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A SEMIGRAPHICAL METHOD FOR THE SOLUTION OF THE BURMESTER PROBLEM

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Abstract

The determination of the planar four-bar linkage intended to guide a rigid body through five finitely-separated configurations is known as the *Burmester Problem*. This is a nonlinear problem that involves finding the roots of a system of four bilinear equations in four unknowns. When solving this problem numerically, for it is evasive to other means, issues such as numerical conditioning, multiplicity of solutions and singularities must be considered. Here we present a *semigraphical method* of solution that reduces the problem to finding the intersections of two contours in the x-y plane of motion. The method not only produces all real solutions, by simple inspection, but also provides information about the numerical conditioning of those solutions.

1 Introduction

The Burmester problem consists of designing a planar four-bar linkage, as shown in Fig. 1, to guide a rigid-body through a series of five prescribed finitely-separated *poses*—positions and orientations.

In the Burmester problem we want to find the *circle points* of the moving body, i.e., points A_0 and A_0^* of Fig. 1 that move on a circle as the body passes through the given poses. For each such a circle, the corresponding center B or B^* is called a *center point*. The study of the *circle-points* is referred to as Burmester theory, named after the 19th century kinematician who first proved that for five given positions, there are a maximum of four circle-points (Burmester 1886). Any two of the circle-points, together with their associated center-points (which define the ground pivots), yields a four-bar linkage. It is known (Bottema and Roth 1979) that at most four such points can be found, and hence, up to six different linkages can be assembled.

Several techniques have been utilized for the planar synthesis problem, such as algebraic methods, complex numbers and matrix methods (Angeles, 1982; Sandor and Erdman, 1984). A commercial package by the name of LINCAGES (Sandor and Erdman, 1984) is available for obtaining solutions to this problem by plotting the circle-point or center-point curves. The method used by Sandor and Erdman consists of breaking the five precision-point problem into four four-position cases for which the loci of the circle-points or the loci of the center-points are plotted, the solutions to the original five-position problem being obtained from the intersections of the four curves. Since these circle- and center point curves are cubic, up to nine intersections could be found; of these, only a maximum of four solutions are usable, the rest being spurious. Numerical techniques known as *continuation methods* have been recently introduced in the solution of this type of problems (Wampler, Morgan and Sommese, 1990).

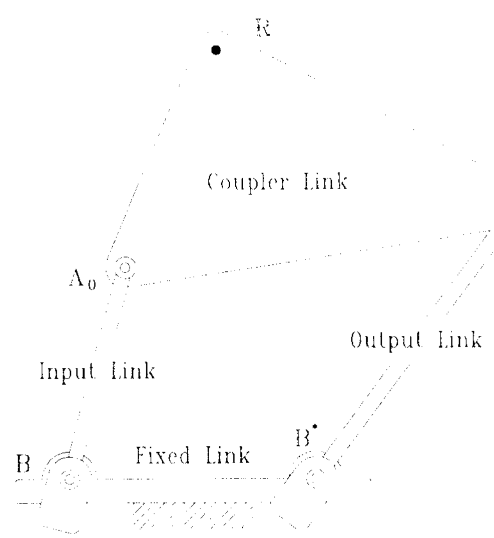

Figure 1: A four-bar linkage

Continuation methods are very powerful and provide all solutions to a given nonlinear problem formulated as a system of polynomial equations. We will present an alternative method in this paper, that consists of reducing the given nonlinear system of equations to only two equations in two unknowns. Here we differ essentially from other approaches that aim at either reducing all equations to only one polynomial equation in one unknown or that handle all equations simultaneously. The two equations that we derive for the case at hand are quadratic, and hence, spurious solutions are eliminated. Each of the two equations then defines a contour in the plane of the two associated variables. In our case, these are the Cartesian coordinates of the circle points in the reference or zeroth configuration. The intersection of the contours thus provides the two circle points that are sought.

Furthermore, the numerical condition of the solutions, i.e., their sensitivity to *small* changes in the problem data, can be readily estimated from the angle at which the contours cross each other. Note that this information is not readily available with other methods. Moreover, the use of suitable plotting software eases tremendously the task of the designer when applying this method. In our solution, we have resorted either to MATHEMATICA, running on an IRIS 4D/210VGX workstation or Autocad, running on a Sun Sparcstation. Under these conditions, the contours are produced in a fraction of a second.

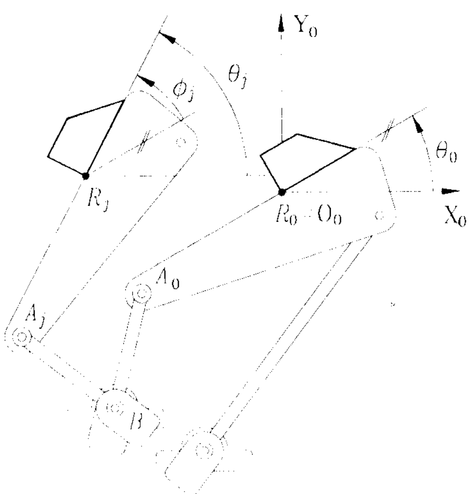

Figure 2: The local coordinate system set at R_0

2 Derivation of the Contour Equations

In Fig. 2, the coupler link is represented by a trapezoidal block, the four-bar linkage being shown at the zeroth pose (R_0, θ_0). It is required to design a four-bar linkage, i.e., to find a pair $(\mathbf{a}_0, \mathbf{b})$ of position vectors of points A_0 and B, respectively, such that the resulting linkage will carry the block rigidly attached to the coupler link through its five prescribed poses, which are given with reference to the coordinate frame O–xy as $\{(R_j, \theta_j)\}_0^4$.

Without loss of generality, let us define our local coordinate frame $\mathcal{F}_0$ with origin at R_0 as shown in Fig. 2, and hence, $\mathbf{r}_0 \equiv \mathbf{0}$.

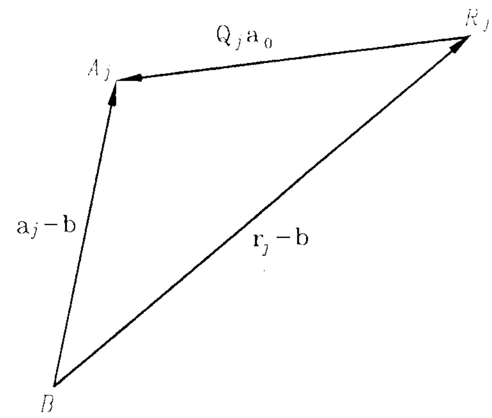

Figure 3: Relations between the position vectors

Referring to Fig. 3, the condition that the distance between $\mathbf{a}_0$ and $\mathbf{b}$ remains constant yields four constraint equations, namely,

$$\|(\mathbf{r}_j - \mathbf{b}) + \mathbf{Q}_j\mathbf{a}_0\|^2 = \|\mathbf{a}_0 - \mathbf{b}\|^2, \qquad \text{for } j = 1, 2, 3, 4 \quad (1)$$

where $\mathbf{Q}_j$ is the rotation matrix involved, i.e.,

$$\mathbf{Q}_j = \begin{bmatrix} \cos\phi_j & -\sin\phi_j \\ \sin\phi_j & \cos\phi_j \end{bmatrix} \quad (2)$$

with $\phi_j \equiv \theta_j - \theta_0$.
After expanding eq. (1) and simplifying, we obtain

$$\mathbf{b}^T(\mathbf{1}-\mathbf{Q}_j)\mathbf{a}_0 + \mathbf{r}_j{}^T\mathbf{Q}_j\mathbf{a}_0 - \mathbf{r}_j{}^T\mathbf{b} + \frac{\mathbf{r}_j{}^T\mathbf{r}_j}{2} = \mathbf{0}, \quad \text{for } j = 1, 2, 3, 4 \quad (3)$$

where $\mathbf{1}$ is the 2×2 identity matrix.

This is a system of four bilinear equations in the four unknowns $\mathbf{a}_0(x, y)$, $\mathbf{b}(x, y)$. In order to solve this system, we reduce it to two quadratic equations by a transformation of variables, as described in (Bottema and Roth, 1979). In fact, the rotation matrix $\mathbf{Q}_j$ can be decomposed as

$$\mathbf{Q}_j = \begin{bmatrix} c_j & -s_j \\ s_j & c_j \end{bmatrix} = c_j\mathbf{1} + s_j\mathbf{E}, \qquad \mathbf{E} \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad (4)$$

where c_j and s_j stand for $\cos\phi_j$ and $\sin\phi_j$, respectively. The first term in eq. (3) thus becomes

$$\begin{aligned} \mathbf{b}^T(\mathbf{1} - \mathbf{Q}_j)\mathbf{a}_0 &= \mathbf{b}^T[(1 - c_j)\mathbf{1} - s_j\mathbf{E}]\mathbf{a}_0 \\ &= (1 - c_j)\mathbf{b}^T\mathbf{a}_0 - s_j\mathbf{b}^T\mathbf{E}\mathbf{a}_0 \\ &= (1 - c_j)u - s_jv \end{aligned} \quad (5)$$

where variables u and v are defined as indicated below:

$$u \equiv \mathbf{b}^T\mathbf{a}_0, \qquad v \equiv \mathbf{b}^T\mathbf{E}\mathbf{a}_0 \quad (6)$$

Substituting eq. (5) into eq. (3) yields

$$(1 - c_j)u - s_jv - \mathbf{r}_j{}^T\mathbf{b} = -\mathbf{r}_j{}^T\mathbf{Q}_j\mathbf{a}_0 - \frac{\mathbf{r}_j{}^T\mathbf{r}_j}{2} \quad (7)$$

or,

$$\begin{bmatrix} 1-c_j & -s_j & -\mathbf{r}_j^T \end{bmatrix} \begin{bmatrix} u \\ v \\ \mathbf{b} \end{bmatrix} = -\mathbf{r}_j^T(\mathbf{Q}_j\mathbf{a}_0+\frac{\mathbf{r}_j}{2}), \quad \text{for } j=1,2,3,4. \tag{8}$$

Equations (8) constitute a linear system of four scalar equations in six unknowns, namely, u, v, the two components of $\mathbf{a}_0$ and the two components of $\mathbf{b}$. These four equations are then used to solve for four of the six unknowns, namely u, v, and the two components of $\mathbf{b}$, in terms of $\mathbf{a}_0$. To this end, we introduce now some definitions, namely, the 4×4 matrix $\mathbf{M}$ and the 4-dimensional vectors $\mathbf{x}$ and $\mathbf{c}$ given as

$$\mathbf{M} \equiv \begin{bmatrix} 1-c_1 & -s_1 & -\mathbf{r}_1^T \\ 1-c_2 & -s_2 & -\mathbf{r}_2^T \\ 1-c_3 & -s_3 & -\mathbf{r}_3^T \\ 1-c_4 & -s_4 & -\mathbf{r}_4^T \end{bmatrix}, \quad \mathbf{x} \equiv \begin{bmatrix} u \\ v \\ \mathbf{b} \end{bmatrix},$$

$$\mathbf{c} \equiv \begin{bmatrix} -\mathbf{r}_1^T(\mathbf{Q}_1\mathbf{a}_0+\frac{\mathbf{r}_1}{2}) \\ -\mathbf{r}_2^T(\mathbf{Q}_2\mathbf{a}_0+\frac{\mathbf{r}_2}{2}) \\ -\mathbf{r}_3^T(\mathbf{Q}_3\mathbf{a}_0+\frac{\mathbf{r}_3}{2}) \\ -\mathbf{r}_4^T(\mathbf{Q}_4\mathbf{a}_0+\frac{\mathbf{r}_4}{2}) \end{bmatrix} \tag{9}$$

Thus, vector $\mathbf{x}$ is now computed as

$$\mathbf{x} = \mathbf{M}^{-1}\mathbf{c} \tag{10}$$

Moreover, eqs. (6) can be rewritten in terms of vector $\mathbf{x}$ as

$$\mathbf{N}\mathbf{x} = \mathbf{0} \tag{11}$$

where $\mathbf{N}$ is a 2×4 matrix and $\mathbf{0}$ is the 2-dimensional zero vector, i.e.,

$$\mathbf{N} \equiv \begin{bmatrix} -1 & 0 & \mathbf{a}_0^T \\ 0 & -1 & \mathbf{a}_0^T\mathbf{E} \end{bmatrix}, \quad \mathbf{0} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{12}$$

Upon substitution of $\mathbf{x}$ as given in eq. (10), into eq. (11), we obtain

$$\mathbf{N}\mathbf{M}^{-1}\mathbf{c} = \mathbf{0} \tag{13}$$

which is the desired system of two equations in two unknowns. Note that these two equations are quadratic in $\mathbf{a}_0$, for matrix $\mathbf{N}$ and vector $\mathbf{c}$ are linear in $\mathbf{a}_0$, whereas matrix $\mathbf{M}$ and hence, its inverse, is constant. Each of the two above equations thus defines a conic section in the x-y plane, the intersection of which, then, provides all real solutions of the problem at hand.

Once $\mathbf{a}_0$ is available from the foregoing calculations, $\mathbf{b}$ can be computed in one of two ways: *i*) by inversion of matrix $\mathbf{M}$, as indicated in eq. (10), which produces not only $\mathbf{b}$, but also u and v values that are not needed; and *ii*) from eqs. (3), which leads to an overdetermined linear system of four equations in two unknowns, of the form

$$\mathbf{B}\mathbf{b} = \mathbf{d} \tag{14}$$

where

$$\mathbf{B} = \begin{bmatrix} [(\mathbf{Q}_1-\mathbf{1})\mathbf{a}_0+\mathbf{r}_1]^T \\ [(\mathbf{Q}_2-\mathbf{1})\mathbf{a}_0+\mathbf{r}_2]^T \\ [(\mathbf{Q}_3-\mathbf{1})\mathbf{a}_0+\mathbf{r}_3]^T \\ [(\mathbf{Q}_4-\mathbf{1})\mathbf{a}_0+\mathbf{r}_4]^T \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} \mathbf{r}_1^T\mathbf{Q}\mathbf{a}_0+\frac{1}{2}\mathbf{r}_1^T\mathbf{r}_1 \\ \mathbf{r}_2^T\mathbf{Q}\mathbf{a}_0+\frac{1}{2}\mathbf{r}_2^T\mathbf{r}_2 \\ \mathbf{r}_3^T\mathbf{Q}\mathbf{a}_0+\frac{1}{2}\mathbf{r}_3^T\mathbf{r}_3 \\ \mathbf{r}_4^T\mathbf{Q}\mathbf{a}_0+\frac{1}{2}\mathbf{r}_4^T\mathbf{r}_4 \end{bmatrix} \tag{15}$$

Thus, following the second approach, $\mathbf{b}$ can be calculated from eq. (14) as its least-square approximation given by

$$\mathbf{b} = (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{d} \tag{16}$$

Note that, by using a least-square approach, rather than picking up any two of the four of eqs. (5), roundoff errors are filtered and a robust solution is possible.

3 Numerical Condition of the Solutions

The conditioning of a solution is understood as the condition number (Golub and Van Loan, 1983) of the Jacobian matrix of the underlying nonlinear equations at that particular solution. Below we show that the condition number of the solutions can be estimated visually from the intersection of the two contours discussed above. Clearly, more precise values of the said numbers can be also obtained from the partial derivatives of the two scalar functions producing the contours.

First, let us recall that the condition number of a matrix, in this case the Jacobian of the two functions providing the contours, with respect to the Cartesian coordinates of the circle points, is the amplification factor of the computed roundoff error with respect to the data roundoff error when solving a linear system of equations associated with that matrix (Golub and Van Loan, 1983). Let the system (13) be represented in the form

$$f_1(x,y)=0, \qquad f_2(x,y)=0 \tag{17}$$

where x and y denote the coordinates of point A_0. The Jacobian of the two functions at hand can be expressed as

$$\mathbf{F} \equiv \begin{bmatrix} (\nabla f_1)^T \\ (\nabla f_2)^T \end{bmatrix}$$

where ∇f_i denotes the gradient of f_i with respect to the two unknowns, i.e.,

$$\nabla f_i \equiv [\partial f_i/\partial x \quad \partial f_i/\partial y]^T$$

It is noted that multiplying any of the two equations (17) by a scalar does not affect the conditioning of the solutions, and hence, we will assume henceforth that each of the two equations has been previously multiplied by a factor rendering its gradient a unit vector in the x-y plane. In order to calculate the condition number of $\mathbf{F}$, which determines the conditioning of the solutions, we calculate its singular values as the positive square roots of the eigenvalues of $\mathbf{F}\mathbf{F}^T$. This matrix is given as

$$\mathbf{F}\mathbf{F}^T = \begin{bmatrix} 1 & \nabla f_1\cdot\nabla f_2 \\ \nabla f_1\cdot\nabla f_2 & 1 \end{bmatrix}$$

and hence, its eigenvalues λ_1 and λ_2 are given by

$$\lambda_1 = 1-\cos\gamma, \quad \lambda_2 = 1+\cos\gamma$$

where γ is the angle at which the contours intersect each other. Thus, the condition number κ of $\mathbf{F}$ can be readily computed as

$$\kappa = \sqrt{\frac{\lambda_2}{\lambda_1}} = \frac{1}{\tan(\gamma/2)} \tag{18}$$

which means that, the closer γ is to $\pm 90°$, the better the numerical conditioning of the associated solution. Clearly, if the two contours are tangent to each other, $\gamma = 0$ and $\kappa \rightarrow \infty$, which means that $\mathbf{F}$ is singular. In this case, small perturbations in the data can produce either a double real solution or no real solution at all, a rather uncomfortable situation!

Table 1: Prescribed positions for Example 1

j	$\mathbf{r}_j$	ϕ_j (deg)
1	(-3.8, -7.8)	-57
2	(-6.4, -7.9)	169
3	(-8.0, 0.5)	-69
4	(-5.0, 2.5)	81

Table 2: Prescribed positions for example 2

j	$\mathbf{r}_j$	ϕ_j (deg)
1	(1.5, 0.8)	10
2	(1.6, 1.5)	20
3	(2.0, 3.0)	60
4	(2.3, 3.5)	90

4 Numerical Examples

We will use two numerical examples to illustrate our method of solution. The first example is taken from (Angeles, 1978), while the second example is taken from (Sandor and Erdman, 1984). The prescribed positions are given in Tables 1 and 2, the contour equations taking on the simple forms displayed below:

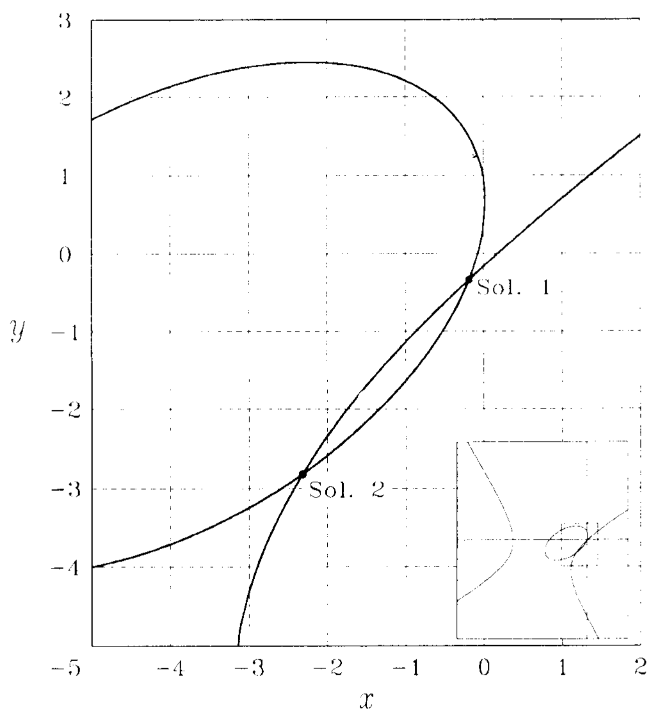

Figure 4: Contours C_1 & C_2 for Example 1

Example 1:

$$-1.548 + 8.503x + 0.6023x^2 - 9.6715y$$
$$-0.6398xy - 0.6342y^2 = 0 \quad (19)$$
$$0.5989 + 7.9304x + 1.0883x^2 - 2.2329y$$
$$-1.2365xy + 1.7281y^2 = 0 \quad (20)$$

Example 2:

$$-3.0217 + 1.0359x - 0.8857x^2 + 5.1383y +$$
$$1.0496xy - 0.9866y^2 = 0 \quad (21)$$
$$5.6722 + 6.7042x + 1.0149x^2 - 0.3737y$$
$$-0.1009xy - 0.0346y^2 = 0 \quad (22)$$

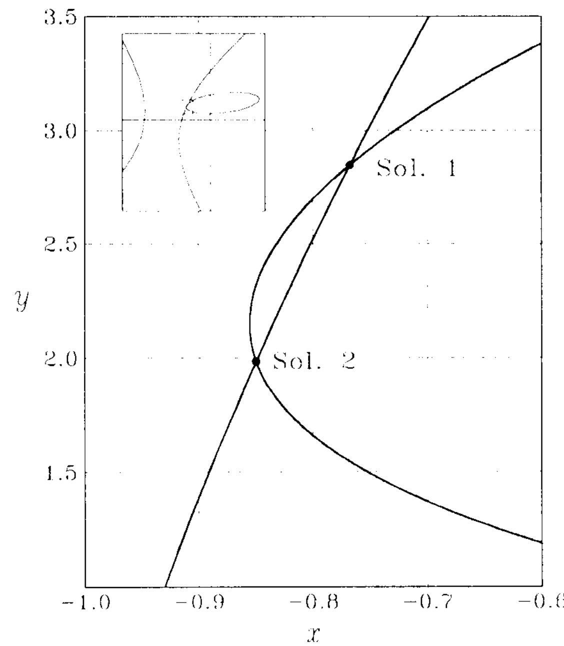

Figure 5: Contours C_1 & C_2 for Example 2

The overlay of the two contours thus obtained is dislayed in Figs. 4 and 5. The calculated condition number of the jth solution of the ith example is denoted by κ_{ij} below:

$$\kappa_{11} = 23.2494, \quad \kappa_{12} = 24.6182$$
$$\kappa_{21} = 228.8696, \quad \kappa_{22} = 155.7128$$

which correspond to the angles γ_{ij} calculated from eq. (18), namely,

$$\gamma_{11} = 23.13°, \quad \gamma_{12} = 22.79°$$
$$\gamma_{21} = 7.56°, \quad \gamma_{22} = 9.63°$$

very much in agreement with the intersection angles of Figs. 4 and 5. The above results show that the data of the second example lead to a problem that is, rooughly, one order of magnitude more ill-conditioned than the second problem. It is worth noting than the results obtained with AutoCAD, for the first example, are very much in agreement with those reported in (Angeles, 1978) using the Newton–Raphson method. However, for the second example, the results shown in Tables 4 and 5, show noticeable differences that may be due to the condition number involved. The morale of these examples is that, if possible, the data should be chosen so as to yield contours that intersect at angles that are as close as possible to 90°.

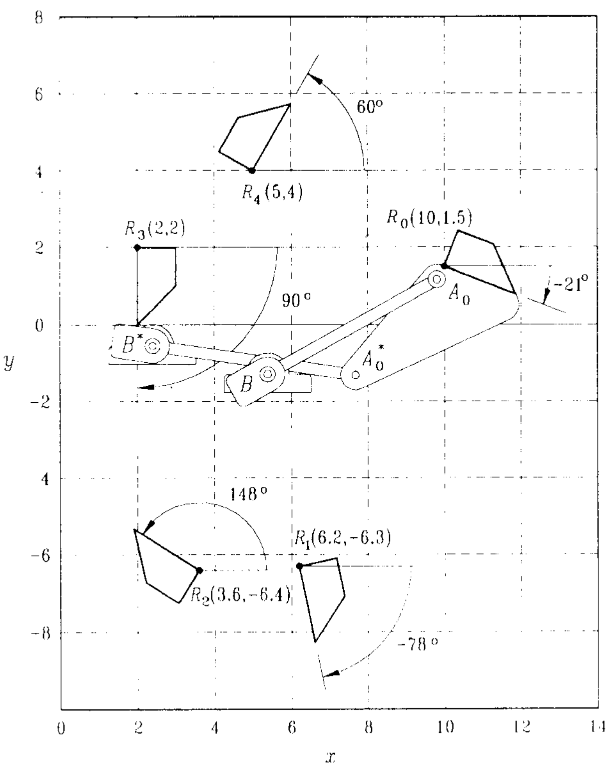

Figure 6: Four-bar linkage for Example 1

Table 3: Solutions for Example 1

Sol. no.	$\mathbf{a}_0$	$\mathbf{b}$
1	(-0.1918, -0.3111)	(-4.6072, -2.7921)
2	(-2.3156, -2.8161)	(-7.6050, -2.0503)

Table 4: Solutions for Example 2

Sol. no.	$\mathbf{a}_0$	$\mathbf{b}$
1	(-0.7676, 2.8467)	(-0.3713, 3.3117)
2	(-0.8498, 1.9847)	(-0.4142, 2.5747)

Table 5: Solutions obtained by Sandor and Erdman

Sol. no.	$\mathbf{a}_0$	$\mathbf{b}$
1	(-0.760, 2.837)	(-0.364, 3.335)
2	(-0.931, 1.936)	(-0.484, 2.515)

Shown in Figs. 6 and 7 are the given poses and the calculated linkages, for Examples 1 and 2, respectively. Note that, for the first example, the data are given in coordinate axes not passing through R_0. Hence, a change of coordinates was introduced here in order to render the data in the required format.

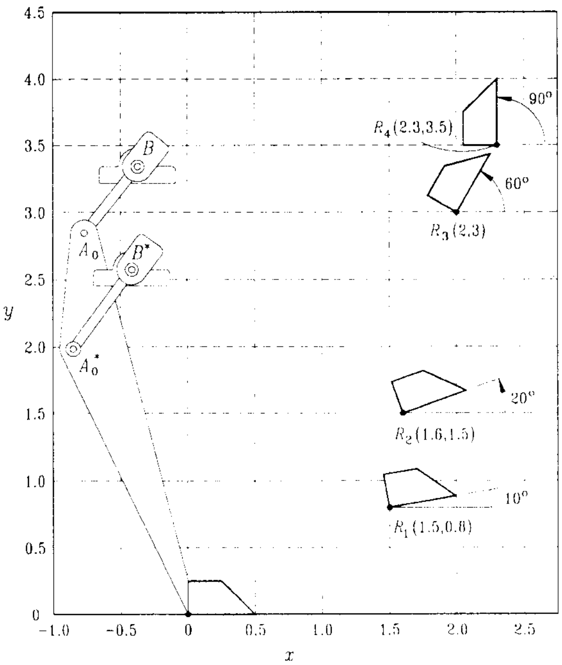

Figure 7: Four-bar linkage for Example 2

5 Conclusions

We derived a method for the determination of all real solutions of the Burmester Problem associated with the synthesis of a planar four-bar linkage meant to guide a rigid body through five finitely separated poses in a plane. The method reduces the original four bilinear equations in four unknowns to only two quadratic equations in two unknowns, while avoiding the introduction of spurious roots. The method is very fast if implemented on modern workstations using advanced mathematical software. As a byproduct, the method produces information on the numerical conditioning of the solutions. Moreover, the method can be used interactively for linkage optimization, if a workstation with multiple window facilities is available. Thus, while one window can display one candidate linkage, another one can display the contours under changes in the prescribed rigid-body poses, for example.

6 Acknowledgements

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