mathbf{a}_{0}||^{2}$$
(13b)

$$\mathbf{a}_{0}^{T}\mathbf{Q}_{j}^{T}\mathbf{E}\mathbf{Q}_{i}\mathbf{a}_{0} = \mathbf{a}_{0}^{T}(c_{j}\mathbf{1} - s_{j}\mathbf{E})(c_{i}\mathbf{E} + s_{i}\mathbf{1})\mathbf{a}_{0} = \mathbf{a}_{0}^{T}[-(c_{j}s_{i} - s_{j}c_{i})\mathbf{1} + (s_{j}s_{i} + c_{j}c_{i})\mathbf{E}]\mathbf{a}_{0}$$

= $(-c_{j}s_{i} + s_{j}c_{i})||\mathbf{a}_{0}||^{2} = -\sin(\phi_{i} - \phi_{j})||\mathbf{a}_{0}||^{2}$ (13c)

Notice that the foregoing relations could have been obtained by pure geometric reasoning, as $\mathbf{E}\mathbf{Q}_i$ is a rotation through an angle $\phi_i + \pi/2$, given that angles of rotation are additive in planar motion. By the same token, $\mathbf{Q}_j^T \mathbf{E}$ is a rotation through an angle $\pi/2 - \phi_j$, while $\mathbf{Q}_j^T \mathbf{E}\mathbf{Q}_i$ is one through an angle $\phi_i + \pi/2 - \phi_j$. Moreover, terms of the form $\mathbf{a}_0^T \mathbf{E}\mathbf{a}_0$ vanish because \mathbf{E} is skew-symmetric.

Further, let $\mathbf{v}_{ij} = -\mathbf{E}\mathbf{r}_i + \mathbf{Q}_j^T \mathbf{E}\mathbf{r}_i + \mathbf{E}\mathbf{r}_j - \mathbf{Q}_i^T \mathbf{E}\mathbf{r}_j$, as appearing in the second line of Eqs. (12), which are now rewritten as

$$\Delta_{ij} = (s_i - s_j - s_{i\bar{j}}) \| \mathbf{a}_0 \|^2 + \mathbf{v}_{i\bar{j}}^T \mathbf{a}_0 + \mathbf{r}_{j}^T \mathbf{E} \mathbf{r}_i = 0, \quad i, j = 1, \dots, m, \quad i \neq j$$
(14)

and represent the loci of A_0 , of position vector \mathbf{a}_0 , namely, a family of circles $\{\mathcal{C}_{ij}\}_{i,j=1;i\neq j}^n$, where $s_{i\bar{j}} \equiv \sin(\phi_i - \phi_j)$. A line \mathcal{L} passing through A_0 and parallel to the direction of sliding of the P joint is shown normal to vector β in Fig. 2. Given that P joints have a direction, but no position, the slider implementing this joint can be placed anywhere on the fixed frame, as long as its sliding direction is parallel to \mathcal{L} . It is common practice to represent the P joint of a PR dyad as a line passing through the centre of the R joint, but this by no means limits the actual implementation of the joint in question. Remarks:

 The foregoing relations have been derived from the condition of the vanishing of the product u^T_j Eu_i in Eq.(11). This product can be shown to be identical to the determinant of a 2 × 2 matrix D, namely,

$$\mathbf{D} = \begin{bmatrix} \mathbf{u}_j & \mathbf{u}_i \end{bmatrix} \tag{15}$$

Obviously, the vanishing of det(D) is equivalent to the linear dependence, and hence, the parallelism of vectors u_i and u_j .

2. The locus of the centrepoints of a PR dyad, for every pair of parallel unit vectors $(\mathbf{u}_i, \mathbf{u}_j)$, is a circle C_{ij} . As one of the anonymous reviewers pointed out, this is a classical result, the circle in question being the "circle of sliders." Indeed, this circle is derived from geometric arguments in [21], although no specific name is given in this reference to the circle. In the same reference, Hall points out that, in the limit, as the three poses defining that circle become infinitesimally separated, the circle becomes the *inflection circle* of curvature theory.

3.1 Four-Pose Case

Upfront, notice that a PR dyad always exists in this case, as we have one single centrepoint curve, the corresponding P joint being equivalent to a R joint with its centre at infinity. This location is given by the single asymptote of the centrepoint curve. Hence, one single P joint is to be expected here.

In this case we have three displacements $\{\mathbf{u}_i\}_{i=1}^3$, and hence, three parallelism conditions, namely, $\mathbf{u}_1 || \mathbf{u}_2, \mathbf{u}_2 || \mathbf{u}_3$ and $\mathbf{u}_3 || \mathbf{u}_1$, as given by Eq. (14) for $(i, j) \in \{(1, 2), (2, 3), (3, 1)\}$. In principle, two of these conditions imply the third. However, if \mathbf{u}_2 happens to vanish, then, while the first two conditions still hold, the third does not necessarily do so. To guarantee the parallelism condition in any event, we use the three equations (14).

Now, the three equations at hand represent, each, a circle in the X_0 - Y_0 plane. It is apparent that we can always find a suitable linear combination of two distinct pairs of the three equations (14) that will yield, correspondingly, two lines. Hence, the parallelism condition leads to one circle C and two lines \mathcal{L}_1 and \mathcal{L}_2 . The geometric interpretation of the problem of finding the point A_0 then allows a straightforward geometric interpretation: the circlepoint sought (a) is the intersection of the two lines and (b) lies on the circle.

Remarks:

- 1. If the coefficient of $\|\mathbf{a}_0\|^2$ in one of Eqs. (14) vanishes, then the resulting equation is already a line. A second line is then obtained by a suitable linear combination of the two circle equations, which will then lead us to the general case.
- 2. If the same coefficient vanishes in two of Eqs. (14), then we need not look for any linear combination to obtain the two lines of the general case.
- 3. If the same coefficient vanishes in the three Eqs. (14), then we have three lines that must be concurrent at a common point.

4. The above statement on the existence of one single circle point A_0 is best explained by noticing that, from the plot of the centrepoint curve, its asymptote can be estimated by inspection—a precise value can be obtained from the equation of the asymptote, of course. If this estimate is plugged into eq.(8), and the second term of the equation is deleted because, as $||\mathbf{b}|| \to \infty$, this term tends to zero, we obtain three linear equations in \mathbf{a}_0 . In the absence of roundoff error, only two are independent⁵, and hence, determine uniquely \mathbf{a}_0 .

Once \mathbf{a}_0 is obtained, solving for $\boldsymbol{\beta}$ is straightforward⁶:

$$\boldsymbol{\beta} = \frac{\mathbf{E}\hat{\mathbf{u}}}{\|\mathbf{E}\hat{\mathbf{u}}\|}, \quad \hat{\mathbf{u}} = \frac{1}{m}\sum_{j=1}^{m}\mathbf{u}_{j}$$
(16)

3.2 Five-Pose Case

Drawing from the case m = 3, we can conclude that a PR dyad is possible in the case at hand if and only if the asymptotes of the four centrepoint curves \mathcal{K}_i are all parallel. Rather than deriving the parallelism condition for the asymptotes, we resort to an alternative approach, based on that introduced in Subsection 3.1.

In this case we have four displacements $\{\mathbf{u}_i\}_{1}^{4}$, and hence, six possible pairs $(\mathbf{u}_i,\mathbf{u}_j)$, for $j \neq i$. It is noteworthy that the parallelism relation is *transitive*; e.g., $\mathbf{u}_1 \| \mathbf{u}_2$ and $\mathbf{u}_2 \| \mathbf{u}_3 \Rightarrow \mathbf{u}_1 \| \mathbf{u}_3$. Hence, the number of independent relations reduces to three. However, and within the spirit of robustness, we use the full six parallelism conditions available.

Similar to the four-pose case, we can always find a suitable linear combination of the first equation with each of the remaining five equations (14) that will yield, correspondingly, five lines. Hence, the parallelism conditions lead to one circle C and five lines $\mathcal{L}_1, \ldots, \mathcal{L}_5$. A geometric interpretation: the circlepoint A_0 sought (a) is the intersection of the five lines and (b) lies on the circle.

A similar discussion for the cases in which the coefficient of $\|\mathbf{a}_0\|^2$ in Eq. (14) vanishes is straightforward, and follows the same line of reasoning as in the four-pose case.

If there is one common point to the five lines and the circle, this point is then the solution A_0 sought. Otherwise, there is no solution. After \mathbf{a}_0 is obtained, β can be found from Eq. (16).

3.3 Summary of Results

In the case of three-pose synthesis, the centrepoint of a PR dyad is located on a circle, which is its locus. In the case of four-pose synthesis, the same centrepoint is found in three circles. If the problem admits a solution, these circles intersect at one common point, thus yielding one unique solution. In the case of five poses, we end up with six circles. If the poses obey the parallelism condition, the six circles intersect at one common point, which thus yields one unique solution for the centrepoint.

4 Synthesis of Linkages with One RP Dyad

This case is handled by means of *kinematic inversion*, i.e., by exchanging the roles of the fixed and the coupler links. Although the concept of kinematic inversion is straightforward, its algorithmic implementation warrants a brief discussion, which is included below.

From the first of Eqs. (5), we can obtain a RP dyad for the four-pose case when the circlepoint of \mathcal{K} goes to infinity. Hence, we may simply compute the asymptote of the cubic curve \mathcal{K} , and determine the P joint. However, rather than resorting to the asymptote, we proceed as in the case of the PR dyad. To this end, we divide both sides of Eq. (2) by $\|\mathbf{a}_0\|$, thus obtaining

$$[\underbrace{(\mathbf{1} - \mathbf{Q}_j^T)\mathbf{b} + \mathbf{Q}_j^T \mathbf{r}_j}_{=\mathbf{r}_j}]^T \frac{\mathbf{a}_0}{\|\mathbf{a}_0\|} + \left(-\mathbf{r}_j^T \mathbf{b} + \frac{\mathbf{r}_j^T \mathbf{r}_j}{2}\right) \frac{1}{\|\mathbf{a}_0\|} = 0 , \quad j = 1, \dots, m$$
(17)

Moreover, we define

$$\boldsymbol{\alpha} = \frac{\mathbf{a}_0}{\|\mathbf{a}_0\|} \tag{18}$$

⁵To account for roundoff error, we recommend to regard the three equations as independent, and compute the unique value of a_0 as their least-square approximation.

⁶One single vector \mathbf{u}_i would suffice. We take the mean value here in order to filter out roundoff error.

When $\|\mathbf{a}_0\| \to \infty$, A_0 goes to infinity, which leads to a RP dyad. Under this condition, the second term of Eq. (17) vanishes. Upon substitution of Eq. (18) into Eq. (17), we obtain $\mathbf{s}_j^T \boldsymbol{\alpha} = 0$, $j = 1, \ldots, m$, where $\mathbf{s}_j \equiv (\mathbf{Q}_j^T - \mathbf{1})\mathbf{b} - \mathbf{Q}_j^T \mathbf{r}_j$. Notice that

$$\mathbf{Q}_j \mathbf{s}_j = -\mathbf{r}_j + (\mathbf{1} - \mathbf{Q}_j)\mathbf{b}$$



Figure 3: Relation between the *j*th pose and the centrepoints, for the RP dyad

Upon comparison of $\mathbf{Q}_j \mathbf{s}_j$ with \mathbf{u}_j , as given by Eq. (10), it is apparent that \mathbf{s}_j represents the displacement of B as seen from a frame with origin at R_j and fixed to the coupler link. We thus define $\mathbf{b}_j \equiv \mathbf{b} + \mathbf{s}_j$ as the position vector of a point B_j , which is the *displaced* centrepoint as seen from the coupler link and as depicted in Fig. 3; in this figure, $\alpha_j \equiv \mathbf{Q}_j \alpha$. Hence, B_j and $B \equiv B_0$ lie on a line \mathcal{L} fixed to the coupler link that is defined by the centrepoint B and the direction of the P joint. At the *j*th pose, this line appears as \mathcal{L}_j , which is illustrated in Fig. 3. Also notice that the unit vector α has a fixed direction in the coupler link, but becomes α_j in the X_0 - Y_0 frame of Fig. 1. Vector \mathbf{s}_j is thus the *j*th displacement vector of the centrepoint in the foregoing *kinematic inversion*. Similar to the PR dyad, if all \mathbf{s}_j ($j = 1, \ldots, m$) are mutually parallel, a RP dyad exists, i.e.,

$$\Gamma_{ij} = \mathbf{s}_i^T \mathbf{E} \mathbf{s}_j = 0, \quad i, j = 1, \dots, m, \quad i \neq j$$
(19)

which expands to

$$\Gamma_{ij} = \mathbf{b}^T (\mathbf{1} - \mathbf{Q}_i) \mathbf{E} (\mathbf{1} - \mathbf{Q}_j^T) \mathbf{b} + [(\mathbf{1} - \mathbf{Q}_i) \mathbf{E} \mathbf{Q}_j^T \mathbf{r}_j - (\mathbf{1} - \mathbf{Q}_j) \mathbf{E} \mathbf{Q}_i^T \mathbf{r}_i]^T \mathbf{b} + \mathbf{r}_i^T \mathbf{Q}_i \mathbf{E} \mathbf{Q}_j^T \mathbf{r}_j = 0$$
(20)

We develop below the first term of Eq. (20) by writing \mathbf{Q}_i in the form $c_i \mathbf{1} + s_i \mathbf{E}$. Hence,

$$\mathbf{b}^{T}(1-\mathbf{Q}_{i})\mathbf{E}(1-\mathbf{Q}_{j}^{T})\mathbf{b} = \mathbf{b}^{T}\mathbf{E}(1-c_{i}-c_{j}+c_{i}c_{j}+s_{i}s_{j})\mathbf{b} + \mathbf{b}^{T}(s_{i}-s_{j}-s_{i}c_{j}+c_{i}s_{j})\mathbf{b}$$
$$= (s_{i}-s_{j}-s_{i}c_{j}+c_{i}s_{j})\|\mathbf{b}\|^{2}$$
(21)

Further, let $\mathbf{t}_{ij} = (1 - \mathbf{Q}_i)\mathbf{E}\mathbf{Q}_j^T\mathbf{r}_j - (1 - \mathbf{Q}_j)\mathbf{E}\mathbf{Q}_i^T\mathbf{r}_i$, as appearing in the second term of Eq. (20), which is now rewritten as

$$\Gamma_{ij} = (s_i - s_j - s_{i\bar{j}}) \|\mathbf{b}\|^2 + \mathbf{t}_{ij}^T \mathbf{b} + \mathbf{r}_i^T \mathbf{Q}_i \mathbf{E} \mathbf{Q}_j^T \mathbf{r}_j = 0, \quad i, j = 1, \dots, m, \qquad i \neq j$$
(22)

and hence, $\Gamma_{ij} = 0$ represents a family of circles as well. Note that different design equations obtained from the geometry of the RP chain can be found in [4].

4.1 Four-Pose Case

In this case we have three displacements $\{s_i\}_{1}^{3}$, and hence, three parallelism conditions, namely, $s_1||s_2$, $s_2||s_3$ and $s_3||s_1$, as given by Eq. (22) for $(i, j) \in \{(1, 2), (2, 3), (3, 1)\}$. In principle, two of these conditions imply the third. However, if s_2 happens to vanish, then, while the first two conditions still hold, the third does not necessarily do so. To guarantee the parallelism conditions in any event, we use the three equations (22), similar to Subsection 3.1

Now, the three equations at hand represent, each, a circle in the X_0 - Y_0 plane. It is apparent that we can always find a suitable linear combination of two distinct pairs of the three equations (22) that will yield, correspondingly, two lines. Hence, the parallelism condition leads to one circle C and two lines \mathcal{L}_1 and \mathcal{L}_2 . The geometric interpretation of the problem of finding the point B then allows a straightforward geometric interpretation: the circlepoint sought (a) is the intersection of the two lines and (b) lies on the circle. Similar remarks to those for PR dyads apply in this case, which need not be repeated here.

Once b is found, solving for α is straightforward:

$$\boldsymbol{\alpha} = \frac{\mathbf{E}\hat{\mathbf{s}}}{\|\mathbf{E}\hat{\mathbf{s}}\|}, \quad \hat{\mathbf{s}} = \frac{1}{m} \sum_{j=1}^{m} \mathbf{s}_j \tag{23}$$

with a similar remark to that in footnote 6.

4.2 Five-Pose Case

Similar to Subsection 3.2, a RP dyad is possible if and only if the asymptotes of all four circlepoint curves \mathcal{M}_i are parallel. Again, rather than deriving the parallelism condition for the four asymptotes, a rather lengthy derivation, we resort to the alternative approach introduced for the PR dyad.

In this case we have four displacements $\{s_i\}_{i=1}^{4}$, and hence, six parallelism conditions, as given by Eq. (22) for $(i, j) \in \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$. As pointed out in previous cases, in order to guarantee the parallelism conditions, we use the six equations (22).

As discussed in Subsection 3.2, we can always find a suitable linear combination of the first equation with each of the remaining five equations (22) that will yield, correspondingly, five lines. Hence, the parallelism conditions lead to one circle C and five lines $\mathcal{L}_1, \ldots, \mathcal{L}_5$. A geometric interpretation follows: the circlepoint B sought (a) is the intersection of the five lines and (b) lies on the circle.

If there is one common point to the five lines and the circles, this point is then the solution B sought. Otherwise, there is no solution. After **b** is obtained, α can be found from Eq. (23).

5 Synthesis of Linkages with One PP Dyad

A PP dyad is a RR dyad with A_0 and B at infinity. In order to find the condition for PP dyads to be solutions to the Burmester problem, we substitute Eqs. (9) and (18) into Eq. (2) and divide its two sides by $\|\mathbf{a}_0\|\|\mathbf{b}\|$. Under the assumption that $\|\mathbf{a}_0\| \to \infty$ and $\|\mathbf{b}\| \to \infty$, we obtain

$$\boldsymbol{\beta}^{T}(1-\mathbf{Q}_{j})\boldsymbol{\alpha}=0, \quad j=1,\dots,m$$
(24)

Now, the conditions on the given poses for the problem to admit one PP dyad are derived upon substituting in the foregoing equation \mathbf{Q}_j by $c_j \mathbf{1} + s_j \mathbf{E}$, as we did when deriving Eqs.(13). With this substitution, Eqs.(24) expand to

$$\cos\gamma - \cos\phi_j \cos\gamma - \sin\phi_j \sin\gamma = 0 , \qquad j = 1, \dots, m$$
⁽²⁵⁾

where γ is the angle between α and β —i.e., between the directions of the two P joints at the *reference* posture of the mechanism, which is defined as that corresponding to the reference pose 0 of the coupler link—with $\cos \gamma = \beta^T \alpha$ and $\sin \gamma = \beta^T \mathbf{E} \alpha$. Equation (25) can be simplified to

$$\cos \gamma = \cos(\phi_j - \gamma)$$
, $j = 1, \dots, m$ (26)

Hence, the conditions sought are readily derived as

$$\phi_j = 0 \quad \text{or} \quad \phi_j = 2\gamma , \qquad j = 1, \dots, m \tag{27}$$

Both solutions verify the synthesis equations (24). The geometric meaning of Eq. (24) is that the angle between α and β at the reference pose equals that between $\mathbf{Q}_j \alpha$ and β , the latter being the angle between the directions of the two P joints at the *j*th pose. Moreover, $\phi_j = 0$ makes \mathbf{Q}_j the identity matrix, which means that the *j*th displacement is a translation. Condition $\phi_j = 2\gamma$ implies a transformation of the pair (α, β) into a pair $(\alpha, \overline{\beta})$ that includes exactly the same angle γ , but this pair is a *reflection* of (α, β) . The second condition for the existence of a PP dyad to be possible is $\phi_j = 0$, for $j = 1, \ldots, m$, which implies that the *m* prescribed rigid-body displacements are translations. Therefore,

The condition for the existence of a PP dyad is that the m prescribed displacements be translations.

6 Numerical Implementation

6.1 Data-Conditioning

Due to roundoff error, the computed solutions of Eqs. (14) and (22) may not be verified within a reasonable tolerance, which then may suggest that a P joint does not exist. Roundoff error in the data is bound to be amplified because of the presence of dimensions. To reduce the effect of data-dimension on roundoff error, dimensionless displacement vectors are desirable. To this end, we introduce a suitable normalization, as described below.



Figure 4: Illustration of vector normalization: (a) determination of the pole P_i ; (b) determination of the pole centroid

As depicted in Fig. 4a, the displacement from the reference pose to the *i*th pose can be considered a pure rotation about the pole P_i . Distances $\{d_i\}_1^m$ between P_i and the centroid C of $\{P_i\}_1^m$ are first obtained. The rms value d_{rms} of these distances is considered as a *characteristic length* for normalization, the displacement vectors being normalized as

$$\rho_i \equiv \frac{\mathbf{r}_i}{d_{rms}}, \quad i = 1, \dots, m \tag{28}$$

with

$$d_{rms} = \sqrt{\frac{1}{m} \sum_{i=1}^{m} d_i^2}, \quad d_i = \|\mathbf{p}_i - \mathbf{c}\|, \quad \mathbf{c} = \frac{1}{m} \sum_{i=1}^{m} \mathbf{p}_i$$
(29)

To find p_i , we refer to Fig. 4a, whence,

$$\mathbf{p}_i = \frac{\mathbf{r}_i}{2} + l_i \mathbf{E} \hat{\mathbf{r}}_i, \quad \hat{\mathbf{r}}_i = \frac{\mathbf{r}_i}{\|\mathbf{r}_i\|}, \quad i = 1, \dots, m$$
(30a)

where, again, we have used the well-known expression for the two-dimensional form of the cross product via matrix \mathbf{E} , with l_i defined as

$$l_i \equiv \frac{\|\mathbf{r}_i\|}{2\tan(\phi_i/2)}, \quad \phi_i \neq 0 \tag{30b}$$

Remarks:

1. Should the set of given poses involve one pure translation, say at the *j*th pose, then P_j goes to infinity, but $\phi_j = 0$ and all the points of the rigid body in question undergo identical translations t_j . In this case, $l_j \to \infty$ and Eqs. (30a & b) would not apply. In order to cope with this case, we declare d_j of Eqs. (29) to be simply

$$d_j = ||\mathbf{t}_j||$$
 (31)

- 2. Should all the *m* displacements be rotations about one and the same point, this pathological case can be readily detected from the vanishing of *d_{rms}*, to a previously established numerical tolerance. In this case, a four-bar linkage is simply not needed, the *m* displacements being reachable by means of one single R joint centred at the common pole.
- 3. A characteristic length, to be worth the name, should be an intrinsic property of the physical phenomenon under study. Other than this, there is no specific criterion to define a characteristic quantity in physics⁷. In our case, the "phenomenon" is simply a set of displacements in the plane. Our claim that this definition is sound is based on the frame-invariance of the quantities involved: the poles or a combination of poles and translation norms, as the case may be; the centroid of the finite poles; and the rms value of the distances from all finite poles to their centroid.
- 4. The case may be that the magnitude of the translations involved are various orders of magnitude the distances d_i in Fig. 4b. The reason why this case arises is an unlucky choice of the point R used to describe the various poses: this point happens to lie "too far" from the region in which the set of poles is concentrated. This case may be detected by a small, yet above the working tolerance, value of d_{rms} when compared to the order of magnitude of the $||\mathbf{r}_j||$ translation norms. An obvious solution to this pathological case is to change the given reference point R to a new one, closer to the said region. How "close" is a matter of engineering judgment. We would recommend to look for values of $||\mathbf{r}_j||$ of the same order of magnitude as d_{rms} . We believe that there must be an optimum location of the point R within the body under guidance that will yield an optimally conditioned set of data values. This issue, however, deserves special attention, and is hence left for further study.

6.2 Solution

In the geometric interpretation of the parallelism conditions, we obtain one circle and two or five lines by manipulating the three or, correspondingly, six circle equations. However, it is not recommended to numerically solve these equations, because of the nature of the operations involved. Indeed, these operations require divisions by the leading coefficients of Eq. (14) or, correspondingly, (22). The leading coefficients are sums and differences of quantities whose absolute values are not greater than unity. The risk of these coefficients attaining unacceptably small absolute values should not be neglected, divisions by such numbers being known to lead to numerical catastrophes [22]. A numerically robust solution should be based on the original circle equations, as illustrated with the examples below.

A key step in the solution procedure proposed here is the computation, in the five-pose case, of the centrepoint coordinates from the four cubic equations (6) or the circlepoint coordinates from the four cubic equations (7). Either of these equations represents an overdetermined system of four nonlinear equations in only two unknowns. The usual approach to solving this system consists in picking up arbitrarily any two of the four equations and solve them with a nonlinear-equation solver. While this approach will work in well-conditioned cases, the risk of ill-conditioning should not be overlooked. By ill-conditioning in the case of nonlinear equations we mean a large condition number of the Jacobian matrix of the nonlinear system, evaluated at the solution. It is shown in [23] that the condition number of the Jacobian matrix of a nonlinear system of two equations in two unknowns, which define two curves in the plane of those unknowns, is given by $1/|\sin \gamma|$, where γ is the angle made by the tangents to the two curves at the intersection point. Hence, if at least two cubics out of the four intersect at a small angle γ , and the designer happens to choose the equations associated with those two curves, the computed solution will be corrupted with an inadmissibly high roundoff error.

Notice that the foregoing discussion applies to cases in which a RR dyad is being sought. In the presence of a dyad with one P joint, the same arguments hold, but this time as applicable to the solution of the overdetermined system (14) or, correspondingly, of system (22).

We propose here to cope with ill-conditioning by using all four curves, within a nonlinear least-square approach. Obviously, in the presence of an intrinsically ill-conditioned problem, all four curves intersect at small angles. In this case, even a least-square approach will not be able to handle the inadmissibly large roundoff error in the computed results.

Finally, the process of computing the least-square approximation to an overdetermined nonlinear system of equations is iterative. If the data are not conditioned as suggested in Subsection 6.1, there is a risk of computing intermediate results, i.e., intermediate error values to that approximation, that are

 $^{^{7}}$ For example, in studying turbulence in ducts of arbitrary cross section, a characteristic length of the duct is usually defined in terms of the dimensions of the cross section when calculating the *Reynolds number*. This length can be the equivalent radius of the circular cross section with the same area as the given one, which amounts to the rms value of the distances of all the points on the perimeter from the centroid of the cross section.

j	\mathbf{r}_{j} [mm]	$\phi_j \; [\text{deg}]$
0	$[0.0, 0.0]^T$	0.0
1	$[-3.1182, 0.3225]^T$	-14.523
2	$[-8.6373, -0.9378]^T$	-21.870
3	$[-14.5056, -2.8354]^T$	-18.945
4	$[-19.4033, -5.1301]^T$	-7.097

Table 1: Five prescribed poses for rigid-body guidance

Table 2: Five dimensionless poses

j	$ ho_j$	$\phi_j ~[\mathrm{deg}]$
0	$[0.0, 0.0]^T$	0.0
1	$[-0.052, 0.005]^T$	-14.523
2	$[-0.144, -0.015]^T$	-21.870
3	$[-0.242, -0.047]^T$	-18.945
4	$[-0.323,-0.085]^T$	-7.097

either inadmissibly large or misleadingly small. Data-conditioning is intended to prevent such problematic situations.

7 Numerical Examples

We provide two examples to illustrate the foregoing synthesis procedure. Prior to embarking on the numerical solution of the synthesis equations, we normalize the equations to render them dimensionless, thereby limiting roundoff-error amplification.

7.1 Example 1

The first example consists of the synthesis of a four-bar linkage guiding its coupler link through the five poses of Table 1. The four cubic centrepoint curves with their asymptotes are shown in Fig. 5; we can see that these asymptotes are parallel to each other, and hence, the problem admits one PR dyad. The six circle equations (14) are displayed below:

$$\Delta_{12} = -0.0061(x^2 + y^2) - 0.9513x - 0.4069y + 5.7107$$
(32a)

$$\Delta_{13} = -0.0032(x^2 + y^2) - 2.5173x - 1.1104y + 13.521$$
(32b)

$$\Delta_{14} = -0.0020(x^2 + y^2) + 4.3143x + 1.9225y - 22.256 \tag{32c}$$

$$\Delta_{23} = 0.0032(x^2 + y^2) - 2.4459x - 1.3278y + 10.886 \tag{32d}$$

$$\Delta_{24} = 0.0060(x^2 + y^2) - 5.7985x - 3.1254y + 26.113$$
(32e)

$$\Delta_{34} = 0.0042(x^2 + y^2) - 4.2511x - 2.2551y + 19.397 \tag{32f}$$

their corresponding circles having one common point, as shown in Fig. 6, which indicates that a PR dyad is possible.

Once the poses are data-conditioned by the method reported in Subsection 6.1, we have the set of dimensionless poses given in Table 2, with the characteristic length $d_{rms} = 59.92$ mm.

Furthermore, the coordinates of the common point of Fig. 6 yield the unique solution \mathbf{a}_0 . With \mathbf{a}_0 known, β is determined from Eq. (16), the results being recorded in the upper part of Table 3.



Figure 5: Four cubic centrepoint curves with their asymptotes parallel to each other



Figure 6: Curve plots to determine the circlerpoint of a PR dyad: (a) the big picture; (b) a zoom-in around the common intersection of all circles



Figure 7: Curve plots to find the centrepoint of a RR dyad, a zoom-in of Fig. 5

		a₀ [mm]	β
PR dyad		$[8.048, -6.372]^T$	$[-0.287, 0.957]^T$
		a ₀ [mm]	b [mm]
RR dyad	#1	$[-11.729, -9.350]^T$	$[-20.921, -17.063]^T$
	#2	$[9.175, -1.512]^T$	$[-15.694, 42.678]^T$
	#3	$[23.799, 9.406]^T$	$[12.964, 9.007]^T$

Table 3: Synthesis of the four-bar linkage of Example 1



Figure 8: The RRRP linkage synthesized for the five prescribed poses of Table 1



Figure 9: The RRRR four-bar linkage synthesized for the five prescribed poses of Table 1

j	\mathbf{r}_{j} [mm]	$\phi_j \; [\text{deg}]$
0	$[0.0, 0.0]^T$	0.0
1	$[-0.4948, 0.0209]^T$	11.0981
2	$[-0.9600, -0.2280]^T$	22.9064
3	$[-1.2645, -0.6955]^T$	34.6220
4	$[-1.3178, -1.2750]^T$	45.9544

Table 4: Five prescribed poses for Example 2

Table 5: Five dimensionless poses for Example 2

j	$oldsymbol{ ho}_j$	$\phi_j \; [\text{deg}]$
0	$[0.0, 0.0]^T$	0.0
1	$[-1.0603, 0.0449]^T$	11.0981
2	$[-2.0571, -0.4885]^T$	22.9064
3	$[-2.7095, -1.4903]^T$	34.6220
4	$[-2.8239, -2.7321]^T$	45.9544

In determining a RR dyad to complete the linkage, the solution of the set of equations (6) and (7) yields the values of a_0 and b, as listed in Table 3. These three solutions correspond to the three intersections in Fig. 5 and indicated in the zoom-out of Fig. 7. A synthesized mechanism, based on solution # 1, is depicted in Fig. 8, where A_{01} and B_1 are the circlepoint and centrepoint of the RR dyad, respectively, while A_{02} is the circlepoint of the PR dyad. Furthermore, the locations of A_{02} and B_1 are also displayed in Figs. 6(b) and 7, respectively. Another mechanism, with two RR dyads, derived from solutions # 1 and # 3, is depicted in Fig. 9.

Apparently, all five prescribed poses are located on one of the two branches of the coupler curve. Therefore, this mechanism is free of branch-defect, with all five poses visited by the coupler link.

The synthesis error, i.e., the least-square error in solving the overdetermined nonlinear system of Eqs. (14) in this example, using the dimensionless poses with data-conditioning, is 2.3×10^{-12} , while the error obtained by the poses without data-conditioning is 2.7×10^{-5} mm. We normalize this error by dividing it by the characteristic length, thereby obtaining the dimensionless synthesis error without data-conditioning as 7.5×10^{-9} . The latter is apparently three orders of magnitude higher than the former, thereby making evident the effect of data-conditioning.

7.2 Example 2

In this example, the synthesis of a four-bar linkage with a RP dyad, guiding its coupler link through the five poses of Table 4, is pursued. Similar to Example 1, the data are first conditioned by means of the characteristic length $d_{rms} = 0.467$ mm, as given in Table 5. The results of this example are recorded in Table 6.

A mechanism was synthesized using solution # 1 for the RR dyad, the RP dyad obtained from the upper part of Table 6. The linkage is depicted in Fig. 10.

8 Conclusions

In this paper the classic Burmester problem is studied by addressing the determination of all four types of dyads, namely, RR, PR, RP and PP, of planar four-bar linkages. A general method for the synthesis of four-bar linkages with four or five prescribed poses is developed. A first contribution of the paper is an alternative approach to the formulation of the Burmester problem, which is coordinate-free and based on an independent computation of the circlepoints and the centrepoints, so as to enhance the robustness



Table 6: Solutions of Example 2

Figure 10: The linkage synthesized for the five prescribed poses

of the solution. As a second contribution, a method to detect the occurrence of PR, RP and PP dyads, which obviates the need of asymptote-derivation, as well as the numerical determination of these dyads, is proposed. A third contribution lies in considering the numerics behind the underlying computations. We derived a robust procedure to find the unique orientation of the prismatic joint. We proved that either the circlepoint of a PR dyad or the centrepoint of a RP dyad is located at the common intersection of six circles in the five-pose case—for the four-pose case, a PR or RP dyad always exists. Moreover, a data-conditioning method was reported here to enhance the accuracy of results.

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