Spatial Four-Bar-Linkages

November 11, 2003

Derivation of the I-O Equation of the RCCC Linkage

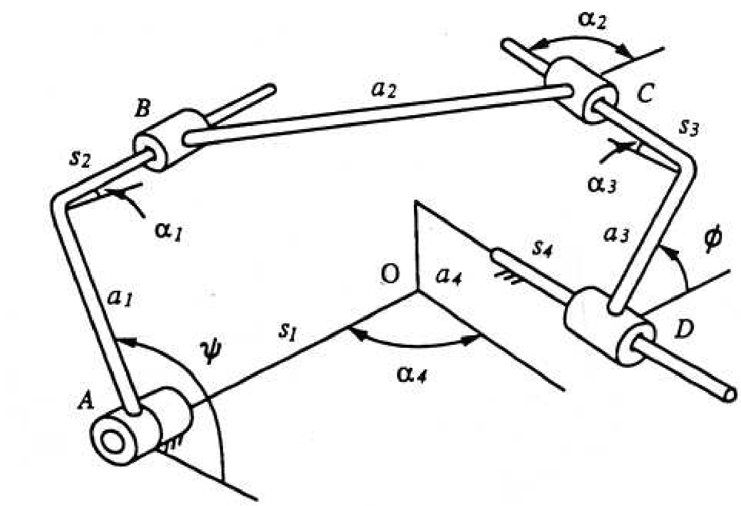

Figure 1: A RCCC linkage for function generation (s_i is b_i in this paper)

The derivation is most easily done by resorting to the *Principle of Transference*: Just put "hats" on all variables and parameters of the I-O equation of the spherical linkage:

$$\hat{k}_1 - \hat{k}_2 c\hat{\theta}_1 - \hat{k}_3 c\hat{\theta}_4 - \hat{k}_4 c\hat{\theta}_1 c\hat{\theta}_4 + s\hat{\theta}_1 s\hat{\theta}_4 = 0 \tag{1}$$

where

$$c\hat{\theta}_i = c\theta_i - \epsilon b_i s\theta_i, \quad s\hat{\theta}_i = s\theta_i + \epsilon b_i c\theta_i \tag{2a}$$

$$c(\cdot) = \cos(\cdot), \quad s(\cdot) = \sin(\cdot) \tag{2b}$$

with θ_i, for $i = 1, 2, 3, 4$, and b_i, for $i = 2, 3, 4$, being the joint variables, while b_1 is a linkage dimension, and ϵ is the *dual unit*, defined as $\epsilon \neq 0$, $\epsilon^2 = 0$.

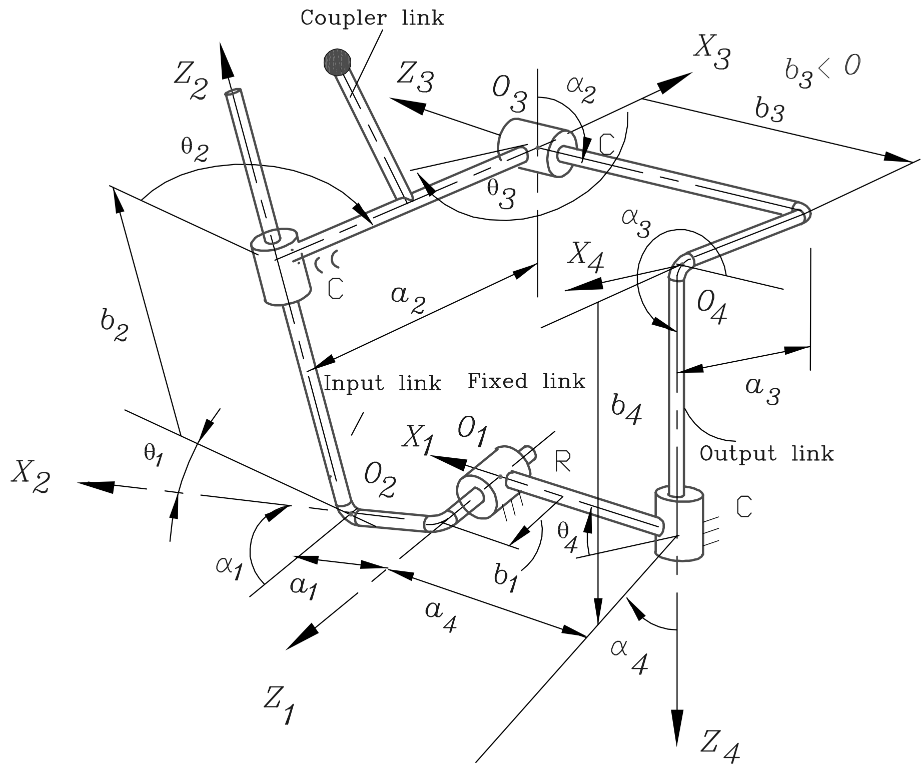

Figure 2: H-D notation associated with the RCCC spatial linkage

Upon substitution of eqs.(2a) into eq.(1), we have

$$\hat{k}_1 - \hat{k}_2(c\theta_1 - \epsilon b_1 s\theta_1) - \hat{k}_3(c\theta_4 - \epsilon b_4 s\theta_4) - \hat{k}_4(c\theta_1 - \epsilon b_1 s\theta_1)(c\theta_4 - \epsilon b_4 s\theta_4)$$
$$+(s\theta_1 + \epsilon b_1 c\theta_1)(s\theta_4 + \epsilon b_4 c\theta_4) = 0 \qquad (3)$$

Since $\{\hat{k}_i\}_1^4$ is a set of dual numbers, we can express them in the standard form

$$\hat{k}_i = k_i + \epsilon b_1 k_{i0}\ , i = 1, 2, 3, 4 \qquad (4)$$

where k_i and k_{i0} are real, nondimensional numbers. In this context, k_i is called the *primal part* and $b_1 k_{i0}$ the *dual part* of $\{\hat{k}_i\}_1^4$. Using expressions (4), we can write eq.(3) in the form:

$$k_1 + \epsilon b_1 k_{10} - [k_2 c\theta_1 + \epsilon b_1(k_{20} c\theta_1 - k_2 s\theta_1)] - [k_3 c\theta_4 + \epsilon b_1(k_{30} c\theta_4 - s_4 k_3 s\theta_4)]$$
$$-[k_4 c\theta_1 c\theta_4 + \epsilon b_1(k_{40} c\theta_1 c\theta_4 - k_4 s\theta_1 c\theta_4 - k_4 c\theta_1 s\theta_4)] + s\theta_1 s\theta_4$$
$$+\epsilon b_1(c\theta_1 s\theta_4 + s_4 s\theta_1 c\theta_4) = 0 \qquad (5)$$

where

$$s_4 \equiv \frac{b_4}{b_1}$$

which can be broken down into two real equations, one for the primal and one for the dual parts, namely,

$$k_1 - k_2 c\theta_1 - k_3 c\theta_4 - k_4 c\theta_1 c\theta_4 + s\theta_1 s\theta_4 = 0 \tag{6a}$$

$$k_{10} - k_{20} c\theta_1 + k_2 s\theta_1 - k_{30} c\theta_4 + s_4 k_3 s\theta_4 - s_4 k_{40} c\theta_1 c\theta_4 + k_4 s\theta_1 c\theta_4 + k_4 c\theta_1 s\theta_4 + c\theta_1 s\theta_4 + s_4 s\theta_1 c\theta_4 = 0 \tag{6b}$$

From eq.(6a) we can obtain θ_4 in terms of θ_1, while from eq.(6b) we obtain s_4 in terms of θ_1, once θ_4 is available. These equations represent the Freudenstein equations for the RCCC linkage.

On the other hand, the dual Freudenstein parameters $\hat{k}_i$, for $i = 1, 2, 3, 4$, are defined as in the spherical case, but this time with hats:

$$\hat{k}_1 \equiv \frac{c\hat{\alpha}_1 c\hat{\alpha}_3 c\hat{\alpha}_4 - c\hat{\alpha}_2}{s\hat{\alpha}_1 s\hat{\alpha}_3}, \quad \hat{k}_2 \equiv \frac{c\hat{\alpha}_3 s\hat{\alpha}_4}{s\hat{\alpha}_3}, \quad \hat{k}_3 \equiv \frac{c\hat{\alpha}_1 s\hat{\alpha}_4}{s\hat{\alpha}_1}, \quad \hat{k}_4 \equiv c\hat{\alpha}_4 \tag{7}$$

where

$$c\hat{\alpha}_i = c\alpha_i - \epsilon a_i s\alpha_i, \quad s\hat{\alpha}_i = s\alpha_i + \epsilon a_i c\alpha_i \tag{8}$$

Upon substitution of eqs.(8) into eqs.(7) we derive

$$\hat{k}_1 = \frac{c\alpha_1 c\alpha_3 c\alpha_4 - c\alpha_2}{s\alpha_1 s\alpha_3 + \epsilon(a_1 c\alpha_1 s\alpha_3 + a_3,\ s\alpha_1 c\alpha_3)} + \epsilon \frac{(-a_1 s\alpha_1 c\alpha_3 c\alpha_4 + a_2 s\alpha_2 - a_3 c\alpha_1 a_3 s\alpha_3 c\alpha_4 - a_4 c\alpha_1 c\alpha_3 s\alpha_4)}{s\alpha_1 s\alpha_3 + \epsilon(a_1 c\alpha_1 s\alpha_3 + a_3,\ s\alpha_1 c\alpha_3)} \tag{9a}$$

$$\hat{k}_2 = \frac{c\alpha_3 s\alpha_4 + \epsilon(a_4 c\alpha_3 c\alpha_4 - a_3 s\alpha_3 s\alpha_4)}{s\alpha_3 + \epsilon a_3 c\alpha_3} \tag{9b}$$

$$\hat{k}_3 = \frac{c\alpha_1 s\alpha_4 + \epsilon(a_4 c\alpha_1 c\alpha_4 - a_1 s\alpha_1 s\alpha_4)}{s\alpha_1 + \epsilon a_1 c\alpha_1} \tag{9c}$$

$$\hat{k}_4 = c\alpha_4 + \epsilon s\alpha_4 \tag{9d}$$

Then, substituting eqs.(4) into eqs.(9a–d), after some simplifications with the aid of computer algebra and the rules of operations with dual numbers, we obtain:

$$k_1 \equiv \frac{c\alpha_1 c\alpha_3 c\alpha_4 - c\alpha_2}{s\alpha_1 s\alpha_3}, \quad k_2 \equiv \frac{c\alpha_3 s\alpha_4}{s\alpha_3}, \quad k_3 \equiv \frac{c\alpha_1 s\alpha_4}{s\alpha_1}, \quad k_4 \equiv c\alpha_4 \tag{10a}$$

$$k_{10} \equiv r_1 \frac{c\alpha_3 c\alpha_4 - c\alpha_1 c\alpha_2}{s\alpha_1^2 s\alpha_3} + r_2 \frac{s\alpha_2}{s\alpha_1 s\alpha_3} + r_3 \frac{c\alpha_2 c\alpha_3 - c\alpha_1 c\alpha_4}{s\alpha_1 s\alpha_3^2} - r_4 \frac{c\alpha_1 c\alpha_3 s\alpha_4}{s\alpha_1 s\alpha_3} \tag{10b}$$

$$k_{20} \equiv r_3 \frac{-s\alpha_4}{s\alpha_3^2} + r_4 \frac{c\alpha_3 c\alpha_4}{s\alpha_3} \tag{10c}$$

$$k_{30} \equiv r_1 \frac{s\alpha_4}{s\alpha_1^2} + r_4 \frac{c\alpha_1 c\alpha_4}{s\alpha_1} \tag{10d}$$

$$k_{40} \equiv -r_4 s\alpha_4, \tag{10e}$$

Interestingly, the expressions for the primal parts of the Freudenstein parameters, as given in eqs.(10a), are identical to those derived for the spherical RRRR linkage. The inverse relations of eqs.(10a) are therefore identical to those of the spherical case, namely,

$$\cos\alpha_1 = \frac{k_3}{\sqrt{E}}, \qquad \sin\alpha_1 = \sqrt{\frac{B}{E}} \tag{11a}$$

$$\cos\alpha_2 = \frac{k_2k_3k_4 - k_1B}{\sqrt{DE}}, \qquad \sin\alpha_2 = \sqrt{\frac{BC}{DE}} \tag{11b}$$

$$\cos\alpha_3 = \frac{k_2}{\sqrt{D}}, \qquad \sin\alpha_3 = \sqrt{\frac{B}{D}} \tag{11c}$$

$$\cos\alpha_4 = k_4, \qquad \sin\alpha_4 = \sqrt{B} \tag{11d}$$

where we introduced the notation

$$\begin{aligned} B &\equiv 1 - k_4^2 \\ C &\equiv B(1 - k_1^2) + k_2^2 + k_3^2 + k_2^2k_3^2 - 2k_1k_2k_3k_4 \\ D &\equiv 1 + k_2^2 - k_4^2 \\ E &\equiv 1 + k_3^2 - k_4^2 \end{aligned}$$

Furthermore, the inverse relations of the eqs.(10b–e) are readily derived, given that these equations are all linear in the remaining unknown parameters $\{r_i\}_1^4$. Upon solving for these parameters symbolically, we obtain

$$r_1 = \frac{\sin\alpha_1}{\sin\alpha_4^2}(k_{03}\sin\alpha_1\sin\alpha_4 + k_{04}\cos\alpha_1\cos\alpha_4) \tag{12a}$$

$$\begin{aligned} r_2 = {} & \frac{k_{01}\sin\alpha_1\sin\alpha_3\sin\alpha_4 + k_{02}(-\cos\alpha_1\cos\alpha_4 + \cos\alpha_2\cos\alpha_3)\sin\alpha_3}{\sin\alpha_2\sin\alpha_4} \\ & + \frac{\sin\alpha_1(-\cos\alpha_3\cos\alpha_4 + \cos\alpha_2\cos\alpha_1)}{\sin\alpha_2}k_{03} \\ & + \frac{\cos\alpha_2\cos\alpha_4(\cos\alpha_1^2 + \cos\alpha_3^2) - \cos\alpha_1\cos\alpha_3(1 + \cos\alpha_4^2)}{\sin\alpha_2\sin\alpha_4^2}k_{04} \end{aligned} \tag{12b}$$

$$r_3 = -\frac{\sin\alpha_3}{\sin\alpha_4^2}(k_{02}\sin\alpha_3\sin\alpha_4 + k_{04}\cos\alpha_3\cos\alpha_4) \tag{12c}$$

$$r_4 = -\frac{k_{04}}{\sin\alpha_4} \tag{12d}$$

or, substituting into (12a–d) relations (11a–d), we derive

$$r_1 = \frac{Bk_{03} + k_3k_4k_{04}}{E\sqrt{B}} \tag{13a}$$

$$r_2 = \frac{\sqrt{B}k_{01}}{\sqrt{C}} + \frac{N_1 k_{02}}{\sqrt{BC}D} + \frac{N_2 k_{03}}{\sqrt{BC}E} + \frac{N_3 k_{04}}{B^2\sqrt{C}DE} \tag{13b}$$

$$r_3 = \frac{k_{02}B + k_2 k_4 k_{04}}{D\sqrt{B}} \tag{13c}$$

$$r_4 = -\frac{k_{04}}{\sqrt{B}} \tag{13d}$$

where we introduce the notation

$$N_1 \equiv -(k_1 k_2 + k_3 k_4)B - 2k_2^2 k_3 k_4$$

$$N_2 \equiv -(k_1 k_3 + k_2 k_4)B - 2k_2 k_3^2 k_4$$

$$N_3 \equiv -2k_2 k_3 k_4^5 - k_1(k_2^2 + k_3^2)k_4^4 + k_2 k_3(3k_2^2 + 3k_3^2 + 4)k_4^3$$
$$+2k_1(k_2^2 + k_3^2 + k_2^2 k_3^2)k_4^2 - k_2 k_3(4k_2^2 k_3^2 + 3k_2^2 + 3k_3^2 - 2)k_4 - k_1 k_2^2(1 + 2k_3^2) - k_1 k_3^2$$