Orthogonal decomposition of A :

$$A = P \begin{bmatrix} \hat{A} & 0 \\ 0 & 0 \end{bmatrix} Q^T$$

need to determine $\perp P$, $\perp Q$ so :

$$P^T A Q = \begin{bmatrix} \hat{A} & 0 \\ 0 & 0 \end{bmatrix}$$

We will illustrate algorithm on:

$$A = \begin{bmatrix} * & * & 0 \\ * & * & * \\ * & * & * \\ \square & * & * \end{bmatrix} \in \mathbb{R}^{4 \times 3}$$

where :

- $*$ denotes arbitrary value
- $\square$ -"- non-zero value
- 0 -"- zero value

Decomposition alg. starts by finding an $\perp P$ so :

$$\underbrace{P^T}_{\in \mathbb{R}^{4\times 4}} \underbrace{A}_{\in \mathbb{R}^{4\times 3}} = \text{an echelon matrix}$$

Echelon matrices

Staircase for a matrix

start of stair-case $\longrightarrow$

$$M_1 = \begin{pmatrix} 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & \square & 0 & * & * \\ 0 & 0 & 0 & \square & * \end{pmatrix}$$

Rules :

staircase never goes up
goes down just enough so all
entries below it, if any, are zero

Echelon matrix : has a staircase with

all steps (if any) $=$ 1 row

e.g.

$M_1 \neq$ echelon

$M_2 =$ echelon

$$M_2 = \begin{pmatrix} \square & * & * & * & * \\ 0 & 0 & \square & * & * \\ 0 & 0 & 0 & 0 & \square \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\perp$ Transformation to echelon form

based on introducing zeros in columns

Putting a zero in a column

Can choose $\perp$ H, so

$$H\begin{bmatrix} a\\ b\\ c\\ \delta\\ \varepsilon\\ f\end{bmatrix} = \begin{bmatrix} a\\ b\\ c\\ \square\\ 0\\ f\end{bmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{- unchanged} \\ \leftarrow \text{non-zero} \\ \\ \leftarrow \text{unchanged}\end{array} \qquad 0 \neq \varepsilon$$

a, b, c, f $\equiv$ unchanged but ε changed to zero

H has structure

$$H = \left[\begin{array}{ccc|cc|c} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & \multicolumn{2}{c|}{\bar{H}_{2\times2}} & 0\\ 0 & 0 & 0 & & & 0\\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\bar{H}$ is the part of H which does the job of transforming $\begin{pmatrix}\delta\\ \varepsilon\end{pmatrix}$ into $\begin{pmatrix}\square\\ 0\end{pmatrix}$. The identity matrices on the block diagonal of H are there so $(a\,b\,c)^T$ and f are unchanged by H.

Take:

$$\bar{H}_{2\times2} = I_{2\times2} - \frac{2uu^T}{\|u\|^2}$$

this is a so-called Givens rotation matrix

with

$$u \triangleq \begin{pmatrix}\delta\\ \varepsilon\end{pmatrix} + e^1 \left\|\begin{pmatrix}\delta\\ \varepsilon\end{pmatrix}\right\| \operatorname{sign}(\delta)$$

$$e^1 \triangleq \begin{pmatrix}1\\ 0\end{pmatrix} \quad ; \quad \operatorname{sign}(\delta) = \begin{cases} 1 & \text{if } \delta \geq 0\\ -1 & \text{if } \delta < 0\end{cases}$$

Does it zero the ε-element?

Check by direct calculation:

notation:

$$a \triangleq \begin{pmatrix}\delta\\ \varepsilon\end{pmatrix}$$

$$\tilde{\delta} \triangleq \delta + \|a\| \operatorname{sign}(\delta)$$

$$uu^T = \begin{pmatrix}\tilde{\delta}\\ \varepsilon\end{pmatrix}(\tilde{\delta}\ \varepsilon)$$

$$= \begin{bmatrix}\tilde{\delta}^2 & \varepsilon\tilde{\delta}\\ \varepsilon\tilde{\delta} & \varepsilon^2\end{bmatrix}$$

$$\|u\|^2 = \delta^2 + \varepsilon^2 + 2|\delta| \left\|\begin{pmatrix}\delta\\ \varepsilon\end{pmatrix}\right\| + \left\|\begin{pmatrix}\delta\\ \varepsilon\end{pmatrix}\right\|^2$$

$$= 2(\|a\|^2 + |\delta|\,\|a\|)$$

$$\bar{H} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}\tilde{\delta}^2 & \varepsilon\tilde{\delta}\\ \varepsilon\tilde{\delta} & \varepsilon^2\end{bmatrix} \cdot \frac{1}{\|a\|^2 + |\delta|\,\|a\|}$$

$$\Rightarrow \quad \overline{H}\begin{pmatrix}\delta\\ \varepsilon\end{pmatrix} \triangleq \begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}\overline{h}_{*1}\\ \underbrace{\overline{h}_{*2}}_{\boxed{\text{second row of } \overline{H}}}\end{pmatrix}\begin{pmatrix}\delta\\ \varepsilon\end{pmatrix}$$

$$\Rightarrow \quad v_2 = \overline{h}_{*2}\begin{pmatrix}\delta\\ \varepsilon\end{pmatrix}$$

$$= \frac{1}{\|a\|^2+|\delta|\|a\|}\left(-\varepsilon\tilde{\delta},\ \|a\|^2+|\delta|\|a\|-\varepsilon^2\right)\begin{pmatrix}\delta\\ \varepsilon\end{pmatrix}$$

$$= \frac{-\varepsilon\delta(\delta+\|a\|\,\mathrm{sign}(\delta))+\varepsilon\|a\|^2+\varepsilon|\delta|\|a\|-\varepsilon^3}{\|a\|^2+|\delta|\|a\|}$$

$$= \frac{-\varepsilon\delta^2-\varepsilon|\delta|\|a\|+\varepsilon\delta^2+\varepsilon^3+\varepsilon|\delta|\|a\|-\varepsilon^3}{\|a\|^2+|\delta|\|a\|}$$

$$\equiv 0\ ! \qquad \boxed{\text{YES}}$$

Is $\overline{H}$ an orthogonal matrix ?

$$\overline{H}^T = I^T - \frac{2}{\|u\|^2}(uu^T)^T = I - \frac{2}{\|u\|^2}uu^T = \overline{H}$$

$$\Downarrow$$

$$\overline{H}^T\overline{H} = \left(I - \frac{2}{\|u\|^2}uu^T\right)\left(I - \frac{2}{\|u\|^2}uu^T\right)$$

$$= I \underbrace{- \frac{2}{\|u\|^2}uu^T - \frac{2}{\|u\|^2}uu^T + \frac{4}{\|u\|^2}\cdot\frac{(uu^T)(uu^T)}{\|u\|^2}}_{=0}$$

$$= I \qquad \boxed{\text{YES}}$$

Algorithm for $\perp$ transformation to echelon form

e.g. $A = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ \square & * & * \end{bmatrix}$ $\neq$ echelon matrix because of the 4-row step for column 1

fix column 1 :

choose $\perp H_1$ so

$$H_1 \underbrace{\begin{pmatrix} * \\ * \\ * \\ \square \end{pmatrix}}_{\boxed{\text{column 1 of A}}} = \begin{pmatrix} * \\ * \\ \square \\ 0 \end{pmatrix}$$

$$\boxed{H_1 = \begin{bmatrix} I_{2\times 2} & 0 \\ 0 & \overline{H} \end{bmatrix}}$$

$\boxed{\text{for the appropriate } \overline{H}}$

Then

$$H_1 A = H_1 [a_{*1}\ a_{*2}\ a_{*3}]$$

$$= [H_1 a_{*1} \,\vdots\, H_1 a_{*2} \,\vdots\, H_1 a_{*3}]$$

by reducing the 4-row step for column 1 to a 1-row step, using a sequence of $\perp$ transformations H_i to zap to zero the bottom 3 entries in column 1

$$\Downarrow$$

$$H_1 A = \begin{bmatrix} * & * & * \\ * & * & * \\ \square & * & * \\ 0 & * & * \end{bmatrix} \triangleq A_1 \neq \text{echelon matrix}$$

changed — may be changed

choose $\perp H_2$ so

$$H_2 \begin{pmatrix} * \\ * \\ \square \\ 0 \end{pmatrix} \Big\} = \begin{pmatrix} * \\ \square \\ 0 \\ 0 \end{pmatrix}$$

$H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \bar{H} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ for the appropriate $\bar{H}$

Then

$$H_2 A_1 = \begin{bmatrix} * & * & * \\ \square & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \triangleq A_2 \neq \text{echelon matrix}$$

choose $\perp H_3$ so

$$H_3 \begin{pmatrix} * \\ \square \\ 0 \\ 0 \end{pmatrix} \Big\} = \begin{pmatrix} \square \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad H_3 = \begin{bmatrix} \bar{H} & 0 \\ 0 & I_{2\times 2} \end{bmatrix}$$

$\}$ indicates those entries which are changed by the orthogonal transformation concerned.

So that:

$$H_3 A_2 = \begin{bmatrix} \square & * & * \\ 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \triangleq A_3$$

Suppose actually that:

$$A_3 = \begin{bmatrix} \square & * & * \\ 0 & * & * \\ 0 & \square & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \text{echelon matrix}$$

zero by chance

because of 2-row step for Col. 2

fix column 2

choose $\perp H_4$ so

$$H_4 \begin{pmatrix} * \\ * \\ \square \\ 0 \end{pmatrix} \Big\} = \begin{pmatrix} * \\ \square \\ 0 \\ 0 \end{pmatrix}$$

Then

$$H_4 A_3 = \begin{bmatrix} \square & * & * \\ 0 & \square & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix} \triangleq A_4$$

still OK, as zeros induced by by earlier H_i not destroyed by later H_i.

Suppose, actually that:

$$A_4 = \begin{bmatrix} \square & * & * \\ 0 & \square & \square \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{echelon matrix}$$

$\boxed{\text{if not use extra } H_i \text{ to zap the offending non-zero entries}}$

Then

$$\underbrace{A_4}_{\underbrace{H_4 A_3}_{\underbrace{H_3 A_2}_{\underbrace{H_2 A_1}_{H_1 A}}}} = \text{echelon matrix}$$

i.e. $\underbrace{H_4 H_3 H_2 H_1}_{}\, A$ = echelon matrix

$\boxed{\text{would be OK if this matrix was orthogonal is it ?}}$

$$(H_4 H_3 H_2 H_1)^T (H_4 H_3 H_2 H_1)$$

$$= H_1^T H_2^T H_3^T H_4^T H_4 H_3 H_2 H_1 = I$$

$\boxed{\text{because } H_i,\ i=1,..,4 \text{ are all } \perp}$

$\Rightarrow$ $\boxed{\text{YES},\ H_4 H_3 H_2 H_2 \triangleq P \text{ is } \perp}$

<u>end of alg.</u> for $\perp$ transf. to echelon form

<u>$\perp$ Decomposition of A</u>

e.g. $A = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ \square & * & * \end{bmatrix}$

① Apply to A the Alg. for $\perp$ transf. to echelon form.

Finds $\perp$ P so $P^T A =$ e.g. $\begin{bmatrix} \square & * & * \\ 0 & \square & \square \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \triangleq \bar{A}$

(2) Consider

$$\bar{A}^T = \begin{bmatrix} \square & 0 & 0 & 0 \\ * & \square & 0 & 0 \\ * & \square & 0 & 0 \end{bmatrix}$$

(3) Apply to $\bar{A}^T$: Alg. for $\perp$ transf. to echelon form

Finds $\perp Q$ so

$$Q^T\bar{A}^T = \underbrace{\begin{bmatrix} \boxed{1} & * & 0 & 0 \\ 0 & \boxed{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

this occurs because the $\perp$ transformation alg. first zeroes the bottom 2 entries in column 1 of $\bar{A}^T$, giving rise to $\boxed{1}$ in $Q^T\bar{A}^T$. Then it zeroes the resulting bottom entry of column 2, giving rise to $\boxed{2}$. The zeros on the right of $\bar{A}^T$ are not destroyed by those transformations, and so appear in $Q^T\bar{A}^T$

(4) Transposing

$$\underbrace{\bar{A}Q}_{P^TA} = \begin{bmatrix} \boxed{1} & 0 & 0 \\ * & \boxed{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So

$$P^TAQ = \left[\begin{array}{cc|c} \boxed{1} & 0 & 0 \\ * & \boxed{2} & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} \hat{A} & 0 \\ \hline 0 & 0 \end{array}\right]$$

$\hat{A}$ = invertible as : $\det[\hat{A}] = \boxed{1}\cdot\boxed{2} \neq 0$

Hence : have found $\perp P$, $\perp Q$ so

$\boxed{\text{invertible}}$

$$A = P\left[\begin{array}{c|c} \hat{A} & 0 \\ \hline 0 & 0 \end{array}\right]Q^T \qquad \text{as required.}$$

The method presented here works for any non-zero A, but two or more of the zero sub-matrices in $\begin{bmatrix} \hat{A} & 0 \\ 0 & 0 \end{bmatrix}$ may be absent, depending on the particular A considered.