Supplement to Section 2.2

The concept inverse to the CPM of a 3D vector is the axial vector of a 3×3 matrix.

The Cartesian decomposition of a $n \times n$ matrix $\underset{\sim}{A}$ is

$$\underset{\sim}{A} = \underset{\sim}{A}_s + \underset{\sim}{A}_{ss} \tag{1}$$

where $\underset{\sim}{A}_s$ is symmetric and $\underset{\sim}{A}_{ss}$ is skew-symmetric.

The reader can readily prove that

$$\underset{\sim}{A}_s = \frac{1}{2}(\underset{\sim}{A} + \underset{\sim}{A}^T), \quad \underset{\sim}{A}_{ss} = \frac{1}{2}(\underset{\sim}{A} - \underset{\sim}{A}^T) \tag{2}$$

Definition: For $n=3$, let $\vec{v}$ be any 3D vector. Then there exists a vector $\vec{a}$ such that

$$\underset{\sim}{A}_{ss}\vec{v} = \vec{a} \times \vec{v} \tag{3}$$

Vector $\vec{a}$ is called the axial vector of $\underset{\sim}{A}_{ss}$:

$$\vec{a} = \text{vect}(\underset{\sim}{A}_{ss}) \tag{4}$$

Theorem 1:

$$\underset{\sim}{A}_{ss} = \text{CPM}(\vec{a}) \tag{5}$$

Proof: trivial. DIY.

Theorem 2: Operations $\text{CPM}(\cdot)$ & $\text{vect}(\cdot)$ are linear, i.e., if $\underset{\sim}{A}$ & $\underset{\sim}{B}$ are 3×3 matrices & $\vec{a}$ & $\vec{b}$ are 3D vectors, while α & β are real numbers, then

$$\text{CPM}(\alpha\vec{a} + \beta\vec{b}) = \alpha\,\text{CPM}(\vec{a}) + \beta\,\text{CPM}(\vec{b}) \tag{6a}$$

while

$$\text{vect}(\alpha\underset{\sim}{A} + \beta\underset{\sim}{B}) = \alpha\,\text{vect}(\underset{\sim}{A}) + \beta\,\text{vect}(\underset{\sim}{B}) \tag{6b}$$

Proof: follows directly from the defn's of CPM(·) & vect(·) plus that of the cross product.

If a_{ij} is the (i,j) entry of $\underset{\sim}{A}$, what is $\vec{a}$?

$$\underset{\sim}{A}_s = \begin{bmatrix} a_{11} & (a_{12}+a_{21})/2 & (a_{13}+a_{31})/2 \\ & a_{22} & (a_{23}+a_{32})/2 \\ \text{sym} & & a_{33} \end{bmatrix} \quad (7a)$$

$$\underset{\sim}{A}_{ss} = \begin{bmatrix} 0 & (a_{12}-a_{21})/2 & (a_{13}-a_{31})/2 \\ & 0 & (a_{23}-a_{32})/2 \\ \text{skew-sym} & & 0 \end{bmatrix} \quad (7b)$$

Further, if a_i denotes the ith component of $\vec{a}$ with a similar definition for v_i,

$$\vec{a}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{bmatrix} a_2 v_3 - a_3 v_2 \\ a_3 v_1 - a_1 v_3 \\ a_1 v_2 - a_2 v_1 \end{bmatrix} \quad (8)$$

As well,

$$\underset{\sim}{A}_{ss}\vec{v} = \frac{1}{2}\begin{bmatrix} (a_{12}-a_{21})v_2 + (a_{13}-a_{31})v_3 \\ -(a_{12}-a_{21})v_1 + (a_{23}-a_{32})v_3 \\ -(a_{13}-a_{31})v_1 - (a_{23}-a_{32})v_2 \end{bmatrix} \quad (9)$$

As per eq. (3), equate eqs. (8) & (9):

$$a_2 = \tfrac{1}{2}(a_{13}-a_{31}), \quad a_3 = -\tfrac{1}{2}(a_{12}-a_{21}), \quad a_1 = -\tfrac{1}{2}(a_{23}-a_{32})$$

or

$$\vec{a} = \frac{1}{2}\begin{bmatrix} a_{32}-a_{23} \\ a_{13}-a_{31} \\ a_{21}-a_{12} \end{bmatrix}; \quad \underset{\sim}{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad (10)$$

Facts: for any $n \times n$ matrix $\underset{\sim}{M}$ and any n-dim vectors $\vec{a}$ & $\vec{b}$,

F1: $tr(\underset{\sim}{M}) = m_{11} + m_{22} + \cdots + m_{nn}$

F2: $tr(\vec{a}\vec{b}^T) = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \vec{a}^T \vec{b}$

F3: If $\underset{\sim}{M} = -\underset{\sim}{M}^T$, then all its diagonal entries vanish, and hence, $tr(\underset{\sim}{M}) = 0$

Theorem 3: if $\underset{\sim}{A} = \underset{\sim}{A}^T$, then $vect(\underset{\sim}{A}) = \vec{0}$

Proof: follows from (10)

Theorem 4: the trace$(\cdot)$ operation is linear, i.e.,

$$tr(\alpha \underset{\sim}{A} + \beta \underset{\sim}{B}) = \alpha\, tr(\underset{\sim}{A}) + \beta\, tr(\underset{\sim}{B})$$

Proof: follows from its definition.

Application: Given the rotation matrix $\underset{\sim}{Q}$, find $\vec{e}$ & ϕ.

$$tr(\underset{\sim}{Q}) = \underbrace{tr(\vec{e}\vec{e}^T) + tr[\cos\phi\,(\underset{\sim}{1} - \vec{e}\vec{e}^T) + tr(\sin\phi\, \underset{\sim}{E})}_{\text{Thm 4}}$$

$$= \underbrace{\vec{e}^T\vec{e}}_{F2} + \underbrace{\cos\phi\,[\underbrace{tr(\underset{\sim}{1})}_{3} - \underbrace{\vec{e}^T\vec{e}}_{1}]}_{\text{Thm 4}} + \sin\phi\, \underbrace{\underbrace{tr(\underset{\sim}{E})}_{0}}_{F3}$$

$$= 1 + 2\cos\phi \qquad (11)$$

$$vect(\underset{\sim}{Q}) = \underbrace{vect(\vec{e}\vec{e}^T) + vect[\cos\phi\,(\underset{\sim}{1} - \vec{e}\vec{e}^T)] + vect(\sin\phi\, \underset{\sim}{E})}_{\text{Thm 2}}$$

$$= \underbrace{\vec{0}}_{\text{Thm3}} + \underbrace{\vec{0}}_{\text{Thm3}} \quad \underbrace{\sin\phi\, vect(\underset{\sim}{E})}_{\text{Thm2}} = \underbrace{(\sin\phi)\,\vec{e}}_{3} \qquad (12)$$

(11) $\Rightarrow$ ϕ in two possible quadrants

(12): A reversal in the sign of $\sin\phi$ leads to a reversal in the sign of all 3 components of $\vec{e}$, which doesn't change the direction of the axis of rotation $\Rightarrow$ for concreteness, assume $\sin\phi > 0 \Rightarrow \phi$ in 1st or 2nd quadrant

$$\Rightarrow \text{If } \cos\phi = \frac{tr(\underset{\sim}{Q})-1}{2} > 0, \text{ then } \phi \text{ in 1st quad}$$

$$\text{If } \quad < 0, \text{ then } \phi \text{ in 2nd}$$

Example: For $\underset{\sim}{Q}_1$ of eq. (2.8a), find $\vec{e}$ and ϕ

$$vect(\underset{\sim}{Q}_1) = \frac{1}{2}\begin{bmatrix} 3/2 \\ \sqrt{3}/2 \\ \sqrt{3}/2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 \\ \sqrt{3} \\ \sqrt{3} \end{bmatrix} \Rightarrow \sin\phi = \|vect(\underset{\sim}{Q}_1)\| = \sqrt{15}/4$$

$$\Rightarrow \phi = 75.52^\circ \text{ or } 104.48^\circ$$

$$\vec{e} = vect(\underset{\sim}{Q}_1)/\|vect(\underset{\sim}{Q}_1)\| = \frac{\sqrt{15}}{15}\begin{bmatrix} 3 \\ \sqrt{3} \\ \sqrt{3} \end{bmatrix}$$

$$tr(\underset{\sim}{Q}_1) = \frac{1}{2} \Rightarrow \cos\phi = -\frac{1}{4} \Rightarrow \phi = 104.48^\circ \quad \underline{Ans.}$$