(15.1) <u>Definition</u> (P) is called convex if:

V = convex

$h(x) = Dx - f \leftarrow$ $\boxed{\text{i.e. } h \text{ is affine}}$

(15.2) <u>Th.</u> [(P) is convex] ($)

$\Rightarrow [\ \pi = \text{convex on } \mathbb{R}^m\ ]$

(15.3) <u>Th</u> (slight extension of Th. (13.3))

If : $f : \mathbb{R}^n \to \mathbb{R}$ is convex on $\mathbb{R}^n$ and

f is differentiable at x:

Then:

$$f(z) \geq f(x) + \nabla f(x)^T [z - x] \ , \ \forall z \in \mathbb{R}^n$$

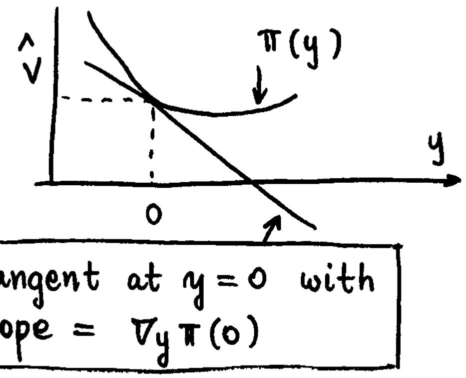
i.e. f lies above its tangent at x.

($) so the situation with π = convex happens for at least one important type of problem

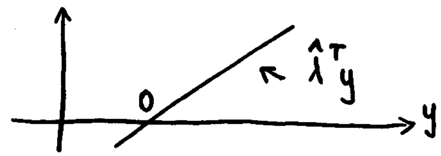
Suppose π is

(i) convex on $\mathbb{R}^m$

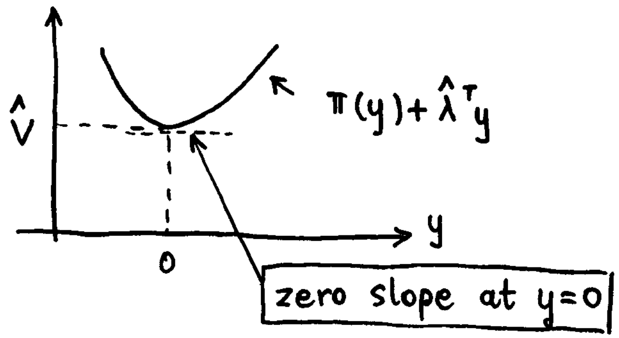
(ii) differentiable at $y=0$



Let $\hat{\lambda} = -\nabla_y \pi(0)$



Then



So:

$$d(\hat{\lambda}) = \min_{y \in \mathbb{R}^m} \pi(y) + \hat{\lambda}^T y \equiv \hat{V} \quad !$$

Hence, we have seen graphically that:

(15.4) **Th** If:

(i) π is convex on $\mathbb{R}^m$
(ii) π is differentiable at $y=0$ } $\boxed{(H)}$
(iii) $\hat{\lambda} = -\nabla_y \pi(0)$

Then: $d(\hat{\lambda}) = \hat{V}$.

$\Downarrow$

So, if $(H) = \text{true}$, then

$d(\hat{\lambda}) = \hat{V}$ for some $\hat{\lambda} \in \mathbb{R}^n$

$\boxed{\text{e.g. for } -\nabla_y \pi(0)}$

Also,

$d(\lambda) \leq \hat{V}$, $\forall \lambda \in \mathbb{R}^m$

$\boxed{\text{Th } (14.2)}$

Hence:

$$\boxed{\max_{\lambda \in \mathbb{R}^m} d(\lambda) = \hat{V}} \leftarrow \boxed{\text{Problem (D)}}$$

$$\hat{V} = \min_{x \in \mathbb{R}^n} \{ V(x) : h(x) = 0 \}$$

$\boxed{\text{constrained optimization} \equiv \text{THE PRIMAL PROBLEM (P)}}$

$$\max_{\lambda \in \mathbb{R}^m} d(\lambda) \leftarrow \boxed{\text{unconstrained optimization}} \leftarrow \boxed{\text{THE DUAL PROBLEM}}$$

$\boxed{\text{we know how to solve it e.g. } \max d = -\min(-d)}$

So: we can find $\hat{V}$ for (P) by unconstrained maximization of (D)

OK, but what we really want is a global minimizer $\hat{x}$ for (P).

(15.5) **Th** If:

$$\left.\begin{array}{ll}\text{(i)} & \pi \text{ is convex on } \mathbb{R}^m \\ \text{(ii)} & \pi \text{ is differentiable at } y=0 \\ \text{(iii)} & \hat{\lambda} \in \underset{\lambda \in \mathbb{R}^m}{\arg\max}\ d(\lambda)\end{array}\right\} \text{— (H)}$$

Then:

$$\arg\min(P) = B \cap F \qquad (\pounds)$$

$$\| \triangle$$

$$\underset{x \in \mathbb{R}^n}{\arg\min} \{V(x) : h(x) = 0\}$$

where:

$$B = \underset{x \in \mathbb{R}^n}{\arg\min} \{V(x) + \hat{\lambda}^T h(x)\}$$

$$F = \{x \in \mathbb{R}^n : h(x) = 0\}$$

(£) This provides a useful mechanism for finding the set of global minimizers for the primal problem from the set of minimizers B, associated with the dual problem. A singleton assumption yields a big simplification.

$\downarrow$

Remark:

Suppose: each set arg min contains exactly one element i.e. is a singleton

$\boxed{\text{often true}}$

$\Downarrow$

$$\arg\min(P) = B \cap F$$

$$\{\hat{x}\} \qquad\qquad \{\tilde{x}\}$$

Then either: $\tilde{x} \notin F$ or $\tilde{x} \in F$

$\Downarrow$

$$\underbrace{\{\hat{x}\} = \{\tilde{x}\} \cap F = \emptyset}_{\boxed{\text{a contradiction}}} \qquad \underbrace{\tilde{x} \in F}_{\boxed{\text{must be true}}}$$

So: $\{\hat{x}\} = \{\tilde{x}\} \cap F = \{\tilde{x}\} \Rightarrow \hat{x} = \tilde{x}$.

(15.6) Duality-based procedure for solving (P)

Assume: • π = convex on $\mathbb{R}^m$ ← OK if (P) is convex

• π = differentiable at $y = 0$

not easy to check but usually OK

• that the singleton hypothesis is valid.

often is OK

Find:

$$\{\hat{\lambda}\} = \arg\max_{\lambda \in \mathbb{R}^m} d(\lambda)$$ ← unconstrained optimization algs. used

$$\{\tilde{x}\} = \arg\min_{x \in \mathbb{R}^n} \{V(x) + \hat{\lambda}^T h(x)\}$$

unconstrained optimization

Then: $\hat{x} = \tilde{x}$

for primal problem (P)