# DESIGN OF STABILIZING FEEDBACK CONTROLS FOR STRONGLY NONLINEAR SYSTEMS


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*Gratefully dedicated to my family*

# Abstract

This work addresses the problem of stabilizing feedback design for strongly nonlinear systems, i.e. systems whose linearization about their equilibria is uncontrollable and for which there does not exist a smooth or even continuous stabilizer. The construction of stabilizing controls for these systems is often further complicated by the presence of a drift term in the differential equation describing their dynamics. This research also considers the development of stabilizing feedback laws for bilinear systems. The relevance of bilinear systems is due to the fact that they result from the linearization of certain nonlinear control systems with respect to the state only. The proposed methodologies yield time-varying feedback controls whose construction is based on Lie algebraic techniques.

Two systematic approaches to the construction of time-varying feedback laws for nonlinear systems with drift are proposed: (1) a continuous time-varying control strategy, partially drawing on the ideas by Coron and Pomet for driftless systems, and (2) a computationally feasible discontinuous time-varying feedback approach based on Lie algebraic techniques.

The continuous time-varying control law is a combination of a periodic time-varying control providing for critical stabilization with an asymptotically stabilizing feedback "correction" term. The periodic control is obtained through the solution of an open-loop, finite horizon, control problem on the associated Lie group which is posed as a trajectory interception problem in the logarithmic coordinates of flows for the system and its Lie algebraic extension. The correction term is calculated to be a control which decreases a Lyapunov function whose level sets contain the periodic orbits of the system stabilized by the time-periodic feedback.

The control strategy proposed in the second stabilization approach comprises two modes. In one mode the control is a smooth static state feedback that guarantees an instantaneous decrease of a chosen control Lyapunov function. This mode is applied whenever there exists a smooth control. The other mode considers a time-varying piece-wise constant feedback control which decreases the control Lyapunov function on average, after a finite period of time. The synthesis of the smooth state feedback is based on the standard Lyapunov approach, thus the emphasis is put on the construction of the time-varying discontinuous control. The Lie algebraic control is composed of a sequence of constant controls whose values are calculated as the solution to a non-linear programming problem. Two approaches to the formulation of the non-linear programming problem are proposed. In the first approach, the formulation of the non-linear programming problem results from the direct application of the Campbell-Baker-Hausdorff formula for composition of flows, while in the second approach, the non-linear programming problem is formulated by posing the original control problem in terms of a relaxed control problem in the associated logarithmic coordinates.

These approaches are general and applicable to a large class of nilpotent systems which do not lend themselves to controllable linearization (be it through state-feedback transformations, or else simply around some operating points).

Concerning the stabilization of homogeneous bilinear systems with unstable drift, two control strategies are proposed. The first method relies on the above mentioned trajectory interception approach, employed in the construction of critically stabilizing controls. However, the trajectory interception problem is now used to generate controls that track an exponentially stable Lie algebraic extension of the system. The second stabilization method can be considered to be a type of sliding mode control, in which the reaching phase feedback law exploits the ideas proposed in the methodology to discontinuous time-varying feedback design.

The computationally feasible approaches proposed in this research necessitated the development of a set of software tools for symbolic manipulation of expressions with Lie brackets. The novel software package constitutes a contribution towards the automated construction of Philip Hall bases, the simplification of any Lie bracket expression, the composition of flows via the Campbell-Baker-Hausdorff formula, the derivation of the Wei-Norman equations for the logarithmic coordinates, and other Lie algebraic manipulations.

Conditions under which the constructed feedback laws render the corresponding systems asymptotically stable are analyzed. The applicability and effectiveness of the proposed approaches is demonstrated through computer simulations of several nonlinear systems, including well known nonholonomic driftless systems, such as the kinematic models of a unicycle and a front-wheel drive car, and systems with drift like the dynamic model of a satellite in the challenging actuator failure condition.

# Résumé

Ce travail adresse le problème du design de rétroaction stabilisatrice pour les systèmes fortement non linéaires, i.e. les systèmes pour lesquelles la linéarisation autour du point d'équilibre est non contrôlable et pour lesquelles il n'existe pas de stabilisateur lisse ou même continu. La construction d'un controlleur stable pour ces systèmes est souvent compliquée par la présence d'un terme de dérive dans l'équation différentielle décrivant leurs dynamiques. Cette recherche considère aussi le développement de loi de rétroaction stabilisatrice pour les systèmes bilinéaires. La pertinence des systèmes bilinéaires provient du fait qu'ils sont le résultat de la linéarisation (par rapport à l'état seulement) de certains systèmes de contrôle non linéaires. Les méthodologies proposées conduisent à des commandes d'asservissement instationnaires dont la construction est basée sur des techniques d'algèbre de Lie.

Deux approches systématiques sont proposées pour la construction de loi d'asservissement instationnaire pour les systèmes non linéaires avec dérive: (1) une stratégie de contrôle continu instationnaire, empruntant partiellement aux idées sur les systèmes sans dérive de Coron et Pomet, et (2) une approche d'asservissement discontinu instationnaire réalisable numériquement et basée sur des techniques d'algèbre de Lie.

La loi continue de contrôle instationnaire est la combinaison d'une commande instationnaire periodique établissant une stabilisation critique et d'un terme de rétroaction stabilisatrice asymptotique. La commande périodique est obtenue par la solution d'une boucle ouverte, à horizon fini, d'un problème d'asservissement sur le groupe de Lie associé; ce dernier étant posé comme un problème d'interception de trajectoire dans les coordonnées logarithmiques de flux du système et de son algèbre de Lie étendu. Le terme de correction est calculé comme étant une commande décroissant une fonction de Lyapunov dont les ensembles de niveau contiennent les orbites périodiques du système stabilisé par la rétroaction périodique dans le temps.

Les stratégies d'asservissement proposée dans la seconde approche de stabilisation comprennent deux modes. Dans l'un des modes, la commande est une rétroaction statique lisse guarantissant une décroissance instantanée de la fonction de Lyapunov sélectionnée. Ce mode est appliqué lorsqu'il existe une commande lisse. L'autre mode considère une commande de rétroaction constante par morceaux et instationnaire décroissant en moyenne la fonction de Lyapunov sur une période de temps fini. La synthèse de retour d'état, lisse, est basée sur l'approche standard de Lyapunov. Ainsi, l'emphase est mise sur la construction de la commande discontinue instationnaire. La commande, par algèbre de Lie, est composée d'une séquence de commande constante dont la valeur est calculée comme étant la solution d'un problème de programmation non linéaire. Deux approches pour la formulation du problème de programmation non linéaire sont proposées. Dans la première approche, la formulation du problème de programmation non linéaire provient de l'application directe de la formule de Campbell-Baker-Hausdorff pour la composition de flux. Dans la seconde approche, le problème de programmation non linéaire est formulé en posant le problème original d'asservissement en terme d'un problème d'asservissement relaché dans les coordonnées logarithmiques associées.

Ces approches sont générales et applicables à une large classe de systèmes nilpotents qui ne se prêtent pas à une linéarisation contrôlable (que ce soit par des transformations de retour d'état ou simplement autour de points d'opération).

Concernant la stabilisation de systèmes bilinéaires homogènes avec dérive instable, deux stratégies de contrôle sont proposées. La première méthode repose sur l'approche d'interception de trajectoire mentionnée plus haut; celle employée dans la construction de commandes de stabilisation critique. Toutefois, le problème d'interception de trajectoire est maintenant utilisé pour générer des commandes suivant un système exponentiellement stable correspondant à l'extension du système original par son algèbre de Lie. La seconde méthode de stabilisation peut être considérer un type de commande par mode glissant dans lequel la phase d'atteinte de la loi de rétroaction exploite les idées proposées dans la méthodologie du design d'asservissement discontinu instationnaire.

Les approches numériquements réalisables proposées dans cette recherche nécessitent le développement d'un ensemble d'outils logiciels pour la manipulation symbolique d'expressions avec des crochets de Lie. Ce nouveau progiciel constitut une contribution vers la construction automatique des bases de Philip Hall, la simplification d'expressions avec des crochets de Lie (qu'elle quelles soit), la composition de flux via la formule de Campbell-Baker-Hausdorff, la dérivation des équations de Wei-Norman pour les coordonnées logarithmique, et pour la manipulation de d'autres expresions en algèbre de Lie.

Les conditions sous lesquelles la construction de loi de rétroaction rendent les systèmes correspondant asymptotiquement stable sont analysées. L'applicabilité et l'efficacité des approches proposées sont démontrées par modèle informatique de plusieurs systèmes non linéaires, incluant des systèmes nonholonomiques sans dérive bien connus tels les modèles cinématiques d'un unicycle et d'une automobile avec traction avant, et des systèmes avec dérives comme le modèle dynamique d'un satellite en présence d'un actuateur défaillant.

# Claims of Originality

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The following novel contributions are made in this dissertation:

- The development of a method for the synthesis of continuous time-varying stabilizing controls, [P2, P3]. The approach is general and applicable to a large class of nilpotent systems with drift that do not lend themselves to successful linearization (be it through state-feedback transformations, or else simply around some operating points).
- The development of two approaches based on Lie algebraic techniques and optimization for the construction of discontinuous time-varying stabilizing feedback controls, [P5, P6, P7, P8]. The proposed methodologies are systematic, computationally feasible, and applicable to a general class of nonlinear systems with drift.
- The development of two Lie algebraic approaches for the synthesis of stabilizing feedback controls for homogeneous bilinear systems, [P1, P4]. Unlike existing methods, the proposed approaches consider systems with unstable drift that cannot be stabilized by any constant control.
- The derivation of sufficient conditions for the existence of the proposed feedback laws.
- The implementation of a software package for symbolic Lie algebraic computations, [P9]. This novel software tool has proved to be very helpful in the construction of the feedback control laws mentioned above. Existing software for the manipulation of Lie algebraic expressions is very specialized and does not provide any of the functionality found in the implemented package.

This research work has been partially reported in the following conference proceedings and journals:

## Publications

[P1] H. Michalska, M. Torres-Torriti. Time-varying stabilising feedback design for bilinear systems. *Nineteenth IASTED International Conference: Modelling, Identification and Control - MIC 2000*, Innsbruck, Austria, February 2000, pp. 93–99.

[P2] H. Michalska, M. Torres-Torriti. Time-varying stabilizing feedback control for nonlinear systems with drift. *Proceedings of the* $40^{th}$ *Conference on Decision & Control*, Orlando, Florida, December 2001, pp. 1767–1768.

[P3] H. Michalska, M. Torres-Torriti. Time-varying stabilizing feedback control for nonlinear systems with drift. Journal paper in preparation.

[P4] M. Torres-Torriti, H. Michalska. Stabilization of bilinear systems with unstable drift. *Proc. of the 2002 American Control Conference*, Anchorage, Alaska, U.S.A., 8-10 May 2002, pp. 3490–3491.

[P5] H. Michalska, M. Torres-Torriti. A geometric approach to feedback stabilization of nonlinear systems with drift. Accepted in *Systems Control Lett.*, Elsevier Science, April 2003.

[P6] H. Michalska, M. Torres-Torriti. Feedback stabilisation of strongly nonlinear systems using the CBH formula. Submitted to *Internat. J. Control*, Taylor & Francis, December 2002.

[P7] H. Michalska, M. Torres-Torriti. A reachable set approach to feedback stabilization of nonlinear systems with drift. Submitted to the $42^{nd}$ *Conference on Decision & Control*, Maui, Hawaii, December 2003.

[P8] H. Michalska, M. Torres-Torriti. Nonlinear programming and the CBH formula in feedback stabilization of nonlinear systems with drift. Submitted to the $42^{nd}$ *Conference on Decision & Control*, Maui, Hawaii, December 2003.

[P9] M. Torres-Torriti, H. Michalska. A software package for Lie algebraic computations. Submitted to *SIAM Rev.*, June 2002.

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*Miguel Torres-Torriti*

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# CHAPTER 1

## Introduction

This work addresses the problem of stabilizing feedback design for strongly nonlinear systems, i.e. systems whose linearization about their equilibria is uncontrollable and for which there does not exist a smooth or even continuous stabilizer. This class of systems includes most controllable driftless systems, such as nonholonomic systems with kinematic constraints (e.g. wheeled robots and carts), as well as some systems with drift, like most nonholonomic systems with acceleration constraints and other underactuated dynamical systems (e.g. satellites, underwater vehicles). Hence, the practical relevance of this research.

In this thesis the emphasis is put on systems with drift; see equation (1.1). From a more theoretical perspective, this work is encouraged by the fact that relatively few general methods exist for the stabilization of systems with a drift term in the differential equation describing their dynamics. So far, most control strategies have been derived exploiting the specific algebraic structure of the model representation, and have thus resulted in feedback laws of limited generality.

If the drift term is unstable, the construction of stabilizing controls becomes a difficult and challenging problem. Moreover, strong controllability assumptions are required in contrast to the case of driftless systems in which the standard Lie algebra rank condition does provide a conclusive answer concerning their controllability. These features, as well as other difficulties arising in the control of strongly nonlinear systems with drift, are explained in more detail in Section 1.4.

The controllability and stabilizability of nonlinear systems has been considerably studied by many authors [**24, 26, 18, 27, 20, 19, 22, 37, 15, 39, 23, 10, 9, 28**]; see also the survey works [**35, 11, 29**]. The fact that for many controllable systems no continuous stabilizing feedback law exists was first pointed out in [**37**]. It was shown in [**32**] that there might exist a *dynamic stabilizer* for some systems for which there does not exist a continuous feedback, and that in general not even the dynamic stabilizer can be defined continuously. Later, a condition that allows one to verify the non-existence of smooth (locally) stabilizing feedback laws for a large class of controllable systems, known as *Brockett's condition*, was given in [**15**] (see also Theorem 1.1 on p. 10). A stronger necessary condition was established in [**16**]. In [**41**], it was shown that Brockett's condition persists when stabilizability by time-invariant continuous feedback is considered, while in [**30**], the condition is shown to persist even when the feedback laws are in the class of discontinuous controls. It is also well known that optimal control problems often result in solutions that can only be implemented in terms of *discontinuous laws*. These results have naturally led to the consideration in the existing literature of mainly two types of control strategies:

- Time-varying feedback.
- Discontinuous feedback.

The application of these feedback laws is also considered here in the context of the general stabilization problem stated in the next section. Following the problem definition, a summary of the motivation to this research is presented in Section 1.2. To further illustrate the practical relevance of systems with drift, some examples of real systems whose model equation has a drift term are presented in Section 1.3. The difficulties arising in the control of this class of systems and the drawbacks of the existing stabilization approaches are explained in sections 1.4 and 1.5. In the light of the discussion in the previous sections, the research objective is stated in Section 1.6. This chapter is concluded with a brief description of the research contributions, an outline of the thesis, and the claims of originality of contributions, in sections 1.7, 1.8 and 1.9, respectively.

The notation and mathematical preliminaries are found in Appendix A on page 194, and follow the standard and basic references indicated therein. The reader is assumed to have some familiarity with the mathematical background presented in Appendix A and will be referred to the appendices where deemed convenient.

## 1.1. Problem Definition

This work addresses the problem of stabilizing feedback design for systems whose dynamics on $\mathbb{R}^n$ is modeled by nonlinear ordinary differential equations of the form:

$$\Sigma: \qquad \dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x)u_i \stackrel{\text{def}}{=} f^u(x) \tag{1.1}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 0, \ldots, m$, are real analytic vector fields, and $u_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$, are the control inputs, with $m < n$. The vector field $f_0$ is the so-called *drift vector field*, while $f_i$, $i = 1, \ldots, m$, are the *input vector fields*.

This research is also relevant to the development of stabilizing control laws for bilinear systems for which $f_i(x) = A_i x$, with $A_i \in \mathbb{R}^{n\times n}$, are linear vector fields. Bilinear systems are of interest since they correspond to the linearization of (1.1) with respect to the state only.

## 1.2. Motivation

The motivation to this research is provided by observing the following:

- The existence of relatively few general methods for stabilization of nonlinear systems with drift.
- The fact that strongly nonlinear systems often result in uncontrollable linearizations about their equilibria and do not satisfy the necessary conditions for smooth [**15**] or even continuous [**41**] stabilizability.
- The lack of computationally feasible methods for the construction of discontinuous feedback and the computational complexity of the feedback laws based on Lie algebraic approaches developed so far.
- The lack of constructive approaches to stabilization of general (higher order $n > 2$) multi-input bilinear systems for which the matrix $A_0 + \sum_{i=1}^{m} u_i A_i$ is unstable for all choices of constant inputs $u_i$.

## 1.3. Some Examples of Systems with Drift

Most real world dynamical systems have a drift term in the equations that characterize them. As a matter of fact, most systems can be locally approximated by a linear system in terms of the well known differential equation $\dot{x} = Ax + Bu$, which is an equation with drift. However, for strongly nonlinear systems a linear approximation is not enough to describe their dynamics, or it simply limits the derivation of global stabilization laws. Thus the richness of a nonlinear model is often desirable, despite analysis difficulties it imposes, and in some cases it is essential to the derivation of adequate control laws. Some examples of strongly nonlinear systems which motivate the development of the proposed approaches are presented below.

### 1.3.1. Underactuated Mechanical Systems

Classical examples are the inverted pendulum and its relatives, such as a unicycle or the Acrobot [**91, 90**]. Other examples include underactuated robotic manipulators, vertical takeoff and landing aircrafts, underactuated hovercrafts and underactuated submersible vessels; see [**88**] for additional references. Most of these systems are described by the following set of Lagrange equations [**88**]:

$$\begin{aligned} M_{11}(q)\ddot{q}_a + M_{12}(q)\ddot{q}_b + F_1(q,\dot{q}) &= B(q)u \\ M_{21}(q)\ddot{q}_a + M_{22}(q)\ddot{q}_b + F_2(q,\dot{q}) &= 0 \end{aligned} \tag{1.2}$$

Here $(q,\dot{q}) = (q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_n)$ denote the local coordinates on the tangent bundle of some configuration manifold $Q$ on which the dynamics of the system is defined. Without loss of generality it is assumed that the generalized coordinates $q = (q_1,\ldots,q_m,q_{m+1},\ldots,q_n) = (q_a,q_b)$, where the components of $q_a$ represent the actuated degrees of freedom, while those of $q_b$ represent the unactuated coordinates. The matrices $M_{ij}(q)$, $i,j = 1,2$, correspond to the components of an $n \times n$ inertia matrix which is symmetric and positive definite for all $q \in Q$, $B(q) \in \mathbb{R}^{m\times m}$ is invertible for all $q \in Q$, and $F_1(q,\dot{q}) \in \mathbb{R}^m$. The vectors $F_2(q,\dot{q}) \in \mathbb{R}^{n-m}$ represent Coriolis and dynamic damping terms. The equations (1.2) can be rewritten in the general form of a system with drift (1.1), and partially linearized by state-feedback, see [**88**] for details.

Models of nonholonomic dynamic systems may also be expressed in terms of a similar set of equations, see [**59**], using the classical approach for the formulation of constrained dynamics introduced in [**178**]:

$$\begin{aligned} M(q)\ddot{q} + F(q,\dot{q}) &= J^T(q)\lambda + B(q)u \\ J(q)\dot{q} &= 0 \end{aligned} \tag{1.3}$$

where $\lambda \in \mathbb{R}^m$ is a vector of constraint multipliers and $u \in \mathbb{R}^r$, with $r \geq n-m$, is a vector of control inputs, and $J(q) \in \mathbb{R}^{m\times n}$ is assumed to be full rank.

Nonholonomic systems are examples of underactuated mechanical systems. However, the general connection between nonholonomic control systems and underactuated systems is not yet fully understood [**49**].

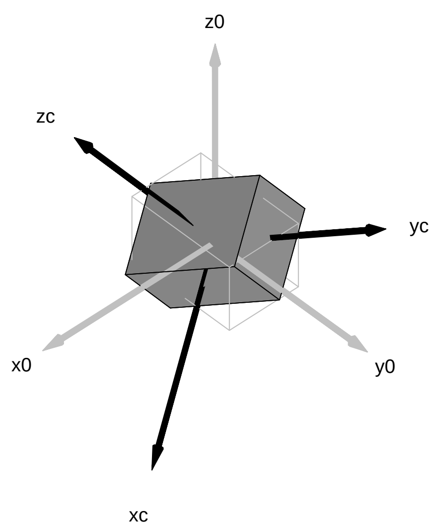
Not all underactuated systems, (1.2), result in uncontrollable linearizations around their operating points or exhibit a dynamics which cannot be linearized by state-feedback transformations. However, when there does not exist any kind of linearization, the stabilization is well recognized to be a difficult problem, such as is that of the rigid body in space described next.

### 1.3.2. Rigid Body Systems in Space

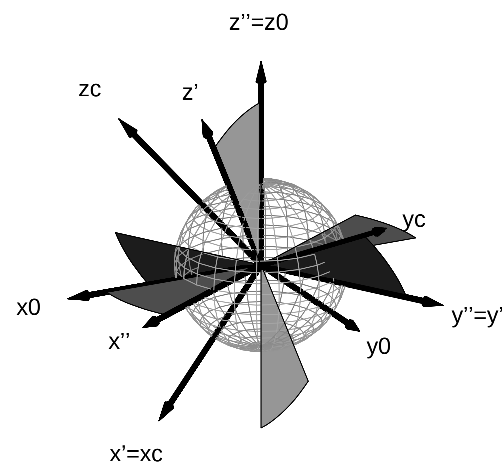
Although the problem of stabilization of *fully actuated* satellites and spacecrafts has been completely solved under different conditions, the stabilization of rigid bodies in space: (a) that are subject to certain disturbances, (b) that must be optimally controlled, or (c) that are underactuated, still constitutes an active area of research. The attitude and angular velocities of an underactuated rigid body in space can be modelled by the following set of equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} \sin(x_3)\,\sec(x_2)\,x_5 + \cos(x_3)\,\sec(x_2)\,x_6 \\ \cos(x_3)\,x_5 - \sin(x_3)\,x_6 \\ x_4 + \sin(x_3)\,\tan(x_2)\,x_5 + \cos(x_3)\,\tan(x_2)\,x_6 \\ a_1\,x_5\,x_6 \\ a_2\,x_4\,x_6 \\ a_3\,x_4\,x_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_1} \\ 0 \\ 0 \end{bmatrix} \tau_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{J_2} \\ 0 \end{bmatrix} \tau_2 \tag{1.4}$$

where $a_1 = (J_2 - J_3)/J_1$, $a_2 = (J_3 - J_1)/J_2$, $a_3 = (J_1 - J_2)/J_3$, and $J_1$, $J_2$, $J_3$ are the principal inertia moments about a set of orthonormal axes $C \stackrel{\text{def}}{=} \{x_c, y_c, z_c\}$ fixed to the rigid body, with the origin at its center of mass, (see Fig. 1.1 (a)). It is assumed that $J_1 \neq J_2$ ($c_3$ is not an axis of symmetry). The state variables $x_1$, $x_2$ and $x_3$ correspond to the orientation of the rigid body locally expressed in terms of the standard Euler angles, $\psi$, $\theta$, and $\phi$ of consecutive clockwise rotations about the orthonormal axes $z_c$, $y_c$ and $x_c$, respectively (see Fig. 1.1 (b)). The state variables $x_4$, $x_5$ and $x_6$ are the absolute angular velocities $\omega_1$, $\omega_2$ and $\omega_3$ measured with respect to $C$. The control inputs to this system are the torques $\tau_1$ and $\tau_2$ applied about the axes $c_1$ and $c_2$, respectively. For details on the model derivation see, for example, [**198**], and references therein.




(a) The global coordinate system $\{x_0, y_0, z_0\}$ and the body coordinate system $\{x_c, y_c, z_c\}$.




(b) Set of rotations in standard Euler angles $\psi$, $\theta$ and $\phi$ about the axis $z_0$, $y_c''$ and $x_c'$, respectively.

FIGURE 1.1. Rigid body in space: the attitude of the rigid body is described by the state variables $x_1$, $x_2$, $x_3$ that correspond to the Euler angles $\psi$, $\theta$, $\phi$ of rotation about the principal axes of the rigid body $z_c$, $y_c$, $x_c$, respectively. The angular velocities $x_4, x_5, x_6$ are also measured with respect to the axes $z_c$, $y_c$, $x_c$.

### 1.3.3. Bilinear Systems

Bilinear systems homogeneous in the state evolving on $\mathbb{R}^n$ are described by the equation:

$$\Sigma_{BS}: \quad \begin{aligned} \dot{x} &= A_0 x + \sum_{i=1}^{m} A_i x u_i \\ &= \underbrace{(A_0 + \sum_{i=1}^{m} A_i u_i)\, x}_{A(u)} \end{aligned} \tag{1.5}$$

where $A_i \in \mathbb{R}^{n\times n}$, $i = 0, \dots, m$, are real valued matrices, $x \in \mathbb{R}^n$ is the state vector, and $u_i \in \mathbb{R}$, $i = 1, 2, \dots, m$, are the control inputs, with $m < n$. The vector field $A_0 x$ corresponds in this case to the drift term, while $A_i x$, $i = 1, \dots, m$, are the input vector fields.

Bilinear systems have received significant attention in the literature as a natural generalization of linear systems theory. However, it has already been suggested by Brockett in [**14**] that bilinear systems should be studied in the context of the more general class of nonlinear systems. Their strongly nonlinear nature has been concealed by their simple *almost* linear equation. In fact, for a constant input, equation (1.5) represents a linear system. It can easily be seen from the second equation in (1.5) that for constant inputs the system is a variable structure linear system, whose dynamics strongly depends on the control parameter $u$ as the Jacobian with respect to the state is given by $A(u_0)$ for a nominal control value $u_0$.

On the other hand, bilinear systems can easily be turned into highly nonlinear systems depending on the class of input functions considered. For instance, while a linear system with linear state feedback is still a linear system, a bilinear system with such a simple feedback law becomes a quadratic system, which can even exhibit chaotic dynamics. An example of such situation is provided by the chaotic third order Lorenz system, which may be regarded as a bilinear system with matrices [**116**]:

$$A_0 = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix} \qquad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

with $\sigma > 0$, $\rho > 0$, $\beta > 0$ and state feedback:

$$u_1(x) = x_1$$

It is well known, see for example [**146**], that under particular values of $\sigma, \rho, \beta$, the Lorenz system has a chaotic behavior. In view of this fact, high-dimensional ($n \geq 3$) bilinear systems may also exhibit complex dynamics, which is not present in linear systems.

Due to their characteristics, bilinear systems constitute suitable description of diverse processes in which the input exerts a multiplicative action on the state, among them:

(i) Biological [**133**], chemical or physical *birth-death processes*, such as population dynamics, molecular reactions and neutron population in fission processes [**130**].
(ii) Dynamics of cars with frictional braking systems [**130**].
(iii) Physiological and ecological processes [**130**].
(iv) Economic systems [**138**].

In most of these processes the state is regulated by the amount of nutrients, enzymes, reactants or particles in biological, chemical or particle systems, respectively, or by the interest rates in the case of economic systems. Additional references to different applications of bilinear systems are found in [**137, 134, 136**] and in the recent survey [**132**].

Another reason to consider bilinear systems is that they often result from the linearization of certain nonlinear systems with respect to the state only [**120, 132**].

## 1.4. Important Features of Strongly Nonlinear Systems with Drift and Difficulties Arising in their Control

The main difficulty of steering systems with drift arises from the fact that, in the most general case of non-recurrent or unstable drift, the system motion along the drift vector field $f_0$ needs to be counteracted by enforcing system motions along adequately chosen Lie bracket vector fields[1] in the system's underlying controllability Lie algebra $L(\mathcal{F})$. Such indirect system motions are complex to design for and can be achieved only through either time-varying open-loop controls or discontinuous

[1]This type of motions are sometimes referred to as *Lie bracket motions*; see explanation in Appendix B, p. 234.

state feedback. To understand why this is so, the main features that characterize the class of strongly nonlinear systems considered here are explained next.

Although the first three properties could be regarded as assumptions about the system, it is important here to stress them as features since they imply that feedback techniques conceived for linear systems cannot be applied, and in this sense, the system is *less linear* than other nonlinear systems which are linearizable by state feedback transformations or simply around some operating points. Together with the first three features, the fourth feature is central in the qualitative characterization of the degree of *strong nonlinearity* the system may exhibit, as it implies that some systems may only have stable closed-loops that are not everywhere smooth, and hence, not locally linear in some neighborhoods of the state space.

The main features of the class of systems considered here are:

(i) **Deficiency in the Number of Controls with Respect to the Dimension of the Space.**

The number $m$ of control variables $u_i$, $i = 1, \ldots, m$, is smaller than the number $n$ of state variables $x_i$, $i = 1, \ldots, n$.

(ii) **Unstable Drift Term.**

The drift term $f_0$ is unstable in the sense that the trajectories of the uncontrolled system obtained by setting all $u_i = 0$ are unstable. It is assumed that at least the origin is an equilibrium point of (1.1) with zero control effort, i.e. $f(0, 0) = 0$.

(iii) **Uncontrollable Linearization and Not Feedback Linearizable.**

The linearization of (1.1) does not satisfy Kalman's controllability condition and there does not exist any linearizing state feedback.

(iv) **Failure to Satisfy Brockett's Necessary Condition for Smooth Static Stabilizability.**

A strongly nonlinear system of the form (1.1) does not satisfy Brockett's necessary condition for the existence of continuously differentiable or even continuous time invariant control laws.

**Theorem 1.1. - Brockett's Necessary Condition for Smooth Static Stabilizability, [15].** *Consider the control system* $\dot{x} = f(x,u)$ *with* $f(x_e, 0) = 0$ *and* $f : x \in \mathbb{R}^n \times u \in \mathbb{R}^m \rightarrow f(x,u) \in \mathbb{R}^n$ *continuously differentiable (denoted by* $f \in \mathcal{C}^1$*) in the neighborhood of* $(x_e, 0)$*, then the system is smoothly stabilizable (in the sense that there exists a feedback* $u \in \mathcal{C}^1 : \mathbb{R}^n \rightarrow \mathbb{R}^m$ *that renders* $(x_e, 0)$ *asymptotically stable), if the following conditions are satisfied:*

(a) *The linearized system does not have uncontrollable modes associated with eigenvalues with positive real part.*

(b) *The system is locally controllable in the sense that there exists a neighborhood* $N$ *of* $(x_e, 0)$ *such that for each* $x_0 \in N$ *there exists a control* $u_{x_0} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ *defined on* $[0, \infty)$ *such that this control steers the solution of* $\dot{x} = f(x, u_{x_0})$ *from* $x(0) = x_0$ *at* $t = 0$ *to* $x(\infty) = x_e$ *at* $t = \infty$*.*

(c) *The image of the mapping* $f$ *is onto an open set containing* 0*.*

The chances that system (1.1) does not satisfy Brockett's condition are certainly increased by the previous feature, (iii), as it implies that the system is invariant with respect to diffeomorphic state transformations, otherwise there would exist a coordinate change that would bring the system to a form that can be stabilized by a smooth (linear) feedback. The failure to satisfy Brockett's condition is in fact a common characteristic exhibited by many systems, among them, driftless systems with $m < n$, some underactuated systems and most nonholonomic systems. Note that condition (c) of Theorem 1.1 implies that system (1.1) is not smoothly stabilizable if the vector space spanned by set of vector fields $\{f_0, f_1, \ldots, f_m\}$ (notice $f_0$ is included) is of dimension smaller than $n$; see [**15**].

(v) **Non-Reversibility of the Trajectories of Systems with Drift.**

The trajectories of systems with drift run backwards in time do not correspond to trajectories of the system (1.1). This fact becomes clear by observing that the backward trajectory can be formally obtained by *reversing the time* with the variable change $t = -s$ applied to (1.1), which implies that the state of the backward system is given by

$$x(T) = x(0) + \int_0^T -f_0(x) - \sum_{i=1}^m f_i(x)u_i ds \tag{1.6}$$

While the latter may be considered as the forward trajectory of a system without drift and sign reversed controls, changing the sign of the right-hand side of (1.1) is not actually possible since there is no control multiplying the drift term. Thus, unlike drift-free systems, – for which every trajectory run backwards is also a trajectory of the system –, system (1.1) cannot be made to behave as if *running backward in time*.

Furthermore, it is easy to see that if $f_0(x) \neq 0$ then the simple control $v(t) = -u(t-T)$ applied on an interval of time of length $T$ does not reverse the action of the control $u(t)$ applied for the same amount of time over a previous interval. By "not being able to reverse the action" it is meant that the resulting trajectory is such that $x(2T-s) \neq x(s)$, $s \in [0,T]$, and not even such that $x(2T) = x(0)$, i.e. the control $v(t)$ does not undo the action $u(t)$ taking the state $x(T)$ back to $x(0)$ as would occur if $f_0(x) = 0$. To see this, consider the states of (1.1) at time $T$ and $2T$, given by:

$$\begin{aligned} x(T) &= x(0) + \int_0^T f_0(x) + \sum_{i=1}^m f_i(x)u_i(t)dt \\ x(2T) &= x(T) + \int_T^{2T} f_0(x) + \sum_{i=1}^m f_i(x)v_i(t)dt \end{aligned}$$

Using the above equations,

$$\begin{aligned} x(2T) - x(0) &= \int_0^T f_0(x) + \sum_{i=1}^m f_i(x)u_i(t)dt + \int_T^{2T} f_0(x) + \sum_{i=1}^m f_i(x)v_i(t)dt \\ &= \int_0^{2T} f_0(x)dt \end{aligned}$$

The last equation follows after applying the change of variables $t = s + T$ to the integral over the interval $[T, 2T]$ and because $v(s+T) = -u(s)$. The integral of the drift term $f_0(x)$ is in general nonzero, and hence, $x(2T) - x(0) \neq 0$. The practical implication of the presence of a drift term is that tracking any desired trajectory or achieving a specific target state will be more complicated, as compared to steering any controllable driftless system, and might not even be possible.

(vi) **Lack of Controllability Results for Systems with Drift.**

As opposed to driftless systems, the controllability of systems of the form (1.1) is in general hard to establish. This is mainly due to the limitations of Chow's theorem, (see Appendix B on p. 234 or [**14**]), as it does not distinguish between positive and negative time, i.e. the submanifold on which the solutions to (1.1) are defined may include points which can *only* be reached by passing backwards along the drift vector field $f_0$. This means that while the reachable set from a given $x_0$ will always be contained in certain manifold, in general, it will be a proper subset of the manifold. Due to the lack of a general criteria to establish the controllability of system $\Sigma$, it will be assumed that $\Sigma$ is *strongly controllable*. A precise definition of the type of controllability assumed here and further comments on the existing criteria to establish some of the controllability properties of (1.1) are given in Appendix B.

## 1.5. Existing Approaches Relevant to the Control of Nonlinear Systems with Drift

The general nonlinear stabilization problem has been considered by many authors, see for instance [**7**], the survey work by Bacciotti [**11**], or the earlier survey by Sontag [**35**] and the references therein.

Due to the aforementioned difficulties in the control of strongly nonlinear systems, classical nonlinear control approaches, such as feedback linearization [**2**] and dynamic inversion [**97**] cannot be applied. Traditional methods based on the construction of a feedback control which decreases certain time-invariant smooth Lyapunov function also fail, as smoothness or analiticity of the feedback law is often required; see for instance [**73**] and references therein. The conceptual appeal and the widespread use of the Lyapunov approach has also encouraged the consideration of non-smooth Lyapunov functions, see [**33**] for some related results, although no constructive stabilization approaches are presented. Moreover, if discontinuous controls are considered, mathematical difficulties will arise since the right-hand side of (1.1) will not be differentiable everywhere, thus the complex techniques developed in [**142**] for the analysis of differential equations with discontinuous right-hand side and other results from differential inclusions [**34**] must be employed. However, these techniques are of limited use in most practical cases.

As opposed to the stabilization of nonlinear driftless systems, fewer control strategies have been proposed for systems with drift that cannot be smoothly stabilized. Driftless representations often arise in the study of nonholonomic systems, which are rather common in practical applications as these are characterized by non-integrable velocity constraints, such as in cars, wheeled robots and rigid bodies in space. Approaches to the stabilization of nonholonomic systems concentrate on either discontinuous or time-varying feedback laws [**180, 49**], as they fail to satisfy Brockett's condition for smooth stabilizability. Although there exist numerous open-loop control strategies for motion planning of nonholonomic systems, and a similar number of feedback laws which make the closed-loop system stable (see [**50**] and the comprehensive survey [**49**]), most methods exploit model or representation specific properties, e.g. [**54, 56**], or consider simple kinematics models, which do not present all the difficulties of a complete dynamic model with drift. Thus, the ideas and approaches applied to the control of driftless systems cannot in general be modified to yield feedback laws that stabilize systems with drift. Some exceptions are those approaches that employ Lie algebraic techniques and that do not rely on the particular model representation. Lie algebraic approaches to stabilization often result in feedback design methodologies which apply to a wider or more general class of systems, see for instance the approach proposed in [**44**].

The few pertinent results which directly address the stabilization problem of systems with drift that fail to satisfy Brockett's condition are summarized in the next sections. As earlier mentioned, such systems can only be stabilized by either time-varying, discontinuous feedback laws or a combination of both. However, most of the approaches consider very specific systems and their particular properties, and therefore they do not provide general control synthesis procedures applicable to the stabilization of (1.1). Rather than classifying the control strategies as being either of time-varying or discontinuous type, it will be more convenient to distinguish the stabilization approaches as:

- Model specific methods.

- General methods for the synthesis of time-varying state feedback laws.
- General methods for the synthesis of discontinuous state feedback laws.

### 1.5.1. Model Specific Methods

The controllability and stabilizability properties of nonholonomic underactuated mechanical systems with drift is considered in [**59, 88**]. Paper [**59**] studies dynamical systems of second order in the configuration vector with nonintegrable velocity constraints, while [**88**] considers the same class of systems, but with nonintegrable acceleration relations. These papers do not present any constructive approach, though. The controllability of a particular third order nonlinear system with time-dependent drift and an open-loop (time-varying) control that steers the system between to arbitrary states is addressed in [**71**]. Path planning for nonholonomic systems with drift in the class of Lagrangian systems with a cyclic coordinate is considered in [**66**] using open-loop bang-bang controls, and an exponentially stabilizing feedback controller is derived in [**65, 67**]. A feedback control law based on time-varying coordinate and input transformations is proposed for the same class of Lagrangian systems in [**70**] and applied to a simple free-flying space robot. Stabilization and tracking control laws for a model with drift of an underactuated surface vessel are proposed in [**85**] and the references therein. The control strategy in [**85**] uses a backstepping approach and exploits the similarities of the model equations to those of systems in chained form to develop a continuous feedback law for trajectory tracking. A numerical procedure for feedback stabilization of nonlinear affine systems with drift is considered in [**90**], and employs a dynamic feedback control that may be regarded as a special class of model predictive control. Relying on the properties of homogeneous systems, combined with time-averaging techniques, a continuous, not everywhere differentiable, time-varying feedback law that locally exponentially stabilizes the attitude of a rigid spacecraft is derived in [**84**]. The control methodology developed in [**84**] is also applied in [**86**] to the stabilization of an underactuated autonomous underwater vehicle. Another approach that takes advantage of model specific properties is presented in [**89**]. The latter is intended for the stabilization of multibody systems in space and uses a composition of open-loop time-varying controls and discontinuous feedback inputs.

An algorithm for the construction of motion primitives, i.e. basic maneuvers, for motion planning of underactuated vehicles that can be modelled as Lagrangian systems on Lie groups, such as airships, hovercrafts, satellites, surface vessels and underwater vehicles is presented in [**61, 62**]. The motion primitives are designed using small-amplitude periodic inputs and can be used for exponential stabilization to a point.

### 1.5.2. General Methods for the Synthesis of Time-Varying State Feedback Laws

The starting point for the systematic study of time-varying stabilization was paper [**55**], which presents an approach relying on ideas that might be traced back to the work in [**32**]. With the exception of [**74**], all the results concerning general methods for the synthesis of time-varying feedback controls focus on driftless systems [**42, 43, 17, 47, 46, 48**]. In [**43**] it was shown that driftless affine systems can be globally asymptotically stabilized by periodic time-varying state feedback laws which are continuous and even smooth. Later, results in [**17**] confirmed that *most* non-smoothly stabilizable systems that are small-time locally controllable (STLC) can in fact be stabilized by continuous time-varying controls. Explicit design procedures for the design of stabilizing laws for driftless affine systems were first proposed by Coron and Pomet in [**42**] and [**48**]. Their approach is based on Pomet's idea of constructing a control which is composed of a periodic time-varying control that critically stabilizes the system, combined with an asymptotically stabilizing "correction" feedback term. In [**46**] these ideas are extended to the stabilization of a class of nonaffine driftless systems using similar techniques relying on bounded feedback design and Lyapunov stability theory. More recently, Morin *et al.*, presented in [**47**] another design method that yields explicit homogeneous time-varying control laws. However, even if the method in [**47**] does not require the solution of a linear partial differential equation (PDE) or the expression for the flow of any vector field that must be calculated before hand, either analytically or numerically, the derivation of control laws by this approach is also very complicated as in the methods of [**42, 48**]. Although the approaches in [**42, 48, 46, 47**] are applicable to rather large classes of controllable driftless systems, they all share as drawback the complexity of the control synthesis procedure or of the control law itself.

Techniques similar to those employed in [**84**] (see Section 1.5.1), relying on the properties of homogeneous systems and on time-averaging methods, are also employed by M'Closkey and Morin in [**74**] to derive exponentially stabilizing time-varying homogeneous feedback laws for a relatively general class of systems of the form (1.1).

Methods that exploit the representation of the system as a system on a Lie group have also been successfully developed. General averaging techniques on Lie groups have been applied to attitude control in [**72**], and a relatively general Lie algebraic approach for the synthesis of smooth time-varying feedback laws, that requires the solution of a trajectory interception problem (see Section 2.5, p. 35) for the flows of the original system and its Lie algebraic extension in terms of the logarithmic coordinates on the associated Lie group, is presented in [**75**]. The latter approach employs sinusoidal controls, however it is not limited to smooth time-varying continuous laws, as the trajectory interception problem can also be solved using discontinuous control laws (this class of controls is not considered in [**75**], though).

### 1.5.3. General Methods for the Synthesis of Discontinuous State Feedback Laws

The approaches providing the most general feedback design methodologies mainly rely on Lie algebraic considerations and the generation of Lie bracket motions; see [**68, 75**]. The paper by Hermes, [**68**], was the first to present systematic procedure for the construction of discontinuous stabilizing controls based on a Lie algebraic approach and the differential geometric formalism. The method in [**68**] requires some modifications for its use in practical applications and has as main drawback the exceedingly complicated feedback synthesis procedure. Nonetheless, it has been successfully applied to the stabilization of the attitude and angular velocity of an underactuated spacecraft in [**64**].

A general approach to motion planning of general driftless affine systems was proposed by Lafferriere in [**44**]. The method in [**44**] is relevant to some of the approaches proposed in this thesis as they draw on similar ideas to develop a systematic procedure for the synthesis of piece-wise constant controls that stabilize (1.1).

Sliding mode control techniques also result in discontinuous controls; see the reviews in [**94, 95, 96**]. The method in [**93**] proposes an explicit feedback law for single-input systems in $\mathbb{R}^n$, however it assumes the existence of a control law such that the system has a stable manifold of dimension

$n-1$. In general this assumption is hard to satisfy, and therefore the method can rarely be applied for the stabilization of (1.1).

### 1.5.4. Drawbacks of the Previously Existing Methods

(i) Although the methods that rely on specific properties of the model yield relatively simple control laws, an evident disadvantage of these methods is that they do not provide sufficiently general procedures for the stabilization of (1.1).

(ii) Most of the approaches, either for time-varying or discontinuous feedback synthesis, consider driftless systems.

(iii) Only two approaches that generate time-varying laws are directly concerned with the stabilization of systems with drift: [**74, 75**]. The disadvantage of [**74**] is mainly due to the fact that the vector fields must be homogeneous and therefore the method is restricted to a subclass of systems (1.1). On the other hand, the Lie algebraic approach described in [**75**] requires an analytic solution of a special trajectory interception problem stated for the original system and its Lie algebraic extension. The condition guaranteeing the interception of trajectories is difficult to satisfy exactly as it is posed in terms of an equality end-point constraint. The computations involved might be exceedingly complicated, especially for large dimensional systems.

(iv) The method proposed in [**68**] is the first method to systematically employ Lie algebraic techniques for the development of control laws. Since then, many approaches conceived specifically for the control of systems which are not smoothly stabilizable have been derived from the basic ideas introduced in [**68**]. However, most of these methods share the drawback of [**68**], in that the motions along the Lie brackets of the system are very difficult to generate. Most approaches become unappealing for practical purposes, especially due to the fact that the complexity of the computations grows rapidly for systems evolving in higher dimensional spaces.

(v) The development of controllers using sliding mode techniques presents two difficulties. From a theoretical point of view, the existence of solutions is harder to guarantee due to the fact that techniques such as those developed in [**142**] for the analysis of differential equations with discontinuous right-hand sides cannot always be applied. From a practical standpoint, these controllers may cause chattering, as the state trajectory *switches* about the sliding manifold. In some cases the undesirable chattering may be smoothed, and even if in certain applications, (e.g. electronic applications such as DC–DC conversion), the laws may be natural to the behaviour the devices can generate, other alternatives, such as the use of piece-wise continuous controls, may in general be preferable since they will result in relatively smoother trajectories.

## 1.6. Research Objective

In the light of the aforementioned difficulties arising in the control of strongly nonlinear systems of the form (1.1), and the disadvantages of the existing methods discussed in the previous section, the main objective of this research is to explore novel and implementable feedback synthesis approaches for the stabilization of such systems to an equilibrium point. Attention is placed on the development of:

(i) Algorithms for the construction of time-varying stabilizing controls for a wide class of systems with drift based on Coron's approach [**42, 43, 48**] of critical orbits for driftless systems.

(ii) Computationally feasible algorithms for the construction of time-varying discontinuous feedback controls for systems with drift.

(iii) Algorithms for the stabilization of bilinear systems with unstable drift.

## 1.7. Contributions of this Thesis

In relation to the research objectives stated above, the contributions of this dissertation are summarized as follows:

(i) The development and analysis of different and novel Lie algebraic based approaches to the synthesis of stabilizing feedback laws for general systems with drift of the type (1.1):

  (a) A continuous time-varying stabilization feedback approach [**77**].

   (b) Two computationally feasible discontinuous time-varying feedback approaches [**79, 80, 82, 83**].

(ii) The development and study of two approaches for the stabilization of bilinear systems:

   (a) A continuous time-varying stabilization feedback approach [**76**].

   (b) An approach relying on the discontinuous time-varying approach for general systems of the form (1.1) and based on the idea of steering the system to a stable manifold [**78**].

(iii) The development of a software package for symbolic computations involving Lie algebraic expressions [**175**].

A brief description of the main approaches is presented next.

### 1.7.1. Continuous Time-Varying Stabilization Feedback Approach

In this approach the proposed control law is a composition of a periodic time-varying control, and an asymptotically stabilizing feedback "correction" term. The time-periodic control is first obtained through a solution of an open-loop, finite horizon control problem on the associated Lie group which is posed as a trajectory interception problem in the logarithmic coordinates of flows, and its purpose is to generate critically stable trajectories for the open-loop system. The correction term is calculated to be a control which decreases a Lyapunov function whose level sets contain the periodic orbits of the system stabilized by the time-periodic feedback.

### 1.7.2. Computationally Feasible Discontinuous Time-Varying Feedback Approach

The proposed feedback law is computed on-line in terms of the repeated solution to an open-loop control problem on the system's associated Lie group, such that the trajectories decrease an arbitrary Lyapunov function, thus achieving asymptotic stability. The open-loop control is constructed as a sequence of constant inputs. Two approaches to the synthesis of the piece-wise constant controls are presented. In both approaches the values of the sequence of piece-wise constant controls are calculated as the solution to a non-linear programming problem. In the first approach, the formulation of the non-linear programming problem results from posing the original control problem in terms of a relaxed control problem (satisficing problem) in the logarithmic coordinates that parametrize the flows on the Lie group associated with the system. In the second one, the formulation of the nonlinear programming problem results from the direct application of the Campbell-Baker-Hausdorff formula for composition of flows.

## 1.8. Thesis Outline

The thesis is organized as follows:

- **Chapter 2: Basic Assumptions, Facts and Preliminary Results.**

  This chapter introduces the main hypotheses and presents some preliminary results essential in the development of the proposed stabilization approaches. Additional mathematical background concerning basic notions of differential geometry, Lie theory, as well as other terminology, are found in Appendix A and Appendix B. The latter motivates the differential geometric approach to the analysis and control of nonlinear systems by means of a geometric interpretation of the Lie product. It also presents some existing results on the controllability of nonlinear systems and a brief overview of the relevant literature.

- **Chapter 3: Continuous Time-Varying Stabilization Feedback Approach.**

  In this chapter a novel approach to the synthesis of time-varying feedback laws that stabilize nilpotent systems with drift is presented, [**77**]. The method partially draws on the ideas of Coron and Pomet, see [**42, 43, 48**], who constructed time-periodic stabilizing controls for systems without drift. The control strategy is based on the combined application of a critically stabilizing control, whose construction relies on Lie algebraic techniques and the solution of an open-loop control problem on a Lie group, and a control "correction" term that provides asymptotic stabilization, whose construction employs standard Lyapunov techniques. The feedback control strategy is shown to be applicable to strongly nonlinear systems that have a nilpotent Lie algebra, and to yield global asymptotic stabilization to a set point under reasonable assumptions. The effectiveness of the approach is demonstrated with an example, previously presented in [**77**].

- **Chapter 4: Discontinuous Time-Varying Feedback Approaches.**

  A novel and computationally tractable approach for the construction of stabilizing discontinuous feedback controls based on Lie algebraic and standard Lyapunov techniques is presented, [**79, 80, 82, 83**]. Compared to the approach proposed in the previous chapter, this approach is of reduced computational complexity, and furthermore it applies to a larger class of systems which do not need to be nilpotent.

  The proposed control law comprises two modes. In one mode the control is a smooth state feedback $u(x)$ that guarantees an instantaneous decrease of a chosen control Lyapunov function $V(x)$. This mode is applied whenever there exists a control $u(x)$ such that $\dot{V}(x, u(x)) < 0$. In the other mode, an open-loop piece-wise constant control $\bar{u}(x,t)$ is applied to achieve a decreases of the control Lyapunov function periodically every finite period of time $T > 0$, i.e. such that $V(x(t+T)) - V(x(t)) < 0$ for any $t \in \mathbb{R}$. The synthesis of $u(x)$ is based on the standard Lyapunov approach, thus the emphasis is put on the construction of $\bar{u}(x,t)$ by means of Lie algebraic techniques.

  Two approaches to the synthesis of the Lie algebraic control $\bar{u}$ are proposed. In both of them the control is composed of a sequence of constant controls whose values are calculated as the solution to a non-linear programming problem. In the first approach, the non-linear programming problem is formulated by posing the original control problem in terms of a relaxed control problem in the associated logarithmic coordinates. The formulation of the non-linear programming problem results, in the second approach, from the direct application of the Campbell-Baker-Hausdorff formula for composition of flows.

  Conditions under which the feedback control strategy renders the equilibrium point of the system globally asymptotically stable are given. The proposed methodology was tested successfully in the stabilization of nonholonomic systems: the unicycle and the front-wheel drive car. The effectiveness of the approach to the stabilization of high dimensional systems is also demonstrated in the attitude and angular velocity stabilization of a satellite in actuator failure mode, [**79, 80, 82, 83**].

- **Chapter 5: Stabilization of Bilinear Systems with Unstable Drift.**

  Two approaches to stabilization of bilinear systems with unstable drift are discussed in this chapter. Both methods make use of the Lie algebraic extension of the system.

  The first method, [**76**], considers the construction of a time-invariant feedback for the extended system, which is a relatively simple task under reasonable assumptions. The original system controls are then obtained as a solution to an open-loop, finite horizon, control problem posed in terms of a finite horizon interception problem of the logarithmic coordinates for flows [**44**]. The open-loop controls so generated are such that the trajectories of the open-loop system intersect those of the controlled extended system after a finite time $T$, independently of their common initial condition. Thus, the "average motion" of the original system corresponds to the motion of the controlled extended system. The speed of convergence of the system trajectory to the desired terminal point is dictated by the static feedback for the extended system.

  The second approach, [**78**], comprises two phases: the *reaching phase* and the *sliding phase*. In the reaching phase the state of the system is steered to a selected stable manifold by employing a suitably designed control Lyapunov function in conjunction with a discontinuous Lie algebraic control. The latter is necessary when there do not exist controls which generate instantaneous velocities decreasing the Lyapunov function. The Lie algebraic control is constructed using the second method proposed in Chapter 4. Conditions are given under which the constructed feedback control renders the stable manifold globally attractive and attainable in finite time. Once the set of stable manifolds is reached, the control is switched to its sliding phase whose task is to confine the motion of the closed-loop system to the latter set, making it invariant under limited external disturbances. Two examples corresponding to different dimension of the stable manifolds are presented to demonstrate the effectiveness of the approach.

- **Chapter 6: A Software Package for Symbolic Lie Algebraic Computations.**

  The computationally feasible approaches proposed in this thesis necessitated the development of a set of software tools for symbolic manipulation of expressions with Lie brackets.

To this end, a software package has been implemented in Maple. The module is called Lie Tools Package (LTP) [**174**], [P7], and among other functions it enables the following automated Lie algebraic manipulations:

- Construction of Philip Hall bases.
- Simplification of any Lie bracket expression.
- Composition of flows via the Campbell-Baker-Hausdorff formula.
- Set up of the logarithmic-coordinates equation.

This chapter discusses the capabilities and features of LTP, and presents some application examples that illustrate the usefulness of the package.

- **Chapter 7: Conclusions and Future Research.**

  The last chapter concludes the thesis with a brief review of main contributions of the research presented in the preceding chapters. Some general remarks concerning the advantages and potential of the proposed approaches are presented. Suggestions on issues for future research are given.

Reference material, which will be cited when needed, is included in several appendices for the reader's convenience:

- Appendix A: Notation and Mathematical Background.
- Appendix B: Controllability of Systems with Drift.
- Appendix C: Useful Theorems and Other Results.

An index of concepts has been included on p. 248 to facilitate finding definitions and symbols, the latter under the keyword *symbols*.

## 1.9. Originality of the Research Contributions

The proposed approaches constitute an original contribution to the stabilization of (1.1) in that:

- The synthesis method for continuous time-varying stabilizing controls, [**77, 81**], is general and applicable to a large class of nilpotent systems with drift which do not lend themselves to controllable linearization (be it through state-feedback transformations, or else simply around some operating points).

- The discontinuous time-varying stabilizing feedback approaches, [**79, 80, 82, 83**], are computationally feasible, and exploit a combination of Lie algebraic and optimization techniques in a novel way.
- The Lie algebraic approaches to the synthesis of stabilizing feedback controls for homogeneous bilinear systems are completely new in the sense that, unlike existing methods, they are applicable to systems with unstable drift which cannot be stabilized by any constant controls, [**76, 78**].
- The software package for symbolic Lie algebraic computations constitutes a novel tool, [**175**]. The existing software for Lie algebraic manipulations are very specialized, e.g. LiE [**171**] and Maple's *liesymm package* [**191**], and do not provide any of the functionality mentioned above.

Sufficient conditions for the existence of the proposed control laws are also given.

# CHAPTER 2

## Basic Assumptions, Facts and Preliminary Results

In this chapter, the main hypotheses concerning system $\Sigma$ of equation (1.1) are stated together with basic definitions and preliminary results essential to the development of the stabilization approaches proposed in the next chapters. Specific assumptions pertinent only to each particular approach will be made in the corresponding chapter.

The familiarity of the reader with the fundamental notions of differential geometry and Lie theory is assumed. For convenience of the reader, the notation and additional mathematical background is included in Appendix A on p. 194. Although, the notation and terminology is as standard as possible the reader is advised to see Appendix A before reading the next sections. An index of concepts is included on p. 248 to facilitate finding definitions and symbols; the latter collected under the keyword *symbols*.

### 2.1. Basic Assumptions

Let $\mathcal{F} \stackrel{\text{def}}{=} \{f_0, \ldots, f_m\}$ be the family of vector fields of system $\Sigma$ defined on $\mathbb{R}^n$. Denote by $L(\mathcal{F})$ the Lie algebra of vector fields generated by $\mathcal{F}$ and by $L_x(\mathcal{F}) \stackrel{\text{def}}{=} \{f(x) \mid f \in L(\mathcal{F})\} \subset \mathbb{R}^n$ the set of vectors resulting from the evaluation of $L(\mathcal{F})$ at $x$. All vector fields $f \in \mathcal{F}$ are assumed to be real, analytic, and complete , i.e. any vector field $f$ is assumed to generate a globally defined one-parameter group of transformations denoted by $\exp(tf)$ acting on $\mathbb{R}^n$; (with the manifold being $\mathbb{R}^n$ this means no finite escape times in forward or backward time). This implies that for all $t_1, t_2 \in \mathbb{R}$, $\exp(t_1 f) \circ \exp(t_2 f) = \exp((t_1 + t_2)f)$, and for all $x \in \mathbb{R}^n$, $x(t) = \exp(tf)x$ satisfies the differential equation $\dot{x} = f$ with initial condition $x(0) = x$. It is a well known fact, see [**158**, p. 95], that if all generators in $\mathcal{F}$ are analytic and complete then all vector fields in $L(\mathcal{F})$ are analytic and complete.

Solutions to system $\Sigma$, starting from $x(0) = x$ and resulting from the application of a control $u$, are denoted by $x(t, x, u)$, $t \geq 0$. The family of piece-wise constant functions, continuous from the right and defined on $\mathbb{R}^m$, is denoted by $\mathcal{P}^m$.

For a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ and a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, $L_f V(x) \stackrel{\text{def}}{=} \nabla V f(x)$ denotes the Lie derivative of $V$ along $f(x)$.

The following hypotheses are assumed to hold with respect to system $\Sigma$.

- H1. The vector fields $f_0, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}^n$ are real, analytic, complete, and linearly independent, with $f_0(0) = 0$, and generate a nilpotent Lie algebra of vector fields $L(\mathcal{F})$, such that its dimension is $\dim L(\mathcal{F}) = r \geq n + 1$.
- H2. The system $\Sigma$ is strongly controllable, i.e. for any $T > 0$ and any two points $x_0, x_f \in \mathbb{R}^n$, $x_f$ is reachable from $x_0$ by some control $u \in \mathcal{P}^m$ of $\Sigma$ in time not exceeding $T$; i.e. there exists a control $u \in \mathcal{P}^m$ and a time $t \leq T$ such that $x(t, x_0, u) = x_f$.

Remark 2.1.

*The analyticity assumption of $f \in \mathcal{F}$ guarantees that all the information about the system is actually contained in $f$ and its derivatives (of all orders) at a given point $x \in \mathbb{R}^n$, see [**36**]. It is a standard assumption as it implies that all the Lie brackets exist and it cannot be relaxed since the basic facts required to prove the results in this thesis also necessitate this assumption.*

*The controllability with piece-wise constant controls is not excessively restrictive as it may seem. In fact it is equivalent to the basic controllability requirement using controls in a rather general class of continuous controls. This is a consequence of Theorem 1 in [**37**] which implies that system $\Sigma$ is completely controllable with continuous controls if and only if $\Sigma$ is completely controllable with piece-wise constant controls.*

A well known consequence of strong controllability is that the condition for accessibility is satisfied, namely: $\operatorname{span} L_x(\mathcal{F}) = \mathbb{R}^n$ for all $x \in \mathbb{R}^n$. In fact, the accessibility condition is also referred to as weak controllability condition, see [**19**], and clearly, if the system is strongly controllable it must also be weakly controllable. Therefore, for the feedback synthesis methods presented in this thesis to be applicable, system $\Sigma$ must obviously satisfy the accessibility condition. However, in contrast to driftless systems, the accessibility of system $\Sigma$ in (1.1) does not in general imply its strong controllability.

## 2.2. Lie Algebraic Extension of the Original System

For steering purposes, it is useful to consider the Lie algebraic extension of system $\Sigma$, denoted by $\Sigma^e$, defined as

$$\Sigma^e: \qquad \dot{x} = g_0(x) + \sum_{i=1}^{r-1} g_i(x) v_i \stackrel{\text{def}}{=} g^v(x) \tag{2.1}$$

where $v \stackrel{\text{def}}{=} [v_1, \ldots, v_{r-1}]$ are referred to as *extended controls*. Solutions to $\Sigma^e$, starting from a point $x(0) = x$ and under the action of a control $v$, are denoted by $x^e(t, x, v)$, $t \geq 0$.

Concerning the system $\Sigma^e$, it is assumed that:

H3.a. The vector fields in $\mathcal{G} \stackrel{\text{def}}{=} \{g_0, \ldots, g_{r-1}\}$ define a basis for $L(\mathcal{F})$.

H3.b. For any $x$ in a sufficiently large neighbourhood of the origin $B(0, R)$, the set of vectors $g_i(x)$, $i = 1, \ldots, r-1$, of the extended system $\Sigma^e$ contains a basis for $L_x(\mathcal{F})$.

Remark 2.2.

- *Due to assumption H1, all the vector fields $f \in L(\mathcal{F})$ generate globally defined one-parameter groups of transformations* $\exp(tf)$. *This holds because it is well known that, see* [**158**, p. 95], *if all generators in $\mathcal{F}$ are analytic and complete then all vector fields in $L(\mathcal{F})$ are also analytic and complete.*
- *The assumption about $\dim L(\mathcal{F}) = r \geq n+1$ is convenient in view of the definition of the extended system $\Sigma^e$. It is worthwhile pointing out that this assumption, as well as hypothesis H3.b, have been introduced primarily to make the control construction and its analysis more transparent to the reader. Hypotheses H1–H3 jointly imply that the vectors $g_i(x)$, $i = 1, \ldots, r-1$, span $\mathbb{R}^n$ for any $x \in B(0, R)$.*
- *By construction, both systems $\Sigma$ and $\Sigma^e$ have the same underlying algebra of vector fields $L(\mathcal{F})$. Additionally, system $\Sigma^e$ not only inherits the strong controllability property of $\Sigma$,*

*but is in fact instantaneously controllable in any direction of the state space. It is hence much easier to steer than $\Sigma$ whose motion along some directions has to be generated by time-varying controls, or else piece-wise constant switching controls.*

## 2.3. Basic Facts and Preliminary Results

Let $\mathcal{R}_{\mathcal{F}}(T,x)$ denote the reachable set of $\Sigma$ at time $T$ from $x$ by piece-wise constant controls; (for a definition of reachable set, see Definition A.56, p. 228 of Appendix A).

It is helpful to introduce the following subsets of *diff*($\mathbb{R}^n$), the group (under composition) of diffeomorphisms on $\mathbb{R}^n$.

$$G \stackrel{\text{def}}{=} \{\exp(t_1 f^{u_{(1)}}) \circ \cdots \circ \exp(t_k f^{u_{(k)}}) \mid u_{(i)} \in \mathbb{R}^m;\ t_i \in \mathbb{R};\ k \in \mathbb{N}\} \tag{2.2}$$

$$G_T \stackrel{\text{def}}{=} \{\exp(t_1 f^{u_{(1)}}) \circ \cdots \circ \exp(t_k f^{u_{(k)}}) \mid u_{(i)} \in \mathbb{R}^m;\ t_i \geq 0;\ \textstyle\sum_{i=1}^k t_i = T;\ k \in \mathbb{N}\} \tag{2.3}$$

where $f^{u_{(i)}} \stackrel{\text{def}}{=} f_0 + \sum_{j=1}^m f_j\, u_{j\,(i)}$, and $u_{j\,(i)}$ are components of $u_{(i)}$. By the strong controllability hypothesis H2, if $G\,x$ and $G_T\,x$ denote the orbits through $x$ of $G$ and $G_T$, respectively, i.e. $G\,x = \{g\,x \mid g \in G\}$ and $G_T\,x = \{g\,x \mid g \in G_T\}$, then $\mathcal{R}_{\mathcal{F}}(T,x) = G_T\,x = G\,x = \mathbb{R}^n$, for any $T > 0$. The group $G \subset$ *diff*($\mathbb{R}^n$) is a subgroup of *diff*($\mathbb{R}^n$), [**20**]. Moreover, by virtue of the results by R. Palais, [**158**], $G$ can be given a structure of a Lie group with Lie algebra isomorphic to $L(\mathcal{F})$. The result of Palais [**158**], interpreted so as to apply to system $\Sigma$, is worth citing:

**Theorem 2.1** (Palais, [**158**], p. 95)**.** *Let $L(\mathcal{F})$ be a finite dimensional Lie algebra of vector fields defined on $\mathbb{R}^n$. Assume that all the generators in $\mathcal{F}$ are analytic and complete. Then there exists a unique analytic, simply connected Lie group $H$, whose underlying group is a subgroup of diff($\mathbb{R}^n$), and a unique global action of this group on $\mathbb{R}^n$ defined as an analytic mapping $\phi : H \times \mathbb{R}^n \ni (h,p) \to h(p) \in \mathbb{R}^n$ which induces an isomorphism between the Lie algebra, $L(H)$, of right invariant vector fields on $H$, and $L(\mathcal{F})$. Precisely, the isomorphism $\phi_L^+ : L(H) \to L(\mathcal{F})$ is constructed by setting:*

$$\phi_L^+(\lambda)(p) = (d\phi_p)_e(\lambda(e)) \quad \textit{for all } \lambda \in L(H),\ p \in \mathbb{R}^n \tag{2.4}$$

*where $e \in H$ is the identity element, and $(d\phi_p)_e$ is the differential of a mapping $\phi_p : H \to \mathbb{R}^n$ at identity. For any $p \in \mathbb{R}^n$, the mapping $\phi_p$ is defined by: $\phi_p(h) = h(p)$.*

This result has strong implications. It can be shown, see [**20**, Thm. 3.2.], that the isomorphism $\phi_L^+$ induces an isomorphism, denoted by $\phi_G^+$, between the groups $H$ and $G$. If for a constant control $u \in \mathbb{R}^m$, $\lambda \in L(H)$ and $f^u$ are related by $\phi_L^+(\lambda) = f^u$, where $f^u$ is the right-hand side of (1.1), then the one-parameter group $\exp(t\lambda)$ maps into the one-parameter group $\exp(tf^u)$ as follows, see [**162**, Thm. 2.10.3, p. 91]:

$$\phi_G^+(\exp(t\lambda)) = \exp(t\phi_L^+(\lambda)) = \exp(tf^u) \quad \text{for all } t \geq 0 \tag{2.5}$$

Each element $\exp(t_1 f^{u_{(1)}}) \circ \cdots \circ \exp(t_k f^{u_{(k)}}) \in G$ can thus be expressed as $\phi_G^+\{\exp(t_1\lambda_1) \circ \cdots \circ \exp(t_k\lambda_k)\}$ where: $\phi_L^+(\lambda_i) = f^{u_{(i)}}$ for all $i = 1, \ldots, k$. It follows that $H$ is given by:

$$\begin{aligned} &H = (\phi_G^+)^{-1}(G) = \\ &\quad \{\exp(t_1\lambda_1) \circ \cdots \circ \exp(t_k\lambda_k) \;|\lambda_i \in L(H),\ \phi_L^+(\lambda_i) \in L(\mathcal{F});\ t_i \in \mathbb{R};\ \ k \in \mathbb{N}\} \end{aligned} \tag{2.6}$$

The above facts, illustrated in Fig. 2.1, allow to reformulate the system $\Sigma$ as a right invariant system $\Sigma_H$ evolving on the Lie group $H$ as follows. If the right invariant vector fields $\eta_i \in L(H)$ are such that $\phi_L^+(\eta_i) = f_i$ for all $i = 0, \ldots, m$, then

$$\Sigma_H : \qquad \dot{S}(t) = \{\eta_0 + \sum_{i=1}^{m} u_i\eta_i\}S(t) \qquad \text{with } S(0) = e; \quad t \geq 0 \tag{2.7}$$

The simplified notation used in the above expression for $\Sigma_H$ deserves explanation. If $T_hH$ denotes the tangent space to $H$ at $h \in H$, then for any $\eta \in T_eH$, the expression $\eta S(t)$ denotes the image of $\eta$ under the map $dR_{S(t)} : T_eH \to T_{S(t)}H$ induced by the map of right translation by $S(t) \in H$: $h \to hS(t)$, for all $h \in H$. In this notation, if $\eta$ is represented by the curve $h(\tau) \in H$, $\tau \geq 0$, then $\eta S(t)$ is represented by the curve $h(\tau)S(t) \in H$, $\tau \geq 0$. Under the assumptions made:

PROPOSITION 2.1. *The system $\Sigma_H$ is strongly controllable on $H$ from the identity $e$, i.e. given any $T > 0$, any $h \in H$ is reachable from $e \in H$ by a trajectory of $\Sigma_H$ using a control $u \in \mathcal{P}^m$, in time not exceeding $T$. There exists a diffeomorphism between trajectories of systems $\Sigma$ and $\Sigma_H$ in the sense that: if $S(t)$, $t \in [0, T]$, is a trajectory of $\Sigma_H$ through $e$, corresponding to a concatenation $u \in \mathcal{P}^m$ of constant controls $u_{(i)} \in \mathbb{R}^m$ defined on intervals of lengths $t_i$, $i = 1, \ldots, j$, respectively,*

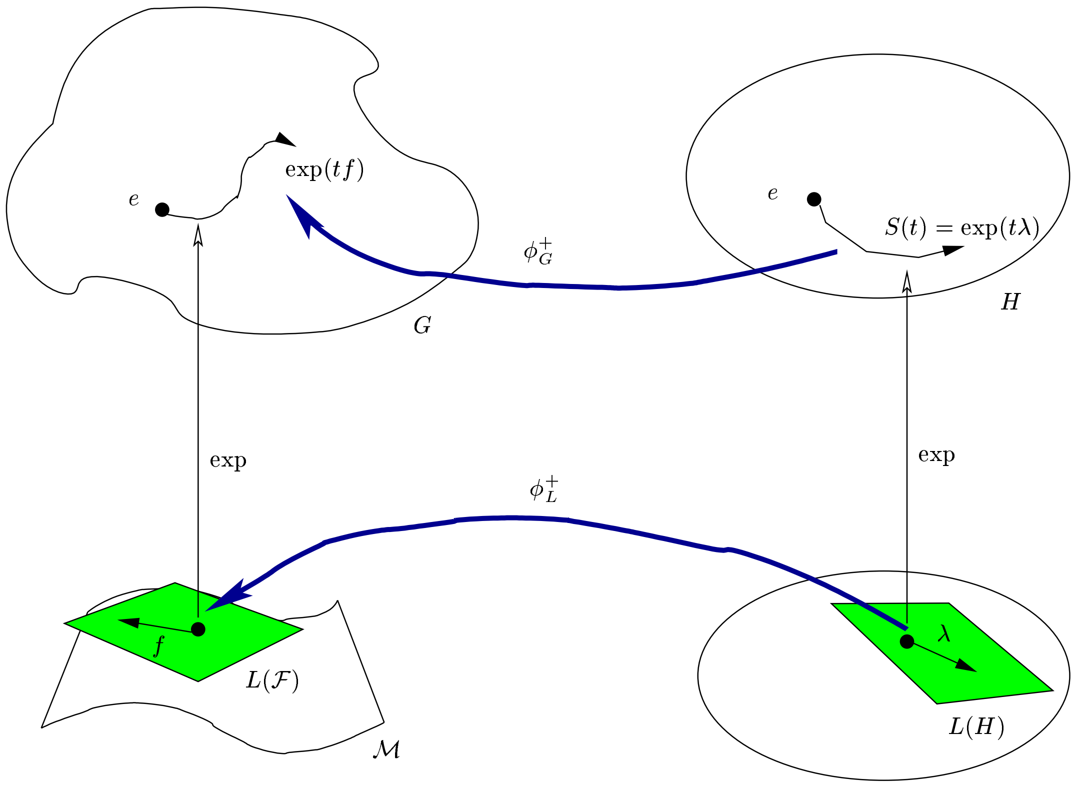



FIGURE 2.1. The exponential mapping and Palais' Theorem: This figure shows an element $f \in L(\mathcal{F})$ and its associated element $\exp(tf) \in G$, which parametrizes the solutions of the system $\dot{x} = f$. By the assumptions on $\mathcal{F}$ and $L(\mathcal{F})$, Palais' Theorem guarantees the existence of a Lie group $H$ and a unique global action that induces an isomorphism $\phi_L^+ : L(H) \to L(\mathcal{F})$, which in turn induces an isomorphism $\phi_G^+ : H \to G$ between the groups and endows $G$ with a Lie group structure.

*then $x(t, x, u) = \phi_G^+(S(t))x$, $t \in [0, T]$, is a trajectory of $\Sigma$ through the point $x$ corresponding to the same piece-wise constant control $u$.*

PROOF. The image of $H_T \stackrel{\text{def}}{=} \{\exp(t_1\lambda_1) \circ \cdots \circ \exp(t_k\lambda_k) \mid \lambda_i \in L(H),\ \phi_L^+(\lambda_i) \in L(\mathcal{F});\ t_i \in \mathbb{R};\ \sum_{i=1}^k t_i = T,\ k \in \mathbb{N}\}$ under the group isomorphism $\phi_G^+$ is $G_T$. Hence $H = (\phi_G^+)^{-1}(G) = (\phi_G^+)^{-1}(G_T) = H_T$, by strong controllability of $\Sigma$. It follows that system $\Sigma_H$ is strongly controllable from the identity because, for any $T > 0$, $\mathcal{R}_{\mathcal{H}}(T, e) = H_T\, e = H\, e = H$ where $\mathcal{R}_{\mathcal{H}}(T, e)$ is the reachable set from identity for system $\Sigma_H$ at time $T$, and $\mathcal{H} \stackrel{\text{def}}{=} \{\eta_0, \ldots, \eta_m\}$. To show that $x(t, x, u)$, $t \in [0, T]$, is a trajectory of $\Sigma$ it suffices to notice that, since $\phi_G^+$ is a group isomorphism corresponding to a Lie algebra isomorphism $\phi_L^+$,

$$\begin{aligned} \phi_G^+ \left(\exp(t_1\lambda_1) \circ \cdots \circ \exp(t_j\lambda_j)\right) x &= \exp(t_1\phi_L^+(\lambda_1)) \circ \cdots \circ \exp(t_j\phi_L^+(\lambda_j))\, x \\ &= \exp(t_1 f^{u_{(1)}}) \circ \cdots \circ \exp(t_j f^{u_{(j)}})\, x \end{aligned} \tag{2.8}$$

□

Similarly to $\Sigma$, the system $\Sigma^e$ can be reformulated on the Lie group $H$ as

$$\Sigma_H^e : \qquad \dot{S}^e(t) = \{\mu_0 + \sum_{i=1}^{r-1} v_i\mu_i\} S^e(t) \qquad \text{with } S^e(0) = e; \quad t \geq 0 \tag{2.9}$$

with $\phi_L^+(\mu_i) = g_i$ for all $i = 0, \ldots, r-1$. Clearly, Proposition 2.1 also holds for $\Sigma_H^e$; i.e. system $\Sigma_H^e$ is strongly controllable from identity, and trajectories of $\Sigma_H^e$ map into trajectories of $\Sigma^e$ according to: $x^e(t, x, v) = \phi_G^+(S^e(t))x$, $t \in [0, T]$.

## 2.4. Solution of Differential Equations on Lie Groups as Products of Exponentials

Since $H$ is analytic, simply connected, and nilpotent ($L(H)$ is nilpotent), it follows that the exponential map on $L(H)$ is a global diffeomorphism onto $H$, see [**162**, Thm. 3.6.2., p. 196]. Let $\{\psi_0, \ldots, \psi_{r-1}\}$ be a basis of $L(H)$ which, without the loss of generality, is ordered in such a way that $\psi_i = \eta_i = \mu_i$, $i = 0, \ldots, m$, and is such that the mapping

$$M : \mathbb{R}^r \ni (t_0, \ldots, t_{r-1}) \to \exp(t_0\, \psi_0) \circ \cdots \circ \exp(t_{r-1}\, \psi_{r-1}) = \prod_{i=0}^{r-1} \exp(t_i\, \psi_i) \in H \tag{2.10}$$

is a global coordinate chart on $H$. The existence of such a basis is guaranteed by the following proposition.

Proposition 2.2. *Under assumption H1, there exists an ordered basis $B = \{\psi_0, \ldots, \psi_{r-1}\}$ such that the mapping $M$ in (2.10) is a global coordinate chart on $H$.*

Proof. Since the exponential map $\exp : L(H) \to H$ is a global analytic diffeomorphism onto $H$, it will only be necessary to show that $M$ is onto and one-to-one. This is because any element

$h \in H$ has a unique representation as a single exponential:

$$h = \exp\left(\sum_{i=0}^{r-1} \theta_i \psi_i\right)$$

where $\theta_i$, $i = 0, 1, \ldots, r-1$, are known as Lie-Cartan coordinates of the first kind. Thus it suffices to show that for each set of coordinates $(\theta_0, \ldots, \theta_{r-1})$, there exists a unique set of coordinates $(t_0, \ldots, t_{r-1})$ which allow to represent $h$ as product of exponentials (2.10), hence making $M$ a global chart on $H$.

To this end, observe that since $L(H)$ is the Lie algebra of the analytic, simply connected, nilpotent Lie group $H$, then $L(H)$ is solvable and thus there exists a chain of ideals $0 \subset \mathcal{I}_{r-1} \subset \mathcal{I}_{r-2} \subset \cdots \subset \mathcal{I}_0 = L(H)$ where each $\mathcal{I}_i$ is exactly of dimension $r - i$. Without the loss of generality, the basis $\{\psi_0, \ldots, \psi_{r-1}\}$ for $L(H)$ can be arranged so that each $\mathcal{I}_i$ is generated by $\{\psi_i, \ldots, \psi_{r-1}\}$. Note that with this ordering, if the number of generators of the $L(H)$ is $m$, ideals $\mathcal{I}_{r-1}, \ldots, \mathcal{I}_{r-m+1}$ are generated by $\{\psi_{r-1}\}, \{\psi_{r-1}, \psi_{r-2}\}, \ldots, \{\psi_{r-1}, \psi_{r-2}, \ldots, \psi_{r-m+1}\}$, respectively, which only contain generators of $L(H)$.

With this arrangement in hand it is possible to show that the mapping $M : \mathbb{R}^r \to H$ is onto and one-to-one.

(1) The mapping $M$ is onto:

Since the exponential map on $L(H)$ is a global diffeomorphism from $L(H)$ onto $H$, see [**162**, Thm. 3.6.2., p. 196], for any $h \in H$ there exists a $z \in L(H)$ such that $h = \exp(z)$, where $z$ can be uniquely written in terms of the basis elements as

$$z = \sum_{i=0}^{r-1} \alpha_i \psi_i \tag{2.11}$$

By the above ordering, application of the CBH formula to the product $\exp(z) \circ \exp(-\alpha_{r-1}\psi_{r-1})$ yields

$$\begin{aligned}
\exp(z) \circ \exp(-\alpha_{r-1}\psi_{r-1}) &= \exp(z - \alpha_{r-1}\psi_{r-1} + R_1) \\
&= \exp(\sum_{i=0}^{r-2} \beta_i \psi_i) \\
&= \exp(z_1)
\end{aligned}$$

where $R_1$ only contains the basis elements $\psi_i$, $i = 0, \ldots, r-2$.

Similarly,

$$\begin{aligned}\exp(z_1) \circ \exp(-\beta_{r-2}\psi_{r-2}) &= \exp(z_1 - \alpha_{r-2}\psi_{r-2} + R_2) \\ &= \exp(\sum_{i=0}^{r-3} \gamma_i \psi_i) \\ &= \exp(z_2)\end{aligned}$$

where now $R_2$ only contains $\psi_i$, $i = 0, \ldots, r-3$.

Repeating this procedure $r$ times one obtains an expression $\exp(z_{r-1}) = I$. Considering this iterative procedure, $\exp(z_{r-1})$ satisfies:

$$\exp(z_{r-1}) = \exp(z) \circ \exp(-\alpha_{r-1}\psi_{r-1}) \circ \exp(-\beta_{r-2}\psi_{r-2}) \circ \cdots \circ \exp(-\zeta_0\psi_0) = I \tag{2.12}$$

Right-multiplying both sides of (2.12) by $\exp(\zeta_0\psi_0), \ldots, \exp(\beta_{r-2}\psi_{r-2}), \exp(\alpha_{r-1}\psi_{r-1})$, (in the expressed order), yields:

$$\exp(z) = \exp(\zeta_0\psi_0) \circ \cdots \circ \exp(\beta_{r-2}\psi_{r-2}) \circ \exp(\alpha_{r-1}\psi_{r-1})$$

Hence any $h = exp(z) \in H$ can be written as the product of exponentials in (2.10) and therefore $M$ is onto.

(2) The mapping $M$ is one-to-one:

To show that $M$ is one-to-one, i.e. that any element $h \in H$ has a unique representation as a product of exponentials (2.10), suppose that:

$$\prod_{i=0}^{r-1} \exp(\alpha_i\psi_i) = \prod_{i=0}^{r-1} \exp(\beta_i\psi_i) \tag{2.13}$$

with $\alpha_i \neq \beta_i$, for some $i \in \{0, 1, 2, \ldots, r-1\}$. Let $i^*$ be the index of the first element $\psi_{i^*} \in B$ such that $\alpha_{i^*} \neq \beta_{i^*}$. Then, pre-multiplying (2.13) by $\exp(-\alpha\psi_i)$, $i = 0, 1, 2, \ldots, i^* - 1$, yields

$$\prod_{i=i^*}^{r-1} \exp(\alpha_i\psi_i) = \prod_{i=i^*}^{r-1} \exp(\beta_i\psi_i) \tag{2.14}$$

Left-multiplication of (2.14) by $\exp(-\alpha_{i^*}\psi_{i^*})$ results in

$$\prod_{i=i^*+1}^{r-1} \exp(\alpha_i\psi_i) = \exp((\beta_{i^*} - \alpha_{i^*})\psi_{i^*}) \prod_{i=i^*+1}^{r-1} \exp(\beta_i\psi_i) \tag{2.15}$$

Applying the CBH formula to both sides of (2.15) one obtains that $\exp(z') = \exp(z'')$, where $z' = \sum_{i=i^*+1}^{r-1} \alpha'_i\psi_i$ and $z'' = (\beta_{i^*} - \alpha_{i^*})\psi_{i^*} + \sum_{i=i^*+1}^{r-1} \alpha''_i\psi_i$ are the resulting exponents of the left-hand and right-hand side of (2.15), respectively. In $z'$ the lowest index is $i^*+1$ and in $z''$ it is $i^*$, but we have that $\exp(z') = \exp(z'')$, i.e. $z' = z''$, since the exponential mapping is a global diffeomorphism on $H$, therefore $z'$ and $z''$ must have expansions in terms of the same Lie brackets and $\beta_{i^*} - \alpha_{i^*} = 0$ so that $\alpha_{i^*} = \beta_{i^*}$, which is a contradiction. Hence, $\alpha_i = \beta_i$, $i = 0, 1, 2, \ldots, r-1$, thus concluding the proof. □

REMARK 2.3. *An algorithmic way to obtain a basis for $L(H)$ which satisfies the above mentioned ordering condition, (requiring that each ideal $\mathcal{I}_i$ is generated by $\{\psi_i, \ldots, \psi_{r-1}\}$), is to employ the construction procedure developed by P. Hall* [**160**].

The fact that (2.10) is a global chart implies that the solutions to $\Sigma_H$ and $\Sigma_H^e$, whose common underlying Lie algebra is $L(H)$, can be expressed as products of exponentials:

$$S(t) = \prod_{i=0}^{r-1} \exp(\gamma_i(t)\psi_i) \qquad \text{and} \qquad S^e(t) = \prod_{i=0}^{r-1} \exp(\gamma_i^e(t)\psi_i) \tag{2.16}$$

where the functions $\gamma_i, \gamma_i^e : \mathbb{R} \to \mathbb{R}$, are referred to as the $\gamma$-coordinates (or logarithmic coordinates) of the flows $S(t)$, $S^e(t)$, respectively, and can be shown to satisfy a set of differential equations of the form, see [**149**]:

$$\Gamma(\gamma)\dot{\gamma} = u^d \qquad \Gamma(\gamma^e)\dot{\gamma^e} = v^d \qquad \gamma(0) = \gamma^e(0) = 0 \tag{2.17}$$

Here $\Gamma(\cdot) : \mathbb{R}^r \to \mathbb{R}^{r\times r}$ is a real analytic, matrix valued function of $\gamma \stackrel{\text{def}}{=} [\gamma_0\ \gamma_1\ \ldots\ \gamma_{r-1}]^T$, the zero initial conditions correspond to $S(0) = S^e(0) = I$, and $u^d \stackrel{\text{def}}{=} [1\ u_1\ \ldots\ u_m\ 0 \ldots\ 0]^T \in \mathbb{R}^r$, and $v^d \stackrel{\text{def}}{=} [1\ v_1\ \ldots\ v_{r-1}]^T \in \mathbb{R}^r$, (where the first component of $u^d$ and $v^d$ corresponds to the drift vector field).

Details concerning the derivation of the $\gamma$-coordinates equation (2.17) in the general setting of free Lie algebras (of indeterminates) are included in Appendix A, p. 222; see also Chapter 6, p. 148.

The solution to equation (2.17) is generally only local unless $\Gamma(\gamma)$ is invertible for all $\gamma$. The invertibility of $\Gamma(\gamma)$ is ensured if a basis for $L(H)$ is constructed as indicated in Proposition 2.2; see also [**149**].

REMARK 2.4. *The representation (2.16) of the solution to equation (2.7) is not unique. The representation in the form of the product of exponentials (2.16) results from the introduction of the Lie-Cartan coordinates of the second kind (2.10) on the group $H$; see* [**149**]. *Alternatively, as pointed out in Proposition 2.2, the solution to (2.7) can be represented using the Lie-Cartan coordinates of the first kind, i.e. it is possible to write*

$$S(t) = \exp\left(\sum_{i=0}^{r-1} \theta_i(t)\psi_i\right)$$

*where $\theta_i : \mathbb{R} \to \mathbb{R}$, $i = 0, 1, \ldots, r-1$, are the "coordinates" of such a solution; see* [**145**].

*It is also known that the solution of (2.7) can be written in terms of a formal series of $\mathcal{C}^\infty$ functions on $H$. The last arises when (2.7) is solved by Picard iteration giving rise to the Peano-Baker formula which exhibits the solution in terms of iterated integrals. Specifically, $S(T)$ defines the Chen-Fliess series of the input $u$; see* [**143**, Theorem III.2, p. 22] *and* [**38**, p. 695]. *The equivalence of the Chen-Fliess series representation and the representation through the product of exponentials of the solutions to (2.7) has been shown in* [**148**] *(see Theorem on p. 328). The product expansion in* [**148**] *uses P. Hall bases.*

## 2.5. Open-Loop Control Problem on the Lie Group

Since the extended system $\Sigma^e$ is instantaneously controllable in any direction of the state space, and thus much easier to steer than $\Sigma$, one possible systematic way to stabilize $\Sigma$ would involve the following steps:

(i) Find a control $v$ that stabilizes $\Sigma^e$.

(ii) Find a control $u$ as a function of $v$ such that the trajectories of $\Sigma$ intersect those of $\Sigma^e$ every $T > 0$ units of time, i.e. such that $x(nT, x, u) = x^e(nT, x, v)$, $n \in \mathbb{Z}_+$.

The first step can be easily solved, for example, using one of the following static state feedback control laws:

- Linearizing feedback $v$, such that $g^v(x) = Ax$:

$$v(x) \quad = \quad Q^{\dagger}(x)\,(Ax - f_0(x)) \tag{2.18}$$

  where $A \in \mathbb{R}^{n\times n}$ is a stable matrix, i.e. whose eigenvalues belong to the left-half of the complex plane, and $Q^{\dagger}(x) = Q(x)^T \left[Q(x)\,Q(x)^T\right]^{-1}$ is the Moore-Penrose right pseudo-inverse of the $n\times(r-1)$ matrix $Q(x) = [g_1(x)\; g_2(x)\; \ldots\; g_{r-1}(x)]$, which is ensured to exist for all $x \in B(0,R)$ because[1] rank $\big(Q(x)\big) = n$ by the assumption that $g_i$, $i = 1,\ldots,r-1$, contains the basis for $L_x(\mathcal{F})$.

- Steepest-gradient *seeking* $v$, such that $g^v(x) = -\nabla V$:

$$v(x) \quad = \quad -Q^{\dagger}(x)\,(\nabla V + f_0(x)) \tag{2.19}$$

  where $\nabla V \overset{\text{def}}{=} \frac{\partial V}{\partial x}$ is the gradient of an arbitrary Lyapunov function $V \in \mathcal{C}^{\infty} : \mathbb{R}^n \to \mathbb{R}_+$ and $Q^{\dagger}(x)$ is as above. This definition of $v$ ensures that the Lyapunov function $V$ has a negative time derivative, namely $\dot{V} = -\|\nabla V\|^2$.

- Lyapunov rate of decrease *shaping* $v$, such that $\dot{V} = -k(x) < 0$:

$$v(x) \quad \overset{\text{def}}{=} \quad \frac{-a(x) - k(x)}{\|b(x)\|^2} b(x)^T, \quad v(0) = 0 \tag{2.20}$$

  where, for some arbitrary Lyapunov function $V \in \mathcal{C}^{\infty} : \mathbb{R}^n \to \mathbb{R}_+$, $k : \mathbb{R}^n \to \mathbb{R}_+$ is some positive definite function such that $k(0) = 0$, $a(x) \overset{\text{def}}{=} L_{g_0}V(x)$, $b_i(x) \overset{\text{def}}{=} L_{g_i}V(x)$, $i = 1,\ldots,r-1$, and $b(x)$ is the row vector $b(x) \overset{\text{def}}{=} [b_1(x)\; \cdots\; b_{r-1}(x)]$. Since $V$ is a Lyapunov function, it is monotonically increasing, and therefore $\nabla V$ only vanishes at $x = 0$. The latter, together with fact that the vector fields $g_i$ span $\mathbb{R}^n$ for all $x \in \mathbb{R}^n$, implies that $L_{g_i}V(x) = \nabla V g_i(x) \neq 0$ for any $x \in \mathbb{R}^n \setminus 0$ and some $i \in \{1,2,\ldots,r-1\}$, i.e.

[1]Since rank$(Q) = n$, $\forall\, x \in W \overset{\text{def}}{=} B(0,R)$, then rank$(Q^T) = n$. The latter implies that $Q^T x \neq 0$, $\forall\, x \in W$, and therefore $\|Q^T x\| = x^T Q Q^T x > 0 \Rightarrow QQ^T > 0$ and $\det(QQ^T) > 0$, thus implying the invertibility of $QQ^T$ for all $x \in W$.

at least one $g_i(x)$ will have a component in the direction of $\nabla V(x)$. Hence, $\|b(x)\| \neq 0$ for all $x \in \mathbb{R}^n \setminus 0$, thus ensuring that the above control is defined for all $x \in \mathbb{R}^n$.

The second step is more complicated as it involves the construction of a discontinuous or a time-varying control $u(x,t)$ such that the trajectories of systems $\Sigma$ and $\Sigma^e$ repeatedly intersect with period $T$, i.e. such that $x(nT) = x^e(nT)$, $n \in \mathbb{Z}_+$. This problem of achieving a *point-wise equivalence* of trajectories of the extended and the original systems will be referred as *trajectory interception problem* (TIP). An approach to the solution of the TIP was first proposed by Michalska in [**75**], drawing on the ideas of Lafferriere [**44**], and consists in re-stating the TIP as a problem of matching, at time $T$, the $\gamma$-coordinates (see (2.16)) that parametrize the flows of systems $\Sigma$ and $\Sigma^e$, i.e. so that $S(T) = S^e(T)$.

Before formally stating the TIP, it is worth mentioning that the extended system $\Sigma^e$ can be stabilized by the sequential application of constant controls $\hat{v} \stackrel{\text{def}}{=} v(x(nT))$ defined on intervals $[nT,(n+1)T)$, $n \in \mathbb{Z}_+$, for some adequately chosen period $T > 0$, and such that the condition $\nabla V g^{\hat{v}}(x(nT)) < \eta \|x(nT)\|^2$ is satisfied for a chosen, strictly increasing and positive definite, function $V \in \mathcal{C}^\infty : \mathbb{R}^n \to \mathbb{R}_+$. The latter is a consequence of Proposition 4.1 on p. 70 and Theorem 4.1 on p. 84 of Chapter 4. This result is useful not only because it implies the stabilizability of system $\Sigma^e$ with piece-wise constant controls, but also because it allows for the following simpler formulationof the TIP in terms of the constant controls $\hat{v}$:

**TIP:** For some $T_{max} > 0$ and a fixed value of the time horizon $T \leq T_{max}$, find control functions $\hat{u}_i(\hat{v},t)$, $i = 1,\ldots,m$, $t \in [nT,(n+1)T)$, $n \in \mathbb{Z}_+$, in the class of functions which are continuous in $\hat{v}$ and piece-wise continuous in $t$, such that for *any initial condition* $x = x(0)$ at time $t_0 = 0$ and *any constant control vector* $\hat{v} \in \mathbb{R}^{r-1}$ the trajectory $x^e(t,x,\hat{v})$ of the extended system $\Sigma^e$ intersects the trajectory $x(t,x,\hat{u})$ of the original system with control $\hat{u} \in \mathbb{R}^m$ precisely at time $T$, so that

$$x(T,x,\hat{u}) = x^e(T,x,\hat{v}) \tag{2.21}$$

Remark 2.5.

- *If an explicit algebraic solution (i.e. a closed form solution) to the TIP can be found, it only needs to be calculated once for one period $T$, since the constant controls $\hat{v}$ can be regarded as parameters of the TIP. This will be shown in Example 3.4 of Chapter 3, p. 58 and Example 5.4 of Chapter 5, p. 120.*
- *The open-loop control problem delivers a feedback control in the sense that the the algebraic solution to the TIP is evaluated at each $t = nT$, $n \in \mathbb{Z}_+$, in terms of the control $\hat{v}$, which depends on $x(nT)$.*
- *It may often be convenient to consider piece-wise constant controls $\hat{u}(\hat{v}, t) \in \mathcal{P}^m$, $t \in [nT, (n+1)T)$ as this facilitates the algebraic solution of the TIP.*
- *Solutions to the TIP with controls in the class of piece-wise continuous functions are clearly guaranteed to exist under hypothesis H2.*

The formalism presented in the previous sections allows to re-state the TIP as a problem of matching the flows of systems $\Sigma_H$ and $\Sigma_H^e$ on the Lie group $H$ at time $T$, i.e. as a flow interception problem (FIP) in which given $\hat{v}$, a control $\hat{u}$ must be found such that $S(T, \hat{v}) = S^e(T, \hat{u})$, and hence $x^e(T, x, \hat{v}) = \phi_G^+(S^e(T, \hat{v}))x = \phi_G^+(S(T, \hat{u}))x = x(T, x, \hat{u})$. The representation of the flows as products of exponentials in (2.16) and the equations (2.17) for the $\gamma$-coordinates that parametrize the flows permit to reformulate the FIP as logarithmic-coordinate interception problem (LCIP) such that $\gamma^e(T) = \gamma(T)$. The LCIP can be stated as follows:

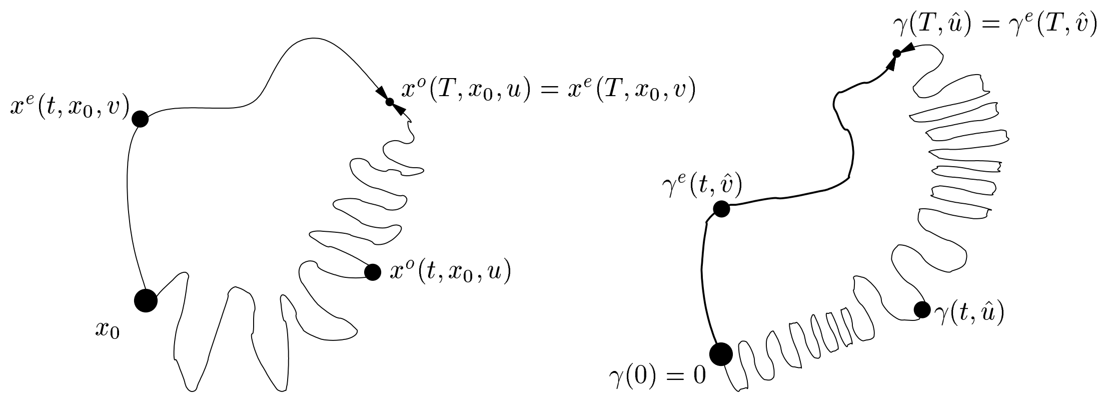
**LCIP:** Consider the two formal "control systems" of equation (2.17) with $u^d = [1\ \hat{u}\ 0\ \ldots\ 0]^T \in \mathbb{R}^r$ and $v^d = [1\ \hat{v}]^T \in \mathbb{R}^r$. For some $T_{max} > 0$ and a fixed value of the time horizon $T \leq T_{max}$, find control functions $\hat{u}_i(\hat{v}, t)$, $i = 1, \ldots, m$, $t \in [nT, (n+1)T)$, $n \in \mathbb{Z}_+$, in the class of functions which are continuous in $\hat{v}$ and piece-wise continuous in $t$, such that for *any constant control vector* $\hat{v} \in \mathbb{R}^{r-1}$:

$$\gamma(T, \hat{u}) = \gamma^e(T, \hat{v}) \tag{2.22}$$

where $\gamma(T, \hat{u})$ and $\gamma^e(T, \hat{v})$ denote the $\gamma$-coordinates at time $T$ corresponding to systems $\Sigma_H$ and $\Sigma_H^e$ with controls $\hat{u} \in \mathbb{R}^m$ and $\hat{v} \in \mathbb{R}^{r-1}$, respectively.

REMARK 2.6. *Reformulating the TIP as an LCIP is advantageous since the LCIP is clearly independent of the initial condition $x = x(0)$ (both the FIP or the LCIP are always solved with initial condition $S(0) = S^e(0) = I$ or $\gamma(0) = \gamma^e(0) = 0$, respectively), although the control functions $\hat{u}(\hat{v}, t)$ must be found in terms of the* parameter $\hat{v}$ *— the value of the extended control for the extended system $\Sigma^e$. On the other hand, the LCIP allows to abstract the problem from the particular vector fields as the LCIP only depends on the specific structure of the Lie algebra $L(\mathcal{F})$.*

Fig. 2.2 (a) illustrates the TIP in which trajectories of the extended and the original systems with control $v$ and $u$, respectively, are matched after a period of time $T$. The equivalent problem as a LCIP independent of the initial condition $x(0) = x_0$ is shown in Fig. 2.2 (b).




(a) Trajectory interception problem.

(b) Logarithmic coordinate interception problem.

FIGURE 2.2. Trajectory interception problem (TIP) for systems $\Sigma$ and $\Sigma^e$ starting from $x_0$ at $t = 0$ with controls $u$ and $v$, respectively, and the corresponding logarithmic coordinate interception problem (LCIP) on the Lie group.

It is worth pointing out that the existence of solutions to the TIP (or LCIP) with controls in the class of smooth functions is guaranteed under hypothesis H2 and the following result of Jurdjevic, [**3**], which establishes that any controllable system with piece-wise controls is also controllable using smooth controls (see part *ii.* of the theorem).

**Theorem 2.2** (Jurdjevic, [**3**], p. 110)**.** *Let $U \in \mathcal{P}^m$ denote the set of admissible controls (in the set of piece-wise constant controls) and denote by $\Sigma(U)$ system $\Sigma$ on a smooth manifold $M$ and with piece-wise constant controls. Then,*

i. *Suppose that* $U$ *is convex and that* $x_f$ *is normally accessible by* $\Sigma(U)$ *from* $x_0$ *in* $T$ *units of time. Then there exists a smooth control* $u(t)$ *defined on an interval* $[0, T]$ *such that the corresponding trajectory* $x(t)$ *of* $\Sigma$ *generated by* $u(t)$ *satisfies* $x(0) = x_0$ *and* $x(T) = x_f$.

ii. *If* $\Sigma(U)$ *is a Lie-determined system, i.e. a system for which the tangent space* $T_x M$ *of each* $x$ *in an orbit of* $\Sigma(U)$ *coincides with* $L_x(\mathcal{F})$*, then any point in the interior of the reachable set of* $\Sigma(U)$ *that can be reached from a point* $x_0$ *is also reachable by a trajectory* $x(t)$ *of* $\Sigma$ *generated by a smooth control* $u(t)$*. If, furthermore,* $\Sigma(U)$ *is controllable, then any two points of the manifold* $M$ *can be connected to each other in a positive time by a trajectory of* $\Sigma$ *generated by a smooth control.*

An approach to the construction of time-varying feedback controls for the stabilization of bilinear systems which is *entirely* based on the solution of the TIP as an LCIP is presented in Chapter 5, p. 111. The reader is advised to consult sections 3.3.1 and 5.3 of this thesis, and especially the examples in section 3.4 and 5.4 for further insight into the TIP and LCIP. Even though chapters 3 and 4 present methods which partially draw on ideas related to the TIP and the open-loop control problem on the Lie group, fair understanding of these topics is assumed and details are omitted with purpose of focusing on the proposed approaches and avoiding unnecessary repetition.

# CHAPTER 3

## Continuous Time-Varying Stabilization Feedback Approach

A novel method is presented for construction of stabilizing time-varying feedback controls for nonlinear systems with drift, [**77**]. The proposed approach draws on the ideas of [**42, 43**], which apply to systems without drift. The method combines the construction of a periodic time-varying critically stabilizing control with the on-line calculation of an additional corrective term to provide for asymptotic convergence to the origin. The periodic control is obtained through a solution of an open-loop control problem on the associated Lie group which is posed as a trajectory interception problem in the $\gamma$-coordinates of flows.

### 3.1. Introduction

Most methods for the construction of control laws that stabilize system $\Sigma$ in equation (1.1) start by finding a Lyapunov function for some type of linearization of $\Sigma$, see for example [**58, 2**], or else assume the existence of a suitable control Lyapunov function which can be forced to decrease to zero by an adequate choice of controls in (1.1), cf. [**73**] and references therein. However, as outlined in Section 1.4, strongly nonlinear systems often result in uncontrollable linearizations. Moreover it is well known that finding suitable control Lyapunov functions is extremely difficult. These facts motivate the use of Lie algebraic methods for the stabilization of (1.1) which, in contrast to the above mentioned approaches, have been less explored.

The novel state-feedback synthesis method presented in this chapter employs a time-periodic feedback which brings into play the Lie brackets of the vector fields $f_i$, $i = 0, 1, \ldots, m$. The method relies on the Lie bracket extension (2.1) of system $\Sigma$ and partially draws on the well known ideas of

Coron and Pomet, see [**42, 43, 48**], who constructed time-periodic stabilizing controls for systems without drift. Under reasonable assumptions, a (critically) stabilizing time-periodic feedback control is first constructed for $\Sigma$ using the extended system $\Sigma^e$ and a specific solution of an open-loop, finite horizon control problem. This open-loop control problem is posed in terms of the $\gamma$-coordinates for flows, [**149, 44**], and its purpose is to generate open-loop controls such that the trajectories of the controlled extended system and the open-loop system intersect after a finite time $T$, independent of their common initial condition.

An additional "correction term" is next determined for the time-periodic stabilizing control to make the aggregated feedback law asymptotically stabilizing for system $\Sigma$. The corrective term is calculated to be a control which decreases a Lyapunov function whose level sets contain the periodic orbits of the system stabilized by the time-periodic feedback.

The synthesis method is general and applicable to a large class of nilpotent systems which do not lend themselves to controllable linearization (be it through state-feedback transformations, or else simply around some operating points).

## 3.2. Problem Definition and Assumptions

**Problem Definition.** *Construct smooth time-varying feedback controls $u_i(x,t) \in \mathcal{C}^\infty : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$, $i = 1, \ldots, m$, such that system $\Sigma$ in (1.1), is Lyapunov asymptotically stable.*

For convenience of notation let $W \stackrel{\text{def}}{=} B(0,R)$, and denote by $\Phi_u(t,x)$ the solution to $\Sigma$ starting from $x \in \mathbb{R}^n$ at time $t = 0$ and resulting from the application of a control $u(x,t) \stackrel{\text{def}}{=} [u_1(x,t) \ \cdots \ u_m(x,t)]^T$. Similarly, $\Phi_v^e(t,x)$ denotes the solution to $\Sigma^e$ starting from $x \in \mathbb{R}^n$ at time $t = 0$ and arising from the application of an extended control $v(x) \stackrel{\text{def}}{=} [v_1(x) \ \cdots \ v_{r-1}(x)]^T$.

By the controllability hypothesis H2 and Theorem 2.2, implied is the existence of a smooth control $w : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ that critically stabilizes system $\Sigma$, i.e. the control $w \in \mathcal{C}^\infty$ is such that for any $x \in W$, $\Phi_w(t,x)$, $t \in \mathbb{R}$, is a *periodic (closed) orbit of system* $\Sigma$, which satisfies $\Phi_w(t+T,x) = \Phi_w(t,x)$ for all $x \in W$ and $t \in \mathbb{R}$.

The construction of the asymptotically stabilizing control $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ relies on the controllability properties of the linearization of system $\Sigma$ about the critical trajectory $\Phi_w(t,x)$. To this end, consider equation (1.1) as satisfied by a trajectory $\Phi_w(t,x)$ of system $\Sigma$:

$$\frac{\partial \Phi_w(t,x)}{\partial t} = f_0(\Phi_w(t,x)) + \sum_{i=1}^{m} f_i(\Phi_w(t,x)) w_i(\Phi_w(0,x),t) \qquad \Phi_w(0,x) = x \tag{3.1}$$

The linearization of $\Sigma$ along the trajectory $\Phi_w(t,x)$ can then be expressed as

$$\dot{\varphi} = \frac{\partial}{\partial x}\left[f_0(\Phi_w(t,x)) + \textstyle\sum_{i=1}^{m} f_i(\Phi_w(t,x)) w_i(\Phi_w(t,x),t)\right]\varphi + \textstyle\sum_{i=1}^{m} f_i(\Phi_w(t,x))\upsilon_i \tag{3.2}$$

where $\upsilon_i = \tilde{w}_i - w_i$, $i = 1,\dots,m$, are the controls and $\varphi(t) = \Phi_{\tilde{w}}(t,x) - \Phi_w(t,x)$.

The approach also relies on the mapping $\Phi_w$ being a $\mathcal{C}^\infty$ diffeomorphism, at least on the set $W$, with the inverse mapping $\Phi_w^{-1}$ of $\Phi_w$ defined as a mapping such that for any $z = \Phi_w(t,x) \in \mathbb{R}^n$, $\Phi_w^{-1}(t,z) = \Phi_w^{-1}(t,\Phi_w(t,x)) = x$, i.e. given a state $z = \Phi_w(t,x)$ of the critically stabilized system, the inverse mapping $\Phi_w^{-1}$ applied to $z$ *retrieves* the starting point $x$ of the periodic orbit which passes through $z$ at time $t$. Note that if $\Phi_w^{-1}$ is applied to a state $\hat{z} = \Phi_u(t,x)$ of the system with a given control $u$, then $\Phi_w^{-1}(t,\hat{z}) = \Phi_w^{-1}(t,\Phi_u(t,x)) = \hat{x}$, where $\hat{x}$ is the starting point of a critically stable trajectory $\Phi_w(t,\hat{x}) = \hat{z}$ passing through $\hat{z}$ at time $t$. In general, $\hat{x} \neq x$, unless e.g. $w = u$.

To show that $\Phi_w$ is a $\mathcal{C}^\infty$ diffeomorphism it will be convenient to consider the variational equation for the evolution of $\Phi_w(t,x)$ obtained by differentiating (3.1) with respect to $x$ and given by:

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial \bar{x}}{\partial x} &= \frac{\partial f_0(\bar{x})}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial x} + \sum_{i=1}^{m} \frac{\partial f_i(\bar{x})}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial x} w_i(x,t) + \sum_{i=1}^{m} f_i(\bar{x})\frac{\partial w_i(x,t)}{\partial x} \\ \frac{\partial \Phi_w(0,x)}{\partial x} &= I \end{aligned} \tag{3.3}$$

where for simplicity of notation $\bar{x} \overset{\text{def}}{=} \Phi_w(t,x)$.

REMARK 3.1. *In (3.3), the trajectory $\Phi_w(t,x)$ can be differentiated with respect to $x$ because the control $w$ is constructed to be a smooth function and because, under assumption H1, the right-hand side of equation (1.1) with control $w$, $f^w(x)$, is differentiable with respect to $x$; see* [**176**, Thm. 11.1.2, p. 491]. *Actually, a stronger result concerning the smoothness of $\Phi_w$ holds, namely $\Phi_w \in \mathcal{C}^\infty$ as by construction $w \in \mathcal{C}^\infty$ and all the vector fields in the definition of the system are analytic.*

Letting $J(t) \stackrel{\text{def}}{=} \frac{\partial \bar{x}}{\partial x}(t)$, equation (3.3) can be expressed as:

$$\dot{J}(t) \quad = \quad A(\bar{x})J(t) + B(\bar{x}) \qquad\qquad J(0) = I \tag{3.4}$$

with

$$\begin{aligned} A(\bar{x}) &\stackrel{\text{def}}{=} \frac{\partial f_0(\bar{x})}{\partial \bar{x}} + \sum_{i=1}^{m} \frac{\partial f_i(\bar{x})}{\partial \bar{x}} w_i(x,t) \\ B(\bar{x}) &\stackrel{\text{def}}{=} \sum_{i=1}^{m} f_i(\bar{x}) \frac{\partial w_i(x,t)}{\partial x} \end{aligned} \tag{3.5}$$

For the construction of the stabilizing control $u$ to be valid, the following hypotheses are necessary in addition to assumptions H1–H3 in Chapter 2:

H4. The origin is an isolated equilibrium state of the unforced system $\dot{x} = f_0(x)$.

H5. There exists a a period $T > 0$ and a critically stabilizing periodic control $w \in \mathcal{C}^\infty : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ satisfying:

$$w(0,t) \quad = \quad 0 \qquad\qquad \forall\, t \in \mathbb{R} \tag{3.6}$$

$$w(x,t+T) \quad = \quad w(x,t) \qquad \forall\, t \in \mathbb{R},\ \forall\, x \in W \tag{3.7}$$

such that for any $x \in W$ the closed-loop trajectory $\Phi_w(t,x)$ is periodic with period $T$, and the linearization (3.2) of system (1.1) along $\Phi_w(t,x)$ is a *uniformly controllable* time-varying system (see definition in [**31**]).

H6. It is assumed that $a \stackrel{\text{def}}{=} \sup_{x \in W} \|A(x)\|$ and $c \stackrel{\text{def}}{=} \sup_{x \in W} \|A(x) + B(x)\|$ are finite, and that $T \in (0, T_{max}]$ with $T_{max}$ satisfying:

$$0 < T_{max} < \frac{1}{a} \log(1 + \frac{a}{c}) \tag{3.8}$$

REMARK 3.2. *It is worth pointing out that assumption H4 can be relaxed, but in this case a more elaborated construction of the periodic stabilizing control and more complex computational issues are involved.*

It follows from [**31**, Thm. 4 and Def. 3] that assumption H5 implies that the time-varying system (3.2) can be **instantaneously** steered from any state to any other state using impulsive controls and that the condition given in [**31**] is equivalent to the following condition (see [**42**]):

$$\operatorname{span}\left\{\left(\mathcal{L}(w)^p f_i\right)(x,t), \quad p \geq 0,\ i = 1,\ldots,m\right\} = \mathbb{R}^n \qquad \forall\, x \in W \setminus 0 \text{ and a given } t \in [0,T] \tag{3.9}$$

where, for $u \in \mathcal{C}^\infty : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$, the operator $\mathcal{L}(u) \in \mathcal{C}^\infty(\mathbb{R}^n;\mathbb{R}^n)$ is defined by[1]:

$$\mathcal{L}(u)X = \frac{\partial X}{\partial t} + \left[f_0 + \sum_{i=1}^{m} f_i u_i, X\right] \qquad \forall\, X \in \mathcal{C}^\infty : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \tag{3.10}$$

where $[\cdot,\cdot]$ denotes the standard Lie bracket, $\mathcal{L}(u)^0 X \overset{\text{def}}{=} X$ and $\mathcal{L}(u)^p X \overset{\text{def}}{=} (\mathcal{L}(u)^{p-1})(\mathcal{L}(u)X)$.

## 3.3. The Synthesis of the Feedback Control

The asymptotically stabilizing feedback for system $\Sigma$ is constructed in two stages as the sum of two control components:

$$\begin{aligned} u(x,t) &\overset{\text{def}}{=} w(x,t) + \Delta u(x,t) \\ w(x,t) &\overset{\text{def}}{=} [w_1(x,t),\ldots,w_m(x,t)]^T \\ \Delta u(x,t) &\overset{\text{def}}{=} [\Delta u_1(x,t),\ldots,\Delta u_m(x,t)]^T \end{aligned} \tag{3.11}$$

where $w \in \mathcal{C}^\infty : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ is the control satisfying (3.6) and (3.7), that renders system $\Sigma$ critically stable, and where $\Delta u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ is a suitable control correction term which additionally provides for asymptotic stabilization. The design of $w$ and $\Delta u$ is discussed next.

### 3.3.1. Construction of the Critically Stabilizing Periodic Control

First, a *critically stabilizing* feedback for the extended system $\Sigma^e$ is defined as

$$v(x) \overset{\text{def}}{=} [v_1(x) \ \cdots \ v_{r-1}(x)]^T = -Q(x)^\dagger g_0(x) \tag{3.12}$$

where $Q(x) \overset{\text{def}}{=} [g_1(x) \ \cdots \ g_{r-1}(x)]$ and $Q(x)^\dagger$ denotes the Moore-Penrose pseudo-inverse of the state dependent matrix $Q$. Since, by assumptions H2 and H4, $\operatorname{span} L_x(\mathcal{F}) = \mathbb{R}^n$ for all $x \in \mathbb{R}^n$, the rank

[1]Here, $\mathcal{C}^\infty(\mathbb{R}^n;\mathbb{R}^n)$ denotes the family of all $\mathcal{C}^\infty$ vector fields $f : \mathbb{R}^n \to \mathbb{R}^n$.

of the matrix $Q(x) = [g_1(x) \ \cdots \ g_{r-1}(x)]^T$ is $n$ for all $x \in \mathbb{R}^n$, and therefore $\text{rank}(Q(x)^T) = n$. Thus $Q^T x \neq 0$, for all $x \in \mathbb{R}^n$, and hence $\|Q^T x\| = x^T Q Q^T x > 0 \Rightarrow QQ^T > 0$ and $\det(QQ^T) > 0$, which implies $Q(x)Q(x)^T$ is invertible for all $x \in \mathbb{R}^n$, so that $Q(x)^\dagger = Q(x)^T[Q(x)Q(x)^T]^{-1}$ is the right inverse of $Q(x)$, i.e. $Q(x)Q(x)^\dagger = I$ for all $x \in \mathbb{R}^n$. The existence of $Q^\dagger$ thus ensures the existence of the critically stabilizing extended control $v$ for all $x \in \mathbb{R}^n$. The properties of the control $v$ are stated in terms of the following proposition.

Proposition 3.1. *The control $v : \mathbb{R}^n \to \mathbb{R}^m$ defined by (3.12) renders the extended system critically stable and is such that for all $x \in W$:*

$$v(\Phi_v^e(t, x_0)) = v(x_0) = \textit{const.} \quad \forall\ t \geq t_0 \tag{3.13}$$

*for any initial condition $x(t_0) = x_0$ satisfying $g_0(x_0) \neq 0$.*

Proof. With $v$ defined as in (3.12), it follows that the trajectories of the extended system $\Sigma^e$ satisfy

$$\dot{x} \ = \ g_0(x) + Q(x)v(x) \ = \ 0 \tag{3.14}$$

which is a stable, trivially periodic system. Moreover, $\Phi_v^e(t, x_0) = \Phi_v^e(t_0, x_0) = x_0$ for all $t \geq t_0$, as $\Phi_v^e$ is a solution of $\dot{x} = 0$. □

The next step involves solving the trajectory interception problem (TIP) described in Section 2.5, p. 35, as an open-loop control problem on the Lie group. The task of the open-loop control problem is to generate the time-varying part of the controls $u_i(x, t)$, $i = 1, \ldots, m$, for the original system $\Sigma$ such that its trajectories and the trajectories of the extended system $\Sigma^e$ intersect periodically with the given frequency $1/T$. To this end, the TIP (2.21) is solved as an LCIP (2.22) only once with respect to the constant controls $\hat{v} = v(x)$, defined in (3.12), and with controls $\hat{u}(\hat{v}, t)$ being sought in the class of smooth functions. The critically stabilizing control $w$ is then defined as a *periodic continuation* of the control $\hat{u}(\hat{v}, t)$. A periodic continuation of a mapping defined some some subinterval of $\mathbb{R}$ is another mapping defined as follows.

**Definition 3.1. - Periodic Continuation.**

*Let* $f : [0,T] \to \mathbb{R}$*,* $T > 0$*, be a mapping defined over an interval* $[0,T]$*. The* periodic continuation *of* $f$ *is a mapping* $g : \mathbb{R}_+ \to \mathbb{R}$ *defined as:*

$$g(t) \quad \stackrel{\text{def}}{=} \quad f(mod(t,T)) \qquad t \geq 0 \tag{3.15}$$

*where* $mod(a,b)$ *is the remainder of* $a/b$*. Note that since* $mod(a,b) = a$ *for* $0 \leq a < b$ *and* $mod(a,b) \in [0,b]$ *for* $a \geq b$*, the periodic continuation* $g$ *in (3.15) satisfies:*

$$g(t+T) = g(t) \qquad \forall\ t \geq 0, \tag{3.16}$$

With the above definition, the critically stabilizing control $w(x,t)$ in terms of the solution $\hat{u}(\hat{v},t)$ to the trajectory interception problem (stated below) is formally defined as follows:

$$w(x,t) \quad = \quad \hat{u}(\hat{v}(x), \text{mod}(t-t_0,T)) \;\; = \;\; \hat{u}(\hat{v}(x_0), \text{mod}(t-t_0,T)) \text{ for all } t \geq t_0 \tag{3.17}$$

where $(x_0, t_0)$ is any point such that the trajectory of $\Sigma$ with control $w$ passes through $x_0$ at time $t_0$.

For clarity of exposition, the TIP (2.21) and LCIP (2.22) are re-stated here as follows:

**TIP:** For a fixed value of the time horizon $T \in (0, T_{max}]$, find a control function $(\hat{v},t) \to \hat{u}(\hat{v},t)$, $\hat{u} : \mathbb{R}^{r-1} \times [0,T] \to \mathbb{R}^m$, whose periodic continuation is in the class of $\mathcal{C}^\infty$ functions, such that for *any initial condition* $x(0) = x_0 \in W$, $x_0 \neq 0$, at time $t_0 = 0$ and *any constant extended control vector* $\hat{v} \in \mathbb{R}^{r-1}$, the trajectory $\Phi^e_{\hat{v}}(t,x_0)$ of the extended system $\Sigma^e$:

$$\dot{x} = g_0(x) + \sum_{i=1}^{r-1} g_i(x)\hat{v}_i, \quad x(0) = x_0$$

intersects the trajectory $\Phi_{\hat{u}}(t, x, \hat{u})$ of the original system $\Sigma$ with control $\hat{u} \in \mathbb{R}^m$:

$$\dot{x} = f_0(x) + \sum_{i=1}^{r-1} f_i(x)\hat{u}_i(\hat{v}, t), \quad x(0) = x_0$$

precisely at time $T$, so that

$$\Phi_{\hat{u}}(T, x) = \Phi^e_{\hat{v}}(T, x) \text{ for all } v \in \mathbb{R}^{r-1} \tag{3.18}$$

REMARK 3.3. *In the above TIP, the control $\hat{u}$, together with its periodic continuation, are sought in the class of $\mathcal{C}^\infty$ functions and not in the class of piecewise continuous functions as in the TIP stated on p. 37 of Chapter 2. This is to insure that $w$ is in fact a smooth function.*

The formalism presented in Chapter 2 allows to re-state the above TIP as a problem of matching the flows of the right-invariant systems $\Sigma_H$ and $\Sigma^e_H$ (see equations (2.7) and (2.9)) on the analytic, simply connected Lie group $H$ at time $T$, i.e. as a flow interception problem (FIP) in which given an extended control $\hat{v}$, a control $\hat{u}$ must be found such that $S(T, \hat{u}) = S^e(T, \hat{v})$, and hence $\Phi^e_{\hat{v}}(T, x) = \phi^+_G(S^e(T, \hat{v}))x = \phi^+_G(S(T, \hat{u}))x = \Phi_{\hat{u}}(T, x)$. Denoting by $\phi^+_L$ the Lie algebra isomorphism introduced in Theorem 2.1, p. 28, which is such that for any $f_i, g_i \in L(\mathcal{F})$, $(\phi^+_L)^{-1}(f_i) = \eta_i \in L(H)$, and $(\phi^+_L)^{-1}(g_i) = \mu_i \in L(H)$, the FIP on the Lie group $H$ is stated here as follows:

**FIP:** Consider the two formal initial value problems:

$$S1: \quad \begin{cases} \dot{S}^e(t) &= (\phi^+_L)^{-1}\left[f_0 + \sum_{i=1}^{r-1} f_i\hat{v}_i\right] S^e(t) \\ S^e(0) &= e \in H \end{cases} \tag{3.19}$$

$$S2: \quad \begin{cases} \dot{S}(t) &= (\phi^+_L)^{-1}\left[g_0 + \sum_{i=1}^{m} g_i\hat{u}_i(\hat{v}, t)\right] S(t) \\ S(0) &= e \in H \end{cases} \tag{3.20}$$

For a fixed value of the time horizon $T \in (0, T_{max}]$, find a control function $(\hat{v}, t) \rightarrow \hat{u}(\hat{v}, t)$, $\hat{u} : \mathbb{R}^{r-1} \times [0, T] \rightarrow \mathbb{R}^m$, whose periodic continuation is in the class of $\mathcal{C}^\infty$ functions, such that for *any constant extended control vector* $\hat{v} \in \mathbb{R}^{r-1}$, the above flows (of the extended and original systems, respectively) intersect at time $T$, i.e.:

$$S(T, \hat{u}) = S^e(T, \hat{v}) \tag{3.21}$$

The result of Wei and Norman, [**149**], establishes that the solution to both initial value problems S1 and S2 has the same general representation as the products of exponentials in (2.16) and that the $\gamma$-coordinates parametrizing the flows satisfy the corresponding differential equations in (2.17). The latter allows to reformulate the following logarithmic-coordinate interception problem (LCIP) such that $\gamma(T) = \gamma^e(T)$, (thus implying $S(T, \hat{u}) = S^e(T, \hat{v})$):

**LCIP:** Consider the two formal "control systems" of equation (2.17):

$$CS1: \quad \dot{\gamma}^e(t) \ = \ \Gamma\left(\gamma^e(t)\right)^{-1} v^d, \quad \gamma^e(0) = 0 \tag{3.22}$$

$$CS2: \quad \dot{\gamma}(t) \ = \ \Gamma\left(\gamma(t)\right)^{-1} u^d, \quad \gamma(0) \ = 0 \tag{3.23}$$

with $u^d = [1 \ \hat{u}(\hat{v}, t) \ 0 \ \ldots \ 0]^T \in \mathbb{R}^r$ and $v^d = [1 \ \hat{v}]^T \in \mathbb{R}^r$. For a fixed value of the time horizon $T \in (0, T_{max}]$, find a control function $(v, t) \rightarrow u(v, t)$, $u : \mathbb{R}^{r-1} \times [0, T] \rightarrow \mathbb{R}^m$, whose periodic continuation is in the class of $\mathcal{C}^\infty$ functions, such that for *any constant extended control vector* $v \in \mathbb{R}^{r-1}$:

$$\gamma(T, u) = \gamma^e(T, v) \tag{3.24}$$

where $\gamma(T, \hat{u})$ and $\gamma^e(T, \hat{v})$ denote the $\gamma$-coordinates at time $T$ of systems $\Sigma_H$ and $\Sigma_H^e$ with controls $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^{r-1}$, respectively.

Remark 3.4.

- *Note that the above TIP, FIP or LCIP must be solved only once, on an time interval* $[0, T]$*, to obtain a control* $\hat{u}(\hat{v}, t)$ *in terms of the the constant controls* $\hat{v}$.

- *Problem LCIP is just a conventional open-loop control problem for the differential system in the logarithmic coordinates $\gamma$ with terminal constraint which is parametrized by $\hat{v}$. Its solution is generally non-unique and can be obtained by using algorithms similar to those found in* [**52, 47**]*, which employ oscillatory functions such as the sine and cosine, with adequately selected frequencies and phase shifts. The analysis of the algorithms in* [**52, 47**] *shows that the open-loop control $\hat{u}$ accomplishing TIP can be sought as linear combinations of such oscillatory functions with coefficients that smoothly depend on the parameter $\hat{v}$.*
- *The existence of smooth open-loop controls $\hat{u}_{open} : t \to \mathbb{R}^m$, $\hat{u}_{open} \stackrel{\text{def}}{=} \hat{u}(\hat{v}, t)$, that solve the TIP, or equivalently LCIP, is guaranteed by the complete controllability hypothesis H2 and Theorem 2.2.*

As mentioned earlier, both FIP and LCIP are independent of the initial condition $x(0) = x_0$. However, the original system controls $\hat{u}(\hat{v}, t)$ must be found in terms of the constant extended controls $\hat{v}$, which depend on the initial condition. In this sense, the periodic continuation $w(x, t)$ of the solution $\hat{u}(\hat{v}, t)$ to LCIP delivers a feedback control which depends on the state $x(nT)$ at discrete intervals of time $t = nT$, $n \in \mathbb{Z}_+$. The following proposition ensures the critically stabilizing properties of the control $w(x, t)$.

PROPOSITION 3.2. *Under hypotheses H1, H2 and H4, the original system $\Sigma$:*

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} f_i(x(t)) w_i(x, t), \qquad x(0) = x_0 \tag{3.25}$$

*with controls $w_i(x, t)$, $i = 1, \ldots, m$, defined by (3.17), has a periodic solution (with period $T$) for any initial condition $x_0 \in B(0, R)$. Furthermore, the time periodic control $w(x, t)$ satisfies (3.6) and (3.7).*

PROOF. Under the controllability assumption H2 and by Theorem 2.2, there exists a $\mathcal{C}^\infty$ control $w$ defined as in (3.17) in terms of a control $\hat{u}(\hat{v}, t)$ that solves the TIP.

By construction, the control $w$ is such that $\Phi_w(t_0 + nT, x_0) = \Phi_v^e(t_0 + nT, x_0)$, for $n \in \mathbb{Z}_+$. Since $\Phi_v^e(t_0 + nT, x_0) = x_0$ for $n \in \mathbb{Z}_+$, then also $\Phi_w(t_0 + nT, x_0) = x_0$ for $n \in \mathbb{Z}_+$ regardless of $(x_0, t_0)$.

The periodicity of $w$ simply follows from the fact that the solution of the TIP is evaluated at discrete moments of time $nT$, $n \in \mathbb{Z}_+$, at the same point $\Phi_w(nT, x_0) = x_0$, $n \in \mathbb{Z}_+$. Condition (3.6) that $w(0,t) = 0$ for all $t \in \mathbb{R}$ is trivially satisfied since, by assumption H4, $x = 0$ is an equilibrium point of the system. □

### 3.3.2. Asymptotically Stabilizing Correction to the Periodic Control

Once the critically stabilizing control $w$ is computed, a corrective control term $\Delta u$, which makes the resulting closed-loop system asymptotically stable, is further introduced while drawing on the idea proposed by Coron and Pomet in [**42, 43**] for systems without drift. The control correction term is found by requiring that the following Lyapunov function:

$$V(x,t) = \frac{1}{2} \left\| \Phi_w^{-1}(t,x) \right\|^2 \tag{3.26}$$

decreases along the trajectories of the closed-loop system using the combined control $u(x,t) = w(x,t) + \Delta u(x,t)$. The existence of the above Lyapunov function relies on the invertibility of $\Phi_w$, which is ensured by the following proposition.

PROPOSITION 3.3. *Let $R_1 < R$ be such that for any $x_0 \in B(0,R_1)$, the trajectory $\Phi_w(t,x_0)$ belongs to $W$ for all $t \geq 0$. Then the mapping $x \to \Phi_w(t,x)$ is a global $\mathcal{C}^\infty$ diffeomorphism of $B(0,R_1)$ onto its image $\{z \in \mathbb{R}^n \mid z = \Phi_w(t,x_0),\ x_0 \in B(0,R_1)\}$.*

PROOF. First, to demonstrate that $\Phi_w$ is a $\mathcal{C}^\infty$ (local) diffeomorphism, the Inverse Function Theorem (see Section C.2 of Appendix C, p. 246) may be applied.

Accordingly, it is sufficient to demonstrate that the strong differential of the mapping $\Phi_w$ with respect to $x$, $\Phi_w'|_x(t,x) : \mathbb{R}^n \to \mathbb{R}^n$, evaluated at any $x \in B(0,R_1)$ and a fixed $t \geq 0$ is an isomorphism, i.e. that the Jacobian $\frac{\partial \Phi_w(t,x)}{\partial x}$ is nonsingular for any $x \in B(0,R_1)$ and $t \geq 0$. For this to hold it is sufficient that

$$\left\| \frac{\partial \Phi_w(t,x)}{\partial x} - I \right\| < 1 \tag{3.27}$$

for any $x \in B(0,R_1)$ and $t \geq 0$, where $\|\cdot\|$ is any induced (operator) matrix norm; see [**176**, Thm. 3.6.1., p. 76].

To show (3.27), note that by (3.4),

$$\begin{aligned} J(t) - J(0) &= \int_0^t A(\bar{x})J(s)ds + \int_0^t B(\bar{x})ds \\ &= \int_0^t A(\bar{x})J(0)ds + \int_0^t A(\bar{x})(J(s) - J(0))ds + \int_0^t B(\bar{x})ds \end{aligned}$$

and since $J(0) = I$, the application of the Gronwall-Bellman Lemma (see Appendix C.1, p. 244) yields:

$$\begin{aligned} \|J(t) - I\| &\leq \int_0^t \|A(\bar{x}) + B(\bar{x})\|ds + \int_0^t \|A(\bar{x})\| \, \|J(s) - I\|ds \\ &\leq \frac{c}{a}(\exp(at) - 1) < 1 \end{aligned} \tag{3.28}$$

where the constants $a$ and $c$ are bounds defined for any $x \in W$ according to assumption H6. Hence, for any $x \in B(0, R_1)$ and any fixed $t \geq 0$ there exists a neighborhood $U(x) \subset B(0, R_1)$ such that $\Phi_w$ is a $\mathcal{C}^\infty$ diffeomorphism on $U(x)$.

Next, it will be demonstrated that the mapping $\Phi_w$ is in fact proper, which will guarantee that $\Phi_w$ is a global diffeomorphism as stated in the assertion of the proposition. To this end, note that, by virtue of continuity of $\Phi_w$, the inverse image $\Phi_w^{-1}(t, C) \stackrel{\text{def}}{=} \{z \in \mathbb{R}^n \mid z = \Phi_w^{-1}(t, x), x \in C\}$ of any closed subset $C \subset W$ is closed. It remains to show that $\Phi_w^{-1}(t, C)$ is bounded whenever $C$ is bounded. To see that the last is true, observe that by continuous differentiability of $\Phi_w$ there exists a constant $M > 0$ such that $\|\Phi_w(t, x)\| \leq M\|x\|$ for any fixed $t \geq 0$ and all $x \in C \subset W$, where $C$ is assumed to be compact. Let $D \subset \mathbb{R}^n$ be a set containing all trajectories passing through $C$: $D \stackrel{\text{def}}{=} \{\Phi_w(t, x) \mid t \geq 0, x \in C\}$. Then $D$ is bounded by $M\|R\|$. By the definition of the inverse image and the fact that all the trajectories are periodic, $\Phi_w^{-1}(t, C) \subset D$, which ends the proof as $\Phi_w$ is shown to be proper[2]. □

[2]A map $f$ is said to be proper if the inverse image of $f$ on a compact set is compact, see definition on p. 48 of [**188**].

The time derivative of $V$ along the trajectories of a closed-loop system employing the combined control (3.11) is given by

$$\begin{aligned}\dot{V}(x,t) &= \frac{\partial V}{\partial t} + \nabla V(\Phi_w(t,x_0),t)\cdot[f_0(x) + Q_o(x)(w+\Delta u)] \\ &= \underbrace{\frac{\partial V}{\partial t} + \nabla V(\Phi_w(t,x_0),t)\cdot[f_0(x) + Q_o(x)w]}_{=0} + \nabla V(\Phi_w(t,x_0),t)\cdot Q_o(x)\Delta u \end{aligned} \tag{3.29}$$

where $\nabla V \stackrel{\text{def}}{=} \frac{\partial V}{\partial x}$ and $Q_o(x) \stackrel{\text{def}}{=} [f_1(x) \ \cdots \ f_m(x)]$. Note that the first two terms of (3.29) are zero because $w$ critically stabilizes the system. Hence, choosing

$$\Delta u = -K\left[\nabla V(\Phi_w(t,x_0),t)\cdot Q_o(x)\right]^T \tag{3.30}$$

yields

$$\dot{V}(x,t) = -K\left\|\left[\nabla V(\Phi_w(t,x_0),t)\cdot Q_o(x)\right]^T\right\|^2 \leq 0 \tag{3.31}$$

To evaluate the gradient of $V(x,t)$ in (3.30) it is convenient to let $h(x,t) = \Phi_w^{-1}(t,x)$, thus $V = \frac{1}{2}h^T h$ and

$$\nabla V(x,t) = \left.\frac{\partial V}{\partial x}\right|_{(x,t)} = h^T\left[\left.\frac{\partial h}{\partial x}\right|_{(x,t)}\right] = \left[\Phi_w^{-1}(t,x)\right]^T\left[\frac{\partial \Phi_w^{-1}(t,x)}{\partial x}\right] \tag{3.32}$$

Employing the fact that $\Phi_w(\Phi_w^{-1}(t,x),t) = x$, whose differential with respect to $x$ is given by

$$\left[\left.\frac{\partial \Phi_w}{\partial x}\right|_{(\Phi_w^{-1}(t,x),t)}\right]\left[\left.\frac{\partial \Phi_w^{-1}}{\partial x}\right|_{(x,t)}\right] = I \tag{3.33}$$

allows to obtain the following expression relating the Jacobians of the *forward* and *inverse* mappings $\Phi$ and $\Phi^{-1}$:

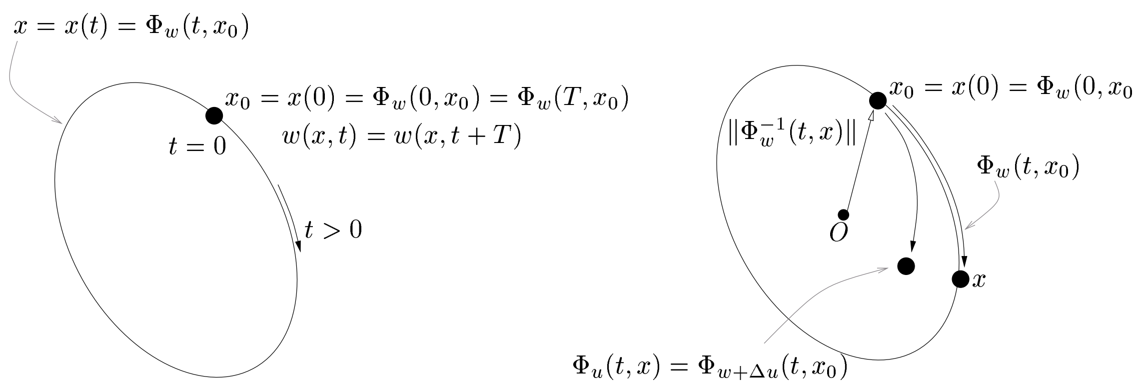
$$\left[\left.\frac{\partial \Phi_w^{-1}}{\partial x}\right|_{(x,t)}\right] = \left[\left.\frac{\partial \Phi_w}{\partial x}\right|_{(\Phi_w^{-1}(t,x),t)}\right]^{-1} \tag{3.34}$$

which substituted in (3.32) yields the final expression for the $\nabla V(x,t)$:

$$\nabla V(x,t) = \underbrace{\left[\Phi_w^{-1}(t,x)\right]^T}_{x_0^T}\left[\left.\frac{\partial \Phi_w}{\partial x}\right|_{(x_0,t)}\right]^{-1} \tag{3.35}$$

Note that since $\Phi_w^{-1}(t,x) = x_0$, the first term in (3.35) is actually the initial state from which the periodic trajectory generated by $w$ would have to start in order for it to reach the state $x$ at time $t$. The second term is the Jacobian of $\Phi_w$ with respect to variations in the initial state $x_0 = \Phi_w^{-1}(t,x)$ corresponding to the current state $x = x(t) = \Phi_w(t,x_0)$ at time $t$. It is also worth pointing out that the purpose of replacing the Jacobian of the inverse mapping $\Phi^{-1}$ in (3.32) by the inverse of the forward mapping $\Phi$ is to make more computationally tractable the actual calculation of the gradient of $V(x,t)$.

The proposed approach is illustrated in Fig. 3.1, that shows the critically stabilized system for which $x_0 = \Phi_w(0,x_0) = \Phi_w(T,x_0)$ in Fig. 3.1.a, and the trajectories of the system with the asymptotically stabilizing correction term $\Delta u(x,t)$ in Fig. 3.1.b. The decrease in $V(x,t)$ is represented in Fig. 3.1.b by the state $\Phi_{w+\Delta u}(t,x_0)$ which is "closer" to the origin as compared to the state $\Phi_w(t,x_0)$ on the closed orbit. The figure also shows that in order to compute the correction term $\Delta u(x,t)$ it is necessary to find the starting point $x_0$ of the critically stable trajectory passing through the current state $x = \Phi_w(t,x_0)$.




(a) Trajectory of the closed-loop system $\Sigma$ with critically stabilizing control $w(x,t)$.

(b) Trajectory of the closed-loop system $\Sigma$ with asymptotically stabilizing corrected control $u(x,t) = w(x,t) + \Delta u(x,t)$.

FIGURE 3.1. The continuous time-varying feedback approach based on the critical stabilization of $\Sigma$ and the calculation of an asymptotically stabilizing corrective control term.

### 3.3.3. The Combined Time-Varying Feedback Control

Under reasonable assumptions, equation (3.31) permits to prove the final stabilization result for the combined time-varying feedback control, as constructed above. The proof uses a similar argument to an analogous result of Coron and Pomet [**42**], valid for systems with no drift, and is included here for completeness.

**Theorem 3.1.** *Under the assumption that hypothesis H1, H2 and H4 are valid, the control (3.11), with* $w(x,t) = \hat{u}(\hat{v},t)$ *the solution to the (LCIP) and* $\Delta u(x,t)$ *given by (3.30), is a locally asymptotically stabilizing feedback law for system (1.1) on* $W_1 \overset{\text{def}}{=} B(0,R_1)$ *for some* $R_1 > 0$ *satisfying Proposition 3.3.*

Proof. Convergence of trajectories to zero (local asymptotic stability) will be shown invoking an extension of the LaSalle's Invariance Principle, see [**25, 40**], which applies to systems with periodic right-hand side, i.e. $f^u(x,t) = f^u(x,t+T)$, under the additional assumptions that:

$$V \in \mathcal{C}^\infty : W_1 \times \mathbb{R} \to \mathbb{R}_+ \tag{3.36}$$

$$V(x,t) > 0 \;\; \forall\, x \neq 0, \quad V(0,t) = 0 \tag{3.37}$$

$$V(x,t+T) = V(x,t) \qquad \forall\, (x,t) \in W_1 \times \mathbb{R} \tag{3.38}$$

Clearly, (3.36)–(3.38) are satisfied by virtue of the definition of the Lyapunov function $V$ in (3.26), the construction of the control (3.17), (which is smooth and, by Proposition 3.2, satisfies (3.6)–(3.7)), and the fact that the mapping $x \to \Phi_w(t,x)$ is a $\mathcal{C}^\infty$ diffeomorphism (by Proposition 3.3). Also note that due to these facts, the time-varying control $u$ in (3.11), is smooth and satisfies (3.6)–(3.7), and therefore, the right-hand side of the closed-loop system $\Sigma$ is periodic.

Since the closure of $W_1$ is compact, positive definiteness and decrescence of $V$ follows from the fact that, by construction, $V(x,t) > 0$ for all $x \neq 0$ and $V$ is continuous and therefore bounded on $W_1$. Thus, by (3.31) and a standard Lyapunov stability result, there exists a ball of radius $B(0,\delta)$ such that if $x_0 \in B(0,\delta)$ then $\Phi_u(t,x_0) \in W_1$ for $t \geq 0$.

To establish that $V$ is decreasing along the solutions of system $\Sigma$ with controls $u = w + \Delta u$ after a period $T$, i.e. that $V$ satisfies:

$$V(\Phi_u(T,x),T) < V(\Phi_u(0,x),0) \qquad \forall\, (x,t) \in W_1 \times \mathbb{R},\;\; x \neq 0 \tag{3.39}$$

note first that for a fixed $x \in W_1 \subset \mathbb{R}^n$ (3.31) implies that $V$ is non-increasing along the solutions of systems $\Sigma$. Hence,

$$V(\Phi_u(T,x),T) \leq V(\Phi_u(0,x),0) \tag{3.40}$$

where the equality occurs if and only if (3.31) is zero for all $t \in [0,T]$. From the definition of $\Delta u$ in (3.30) and the right-hand side of (3.31), observe that the equality occurs if and only if:

$$\Delta u_i = \nabla V(\Phi_u(t,x),t) f_i(\Phi_u(t,x)) = 0 \qquad \forall\, i = 1,\ldots,m,\ \forall t \in [0,T] \tag{3.41}$$

It will now be shown that the origin is the largest invariant set $M = \{0\}$ contained in the set $E = \{x : \dot{V}(x,t) = 0, x \in W_1\}$. Thus, proceeding as in [**22**], assume that(3.40) is an equality, then (3.41) implies that $u(t,x) = w(t,x)$ for all $t \in [0,T]$ and

$$\Phi_u(t,x) = \Phi_w(t,x) \qquad \forall\, t \in [0,T] \tag{3.42}$$

Let $X$ be in $\mathcal{C}^\infty : \mathbb{R}^n \to \mathbb{R}^n$. For any such $X$ one has

$$\begin{aligned}
\frac{d}{dt}\left(L_X V(\Phi_w(t,x),t)\right) &= \frac{\partial L_X V}{\partial t}(\Phi_w(t,x),t) + \left(L_{f^w} L_X V\right)(\Phi_w(t,x),t) \\
&= (L_{\mathcal{L}(u)X} V)(\Phi_w(t,x),t) + L_X \left(L_{f^w} V\right)(\Phi_w(t,x),t) \qquad (3.43) \\
&= (L_{\mathcal{L}(u)X} V)(\Phi_w(t,x),t)
\end{aligned}$$

where $\mathcal{L}(u)$ is the operator defined in (3.10), and $f^w \stackrel{\text{def}}{=} f_0 + \sum_{i=1}^m f_i w_i$ is the right-hand side of system $\Sigma$. The second expression in (3.43) is obtained by invoking the relation satisfied by the Lie derivative operator $L_f L_g - L_g L_f = L_{[f,g]}$ (see Definition A.49 on p. 215 of Appendix A) and the definition of the operator $\mathcal{L}(u)$ in (3.10). The last expression in (3.43) results from the fact that $\dot{V}(x,t) = L_{f^w} V(x,t) = 0$ along the trajectories $\Phi_w(t,x)$ of the critically stable system.

Similarly, by induction on $p$, it is possible to confirm that

$$\frac{d^p}{dt^p}\left(L_X V(\Phi_w(t,x),t)\right) = (L_{\mathcal{L}(u)^p X} V)(\Phi_w(t,x),t) \tag{3.44}$$

By (3.44), (3.42) and (3.41):

$$
\begin{aligned}
&(L_{\mathcal{L}(u)^p f_i} V)(\Phi_w(t,x),t) = \\
&\qquad \frac{\partial V}{\partial x}(\Phi_w(t,x),t)\mathcal{L}(u)^p f_i(\Phi_w(t,x)) = 0 \qquad i = 0,1,\dots,m, \quad p \geq 0, \quad \forall\ t \in (0,T)
\end{aligned}
\tag{3.45}
$$

and hence, by (3.9), there exists $t^* \in (0,T)$, such that

$$
\frac{\partial V}{\partial x}(\Phi_w(t^*,x),t^*) = 0 \tag{3.46}
$$

By virtue of (3.35) and because $\Phi_w^{-1}(t^*,\cdot)$ is a diffeomorphism, equation (3.46) implies that $\Phi_w^{-1}(t,\Phi_w(t^*,x)) = x = 0$. Hence, the $M = \{0\}$ is the largest invariant set in $E$. Lyapunov asymptotic stability of the closed-loop system $\Sigma$ with control (3.11) then follows by the LaSalle's Invariance Principle for time-varying periodic systems in Theorem 14.7, p. 64 of [**40**]. □

Remark 3.5.

(i) *The evaluation of the time-periodic control $w(x,t)$ is carried out off-line. From an implementation point of view, the continuity of $w(x,t)$ with respect to the time variable is not important, as shown by the example presented next. In many cases, finding a solution to the LCIP is easier in the class of functions which are only continuous in $x$ and piece-wise constant in $t$. It should be noted, however, that in the latter case, the Lyapunov function (3.26) is differentiable almost everywhere, with the exception of the points of discontinuity of $w$ (where the correction term $\Delta u$ would then fail to be defined). Under reasonable assumptions, the latter should not inhibit the stabilizing properties of the resulting combined time-varying control.*

(ii) *The evaluation of the correction term $\Delta u$ must be performed on-line and involves the following:*

   (a) *The reconstruction of $x_0$ according to $\Phi_w^{-1}(x,t) = x_0$ which, in theory, should be done at each point $(x(t),t)$ along the controlled system trajectory, and in practice can be done only discretely in time.*

   (b) *The calculation in (3.35) of the Jacobian of $\Phi_w(x,t)$ and its inverse.*

*From a numerical point of view, both of the above tasks are non-trivial and require efficient numerical techniques, see the example presented next for one possible approach to these on-line calculations.*

## 3.4. Example

As an example consider the following single-input dynamical system $\Sigma$, on $\mathbb{R}^3$, described by:

$$\Sigma : \qquad \dot{x} = f_0(x) + f_1(x)u_1 \stackrel{\text{def}}{=} f(x,u) \tag{3.47}$$

where,

$$f_0(x) = \begin{bmatrix} -x_2 + x_3^2 \\ -x_3 \\ 0 \end{bmatrix} \qquad f_1(x) = \begin{bmatrix} 0 \\ 2x_3 \\ 1 \end{bmatrix}$$

It can be verified that system (3.47) fails to satisfy Brockett's conditions for smooth stabilizability given in Theorem 1.1 on p. 10. In fact, the system has an uncontrollable linearization and moreover, for any $\epsilon \neq 0$, points of the form $[-x_2 \ \epsilon \ 0]^T$ in a neighbourhood of 0 do not belong to the image $f$. This is because $f(x,u) = [-x_2 \ \epsilon \ 0]^T$ implies that $x_3 = 0$ and $u = 0$, but then $f(x,u) \equiv 0$. Consequently, system (3.47) cannot be asymptotically stabilized to the equilibrium point $x_e = 0$ by a $\mathcal{C}^1$ static state feedback.

However, this system satisfies the LARC and is nilpotent with $\dim L(\mathcal{F}) = 4$ as shown by the following Lie bracket multiplication table:

| | $f_0$ | $f_1$ | $f_2$ | $f_3$ |
|---|---|---|---|---|
| $f_0$ | 0 | $f_2$ | $f_3$ | 0 |
| $f_1$ | | 0 | 0 | 0 |
| $f_2$ | | | 0 | 0 |
| $f_3$ | | | | 0 |

with

$$f_2 \stackrel{\text{def}}{=} [f_0, f_1] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad f_3 \stackrel{\text{def}}{=} [f_0, [f_0, f_1]] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{3.48}$$

Letting $\mathcal{G} = \{g_0, g_1, g_2, g_3\} = \{f_0, f_1, f_2, f_3\}$ the extended system $\Sigma_e$ is given by:

$$\Sigma_e : \qquad \dot{x} = g_0(x) + \sum_{i=1}^{3} g_i(x) v_i$$

for which the vector fields $g_i = f_i$, $i = 2, 3$, correspond to Lie brackets generated by $f_0$ and $f_1$.

It can be verified that the $\gamma$-coordinate equation (2.17) for the extended system is given by:

$$\dot{\gamma}^e(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\gamma_0^e(t) & 1 & 0 \\ 0 & \frac{\gamma_0^e(t)^2}{2} & -\gamma_0^e(t) & 1 \end{bmatrix} v^d \tag{3.49}$$

with initial conditions $\gamma_i^e(0) = 0$, $i = 0, 1, 2, 3.$ , and $v^d = [1\ \hat{v}]^T$.

The control that produces periodic trajectories for the extended system (generates trivially periodic orbits) can be found by solving:

$$\dot{x}|_{x_0} = f_0(x_0) + Q(x_0)\,\hat{v} = 0 \tag{3.50}$$

where

$$Q(x_0) = [f_1(x_0)\ f_2(x_0)\ f_3(x_0)]$$

$$\hat{v} = [\hat{v}_1\ \hat{v}_2\ \hat{v}_3]^T \text{ and } x_0 = x(0)$$

so that,

$$\hat{v} = -Q(x_0)^{-1} f(x_0) \tag{3.51}$$

which is constant for a given initial condition $x_0$, allowing an easy integration of the equation in the $\gamma$-coordinates. For $\Sigma^e$, the control vector is calculated to be $\hat{v} = v(x_0) = [0,\ x_3(0),\ x_2(0) - x_3(0)^2]^T$, and the integration of (3.49) at time $t = T$ yields:

$$\begin{aligned}
\gamma_0^e(T) &= T \\
\gamma_1^e(T) &= 0 \\
\gamma_2^e(T) &= \int_0^T -\gamma_0^e(t)\gamma_1^e(t) + \hat{v}_2\, dt \;=\; x_3(0)T && (3.52) \\
\gamma_3^e(T) &= \int_0^T \frac{\gamma_0^e(t)^2}{2}\hat{v}_1(t) - \gamma_0^e(t)\hat{v}_2 + \hat{v}_3\, dt && (3.53) \\
&= (x_2(0) - x_3(0)^2)T - x_3(0)\frac{T^2}{2}
\end{aligned}$$

Hence,

$$S^e(t) = \prod_{i=0}^{3} \exp(\gamma_i^e(t)g_i) = \exp(\gamma_0^e(t)g_0)\exp(\gamma_2^e(t)g_2)\exp(\gamma_3^e(t)g_3)$$

Using the CBH formula (see equation (A.13), p. 219 of Appendix A),

$$S^e(t) = \exp\left(\gamma_0^e(t)g_0 + \gamma_2^e(t)g_2 + \left(\frac{\gamma_0^e(t)\gamma_2^e(t)}{2} + \gamma_3^e(t)\right) g_3\right) \tag{3.54}$$

A particular piece-wise constant solution to the TIP defined as a sequence of constant controls $\bar{w} = [w_1, w_2, w_3]$, each applied to $\Sigma$ for $\epsilon$ units of time with $\epsilon = T/3$, is found by employing the CBH formula (A.13) to obtain the expression for $\sum_{i=0}^{3} c_i(\bar{w}, \epsilon)g_i$ in

$$S(T) = \prod_{i=0}^{3} \exp\left(\epsilon f^{w_i}\right) = \exp\left(\sum_{i=0}^{3} c_i(\bar{w}, \epsilon)g_i\right) \tag{3.55}$$

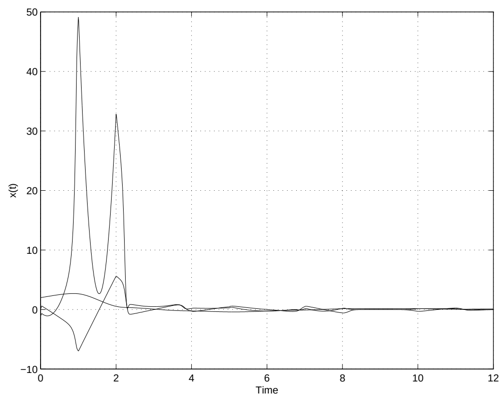
such that $S(T) = S^e(T)$, where $f^{w_i}$, $i = 1, 2, 3$, is the right-hand side of $\Sigma$, (1.1), with constant control $w_i$. Equating the coefficients $c_i$ of $g_i$, $i = 0, 1, 2, 3$, in (3.55) to those of $g_i$ in (3.54) and then solving for the constant controls $w_i$ yields the sequence of controls $\bar{w}(v(x_0), T) = \{w_1(v(x_0), T), w_2(v(x_0), T), w_3(v(x_0), T)\}$. The periodic continuation of $w(v(x_0), t + nT) = \bar{w}(v(x_0), T)$ which solved the TIP is thus found to be:

$$w(v(x_0), t + nT) = \begin{cases} \frac{9x_3(0)}{2T} + \frac{27}{T^2}\left(x_2(0) - x_3(0)^2\right) & nT \le t < nT + \epsilon \\ -\frac{54}{T^2}\left(x_2(0) - x_3(0)^2\right) & nT + \epsilon \le t < nT + 2\epsilon \\ -\frac{9x_3(0)}{2T} + \frac{27}{T^2}\left(x_2(0) - x_3(0)^2\right) & nT + 2\epsilon \le t < nT + 3\epsilon \end{cases} \tag{3.56}$$

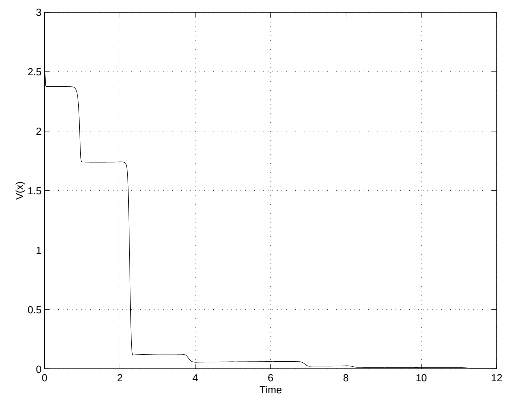
$$\text{with } \epsilon = \frac{T}{3}, \quad \text{forall } n \in \mathbb{Z}_+$$

Remark 3.6. *Notice that the above solution corresponds to solving the TIP as a flow interception problem in which controls u and v for the original and extended system are sought such that* $S(T) = S^e(T)$. *The above problem could also have been solved as a logarithmic coordinate interception problem by integrating the* $\gamma$*-coordinates equation (2.17) corresponding to system* $\Sigma$ *with control* $u^d = [1\ \bar{w}\ 0\ \ldots\ 0]^T \in \mathbb{R}^r$*, and then equating the resulting* $\gamma$*-coordinates at time* $T$ *to those of* $\Sigma^e$ *given in (3.52).*

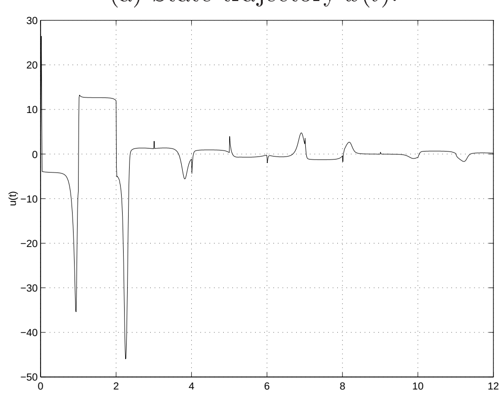
The simulation results obtained by applying the proposed stabilizing feedback control calculated with $T = 3$, $K = 100$ to system (3.47) starting from $x_0 = [2\ \ -1\ .1]^T$ are shown in Figure 3.2.



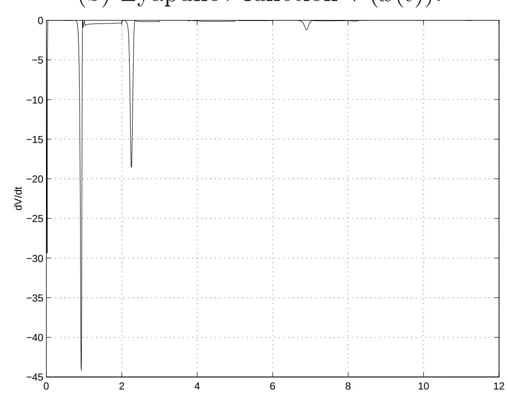

(a) State trajectory $x(t)$.




(b) Lyapunov function $V(x(t))$.




(c) Stabilizing feedback $u(x, t)$.




(d) Lyapunov function time derivative $\dot{V}(t)$.

Figure 3.2. Results for the stabilization of the closed-loop system (3.47) with the control (3.11).

It is worth noting that efficient stabilization is achieved despite the fact that the time periodic, critically stabilizing control $w$ is only piece-wise continuous (and not continuously differentiable, as required by the TIP formulation).

Stabilization is also achieved despite that hypothesis H4 is not satisfied.

The numerical evaluation of the gradient involves “retrieving” the starting point $x_0$ at time $t$ of the corresponding orbit generated with the control $w$ which passes through the current state $x(t)$. The problem of numerically finding $x_0$ is solved here by seeking:

$$x_0 = \arg\min_{x_0 \in \mathbb{R}^n} \|x(t) - \Phi_w(t, x_0)\|^2 \tag{3.57}$$

The Levenberg-Marquardt modification of the Gauss-Newton method is employed as minimization procedure.

The gradient $\nabla V$ is calculated here by using finite difference approximations to the partial derivatives needed.

# CHAPTER 4

## Discontinuous Time-Varying Feedback Approaches

This chapter presents two related approaches to the construction of stabilizing feedback controls for strongly nonlinear systems, [**79, 80, 82, 83**]. The class of systems of interest includes systems with drift which are affine in control and which cannot be stabilized by continuous state feedback. The approaches are independent of the selection of a Lyapunov type function, but require the solution of a nonlinear programming *satisficing problem* which is stated in terms of the logarithmic coordinates of flows, or in terms of expressions resulting from the composition of flows using the Campbell-Baker-Hausdorff (CBH) formula. As opposed to other approaches, point-to-point steering is not required to achieve asymptotic stability. Instead, the flow of the controlled system is required to intersect periodically a certain reachable set in the space of the logarithmic coordinates.

### 4.1. Introduction

This chapter presents two approaches to the design of feedback stabilizing controls for systems with drift of the form of system $\Sigma$ in equation (1.1).

The proposed approaches are a combination of a smooth and a discontinuous time-varying state feedback law. In most applications the control law is sought in the class of functions with some desirable degree of smoothness. As earlier mentioned, systems of the type of system $\Sigma$ may however not be smoothly stabilizable, and therefore, time-varying or discontinuous feedback laws are required. Since time-varying and discontinuous control laws are: (a) harder to design, (b) significantly more complex than smooth static feedback laws and (c) result in highly oscillatory motions, a reasonable approach is to apply a general smooth feedback control as much as possible (i.e. for all those states where the smooth feedback can be defined), in combination with a discontinuous time-varying control. What is exactly meant by "as much as possible" will become clear soon, once the general smooth feedback law is presented.

The construction of the smooth feedback law is based on standard Lyapunov techniques, cf. [**24, 26**], and considers an arbitrary Lyapunov function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$. The smooth feedback law is defined as

$$u(x) \quad \stackrel{\text{def}}{=} \quad \frac{-a(x) - k(x)}{\|b(x)\|^2} b(x)^T \tag{4.1}$$

for all $x \notin E^b \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid b(x) = 0\}$, where $k(x) : \mathbb{R}^n \rightarrow \mathbb{R}+$ is a $\mathcal{C}^p$, $p \geq 1$, positive definite function such that $k(0) = 0$ and

$$\begin{aligned} u(x) \quad &= \quad [u_1(x) \; u_2(x) \; \cdots \; u_m(x)]^T \\ a(x) \quad &\stackrel{\text{def}}{=} \quad L_{f_0} V(x) \\ b(x) \quad &\stackrel{\text{def}}{=} \quad [L_{f_1} V(x) \; L_{f_2} V(x) \; \cdots \; Lf_m V(x)] \end{aligned}$$

In the above expressions, $L_{f_i} V \stackrel{\text{def}}{=} \frac{\partial V}{\partial x} f_i$ denotes the Lie derivative of the chosen Lyapunov function $V$ along the vector field $f_i$.

It can be verified that the time derivative of the Lyapunov function along trajectories the closed-loop system with feedback control (4.1) satisfies:

$$\dot{V} = -k(x) < 0 \qquad \forall \; x \notin E^b \tag{4.2}$$

and hence, (4.1) guarantees the decrease of $V$ for all $x \notin E^b$. Clearly, the feedback (4.1) can be applied as long as the system's state is not in the set $E^b$, in which the input is singular. Under the assumption that the system $\Sigma$ fails to satisfy Brockett's condition for smooth stabilizability, it will be impossible to find a smooth function $V$ such that the set $E^b$ is empty, and therefore, a time-varying or discontinuous feedback law will inevitably be required instead of the smooth control (4.1). Since the design of (4.1) is based on the well known Lyapunov approach, the attention in this chapter is placed on the design of discontinuous time-varying control laws. The interested reader is referred to [**24, 11, 73, 6, 2**] for details on different results concerning the standard Lyapunov techniques and the control law (4.1). It is worth pointing out that universal formulas which have the form of (4.1) for stabilization of systems of the type $\Sigma$ are studied in [**73**] based on ideas that can be traced back to the approaches described in [**11**] and other references therein.

In the above context, the contributions of this chapter can be described as follows. Two approaches to the stabilization of general systems of the form (1.1) are presented. The methods are based on:

- **Method 1:** The explicit calculation of a reachable set of desirable states for the controlled system. Such reachable set is determined when the system $\Sigma$ is reformulated as a right-invariant system on an analytic, simply connected, nilpotent Lie group and once the stabilization problem is re-stated accordingly. The reformulation allows for the time-varying part of the stabilizing feedback control to be derived as the solution of a nonlinear programming problem for steering the open-loop system $\Sigma$ to the given reachable set of states.
- **Method 2:** The generation of system motions along vector fields which span the controllability Lie algebra of system $\Sigma$. This is accomplished by composing flows arising from the application of piece-wise constant controls and the use of the Campbell-Baker-Hausdorff (CBH) formula. The values of the controls are calculated as the solution of a nonlinear programming problem which guarantees that a given Lyapunov type condition is satisfied.

The construction of the feedback laws does not require numerical integration of the model differential equation and is independent of the choice of the Lyapunov type function.

The proposed approaches might prove useful for the construction of feedback laws with a reduced number of control discontinuities and for the development of computationally feasible methods toward the design of smooth time-varying stabilizing feedback. This is because, in Method 1, the system is shown to remain asymptotically stable provided that the state of the system traverses the sets of desirable states periodically in time. Since the sets of desirable states are typically large, the latter condition leaves much freedom for improved design. A similar argument is also valid for Method 2 in which the requirement is that the controls periodically generate motions that satisfy a given Lyapunov type condition.

Examples are presented in the last sections of this chapter. The examples employ the software package described in Chapter 6 that was developed in Maple to facilitate the tedious symbolic Lie algebraic calculations involved and that would otherwise be very difficult to carry out by hand.

The approaches presented are based on a different concept than those used in [**68, 64, 72**] and compares favourably with the previous methods proposed in [**75, 77**]. The advantages of the proposed approaches are in that:

- They do not require the exact analytic solution of the trajectory interception problem for the flows for the original and extended system, as in [**75**].
- They also avoid the expensive on-line computation of the value and the gradient of the Lyapunov function whose level sets are the trajectories of the critically stabilized system, as required by the method presented in Chapter 3, [**77**].
- Furthermore, they should be easier to apply to systems with larger dimension of the state or controllability Lie algebra than the methods in [**68, 64**].

## 4.2. Problem Definition and Assumptions

**Problem Definition.** *The objective is to construct time-varying feedback controls* $u_i(x,t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$, $i = 1, \ldots, m$, *which globally stabilize system* $\Sigma$ *to the origin.*

It is assumed that system $\Sigma$ satisfies hypotheses H1–H3 stated in Chapter 2, p. 26.

The methods presented here employ an arbitrary Lyapunov type function $V : \mathbb{R}^n \to \mathbb{R}_+$ which is only required to satisfy the following conditions:

H7.a. $V$ is twice continuously differentiable with $V(0) = 0$. Additionally, there exists a constant $\zeta > 0$, such that for all $x \in \mathbb{R}^n$, $\|\nabla V(x)\| \geq \zeta \|x\|$.

H7.b. $V$ is positive definite and decrescent, i.e. there exist continuous, strictly increasing functions $\alpha(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ and $\beta(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, with $\alpha(0) = \beta(0) = 0$, such that for all $x \in \mathbb{R}^n$, $\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|)$.

Remark 4.1. *The function $V$ is not a Lyapunov function in the usual sense in that it will be allowed to increase instantaneously along the trajectories of the stabilized system. However, in the sequel, the function $V$ will still be referred to as a Lyapunov function.*

## 4.3. Stabilizing Feedback Control Design

Generally, a control Lyapunov function which leads to a smooth feedback control law for system $\Sigma$ is not guaranteed to exist. An alternative idea for the construction of a stabilizing feedback relies on achieving a periodic decrease in an arbitrarily imposed Lyapunov function $V$, through the action of a time-varying control. Periodic decrease is defined in terms of the condition $V(x(t_0 + kT)) - V(x(t_0 + (k-1)T)) < 0$, which is required to hold for all $k \in \mathbb{N}$ and where $x(t)$, $t \geq t_0$, is the trajectory of the closed-loop system and $T > 0$ is the "period" of decrease.

The following sections present two different methods to construct a feedback control which guarantees a periodic decrease in $V$. It is convenient for sake of clarity to motivate the basic idea underlying both approaches as follows, before presenting a rigorous exposition of each of them.

The traditional Lyapunov approach to the problem of stabilizing system $\Sigma$, in which a control $u : \mathbb{R}^n \ni x \to u(x) \in \mathbb{R}^m$ is sought such that a given control Lyapunov function $V$ decreases instantaneously along some vector field $f^u \in \operatorname{span} \mathcal{F}$, cannot be applied as there might not exist a control $u$ such that $\dot{V} = \nabla V f^u(x) < 0$ for some $x \in \mathbb{R}^n$. However, if vector fields $\bar{f} \in \operatorname{span} L_x(\mathcal{F})$ are also considered, where by the controllability hypothesis H2 and H3, $\operatorname{span} L_x(\mathcal{F}) = \mathbb{R}^n$, for all $x \in B(0, R)$, then it is always possible to find some motion within $B(0, R)$ along $\bar{f}$ such that $\dot{V} = \nabla V \bar{f} < 0$, thus resulting in a decrease in $V$, see Fig. 4.1. The latter condition does not imply an instantaneous decrease, as $\bar{f}$ is realized through switching controls $\bar{u} \in \mathcal{P}^m$ applied to $\Sigma$, but it will be shown that for a suitably selected time interval $T > 0$ and for any $t \in \mathbb{R}_+$ this condition ensures that $\int_t^{t+T} \dot{V} d\tau = \int_t^{t+T} \nabla V f^{\bar{u}} d\tau = V(x(t+T)) - V(x(t)) < 0$, implying a periodic decrease in $V$. The core problem is thus to devise a method for the generation of motions along $\bar{f} \in \operatorname{span} L(\mathcal{F})$. The two methodologies proposed to produce such motions draw on the following ideas:

- **Method 1: based on the solution of a relaxed interception problem in the $\gamma$-coordinates that parametrize the flows of system $\Sigma$.** This method exploits the fact that the solutions of both system $\Sigma$ and its Lie algebraic extension $\Sigma^e$, in equation (2.1), can be written as products of exponentials of the basis elements of the Lie algebra $L(\mathcal{F})$. The solution of system $\Sigma$ as a product of exponentials is given by $x(T) = \prod_{i=0}^{r-1} \exp(\gamma_i(\bar{u})g_i)x(0)$, while a similar representation, $x^e(T) = \prod_{i=0}^{r-1} \exp(\gamma_i^e(v)g_i)x(0)$ exists for system $\Sigma^e$. Therefore, the stabilization problem can be solved by first finding an extended control $v$ that stabilizes $\Sigma^e$ and that gives rise to $\gamma$-coordinates $\gamma_i^e$, and then finding a control $\bar{u} \in \mathcal{P}^m$ such that the $\gamma$-coordinates $\gamma_i$ of the original system match those of the extended system $\Sigma^e$ at time $T$, i.e. $\gamma_i(T) = \gamma_i^e(T)$, thus implying that $x(T) = \prod_{i=0}^{r-1} \exp(\gamma_i(\bar{u})g_i)x(0) = \prod_{i=0}^{r-1} \exp(\gamma_i^e(v)g_i)x(0) = x^e(T)$ and that the trajectories of the system $\Sigma$ *follow* those of the stable system $\Sigma^e$ at least periodically.

- **Method 2: based on the computation of motions $\bar{f} \in$ span $L(\mathcal{F})$ employing the CBH formula.** In this method a sequence $\bar{u} = [u_{(1)}, u_{(2)}, \ldots, u_{(s)}]$ of constant controls $u_{(i)} \in \mathbb{R}^m$, $i = 1, \ldots, s$, where $s$ is the number of switches, is applied to system $\Sigma$. Each control $u_{(i)}$ is applied for an equal interval of time $\varepsilon$, such that $T = s\varepsilon$. The trajectory of system $\Sigma$ at time $T$ is expressed as the concatenation of trajectories resulting from the application of each control $u_{(i)}$, $i = 1, \ldots, s$, given by $x(T) = \exp(\varepsilon f^{u_{(1)}}) \circ \cdots \circ \exp(\varepsilon f^{u_{(s)}})x(0)$ and can be written in terms of a single exponential $\exp(T\bar{f}) = \exp\left(T \sum_{i=0}^{r-1} c_i(\bar{u})g_i\right)$, where the scalar coefficients $c_i$ are obtained using the CBH formula and the vector fields $g_i$ form a basis of $L(\mathcal{F})$. A desired direction of system motion $\bar{f}$ can then be realized to decrease $V$ by adequately "selecting" the coefficients $c_i$.

Notice that in the first method the trajectory at time $T$ is given by an expression of the form $x(T) = \prod_{i=0}^{r-1} \exp(b_i g_i)x(0)$, while in the second method $x(T) = \exp(\sum_{i=0}^{r-1} a_i g_i)x(0)$. The latter representation can be associated with the Magnus representation [**145**] (where $a_i$ are Lie-Cartan coordinates of the first kind), while the former is known as Wei-Norman representation [**149**] (where $b_i$ are Lie-Cartan coordinates of the second kind).

The concepts presented in Chapter 2 are essential in the reformulation of system $\Sigma$ as a right-invariant system on the analytic, simply connected, nilpotent Lie group $H$. This formalism is convenient for the development of the proofs concerning the properties of the time-varying discontinuous control $\bar{u} \in \mathcal{P}^m$ obtained by either of the two methods.

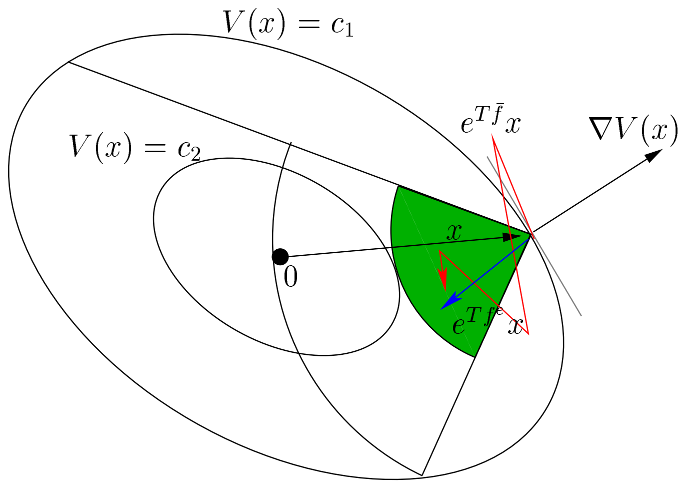



FIGURE 4.1. Cone of directions in which the Lyapunov function $V$ decreases (instantaneously along the extended system trajectory $e^{Tf^e}x$ and after a period $T$ along the original system trajectory $e^{T\bar{f}}x$).

### 4.3.1. Method 1: Stabilizing Feedback Design in the $\gamma$-Coordinates

In Method 1, the construction of a feedback control which guarantees a periodic decrease in $V$ is more conveniently achieved by re-stating the stabilization problem on the Lie group $H$, as explained in Chapter 2. Since $\Sigma^e$ is instantaneously controllable in any direction of the state space, it is helpful to employ the extended system as the first instrument to achieve such a decrease. To this end, for any $x \in B(0,R)$ let a set $U^e(x)$ of admissible extended controls be introduced as follows:

$$U^e(x) \quad \stackrel{\text{def}}{=} \quad \left\{v \in \mathbb{R}^{r-1} \mid \nabla V g^v(x) < -\eta\|x\|^2,\ \|v\| \leq M\|x\|\right\} \tag{4.3}$$

where the constant $R > 0$ is sufficiently large to accommodate for all initial conditions of interest, and $M > 0$ is to be chosen later. The set $U^e(x)$ translates into a reachable set of states of the extended system $\Sigma^e$ at time $T$, $\mathcal{R}_{\mathcal{G}}(T, x, U^e(x))$:

$$\mathcal{R}_{\mathcal{G}}(T, x, U^e(x)) \stackrel{\text{def}}{=} \{z \in \mathbb{R}^n \mid z = x^e(T, x, v),\ v \in U^e(x)\} \tag{4.4}$$

where $x^e(T, x, v)$ denotes the trajectory of $\Sigma^e$ emanating from $x$ at time $t = 0$ and resulting from the application of the control $v$ over the time interval $[0, T]$. The reason for introducing $U^e(x)$ is explained in terms of the following result, Proposition 4.1, which is analogous to that of Proposition 4.5 for Method 2, but applies to the extended system $\Sigma^e$ rather than system $\Sigma$. It will be shown later in

Proposition 4.3 that an equivalent condition to that of Proposition 4.1 also holds for the original system $\Sigma$, i.e. that Proposition 4.5 for Method 2 is also ensured by the control $\bar{u} \in \mathcal{P}^m$ obtained using Method 1. Even if Proposition 4.3 for Method 1 and Proposition 4.5 for Method 2 state the same result, their proofs differ. For this reason, and for clarity of exposition, it is deemed convenient to repeat the proposition statements.

PROPOSITION 4.1. *Under hypotheses H1–H3 and H7 there exists a time horizon $T_{max} > 0$ such that for all $t > 0$ and $T \in [0, T_{max}]$:*

$$V(z) - V(x) \leq -\frac{\eta}{2}\|x\|^2 T \tag{4.5}$$

*for all $z \in \mathcal{R}_{\mathcal{G}}(T, x, U^e(x))$.*

PROOF. Since $V$ is twice continuously differentiable, $g^v$ is analytic and linear in $v$, and $g_0(0) = 0$, then $\nabla V$ and $g^v$ are Lipschitz continuous on $B(0, 2R)$, uniformly with respect to $v = [v_1, \ldots, v_{r-1}]^T$ satisfying $\|v\| \leq M\|x\|$. Hence, there exists a $K > 0$ such that:

$$\|g^w(y) - g^v(x)\| \leq K\|y - x\| \quad \text{and} \quad \|\nabla V(y) - \nabla V(x)\| \leq K\|y - x\|$$

for all $x \in B(0, R)$, $y \in B(0, 2R)$, and for any constant $v$ and $w$ such that $\|v\| \leq M\|x\|$ and $\|w\| \leq M\|x\|$.

Let $x^e(t) \stackrel{\text{def}}{=} x^e(t, x, v)$, $t \geq 0$. First, it is shown that there exists a $T_1 > 0$ and a constant $K_1 > 0$ such that

$$\|x^e(s) - x\| \quad \leq \quad \|x\| \left(\exp(K\, s) - 1\right) \tag{4.6}$$

and

$$\|x^e(s)\| \leq K_1\|x\| \tag{4.7}$$

for all $s \in [0, T_1]$ such that $x^e(s) \in B(0, 2R)$.

To this end it suffices to notice that

$$\|x^e(s) - x\| \;\leq\; \int_0^s \|g^v(x)\| d\tau + \int_0^s \|g^v(x^e(\tau)) - g^v(x)\| d\tau \;\leq\; K\,\|x\|\,s + \int_0^s K\|x^e(\tau) - x\| d\tau$$

which, by the application of the Gronwall-Bellman lemma in Appendix C.1, p. 244, yields inequality (4.6).

It can be shown that if $T_1$ is chosen so that $(\exp(K\,T_1) - 1) \leq \frac{1}{2}$ then (4.6) holds for $s \in [0, T_1]$. By contradiction, suppose that there exists an $s_1 < T_1$ such that $\|x^e(s_1)\| = 2R$. It follows that $2R \leq \|x\| + \|x^e(s_1) - x\| \leq R + \|x\|\,(\exp(K\,s_1) - 1) \leq \frac{3}{2}R$ which is false, and hence (4.6) is valid for $s \in [0, T_1]$. Inequality (4.7) follows from (4.6) since

$$\|x^e(s)\| \;\leq\; \|x^e(s) - x\| + \|x\| \;\leq\; \|x\| \exp(K\,s) \;\leq\; K_1\|x\|$$

with $K_1 = \exp(KT_1)$.

Now,

$$\begin{aligned} V(x^e(T)) - V(x) \quad &\leq \quad \nabla V(x) g^v(x) T + \int_0^T \|\nabla V(x^e(s)) g^v(x^e(s)) - \nabla V(x) g^v(x)\|\, ds \\ &\leq \quad -\eta\,\|x\|^2\, T + \bar{K}\|x\| \int_0^T \|x^e(s) - x\|\, ds \end{aligned} \tag{4.8}$$

because

$$\begin{aligned} \|\nabla V(x^e(s)) g^v(x^e(s)) - \nabla V(x) g^v(x)\| \quad &\leq \quad \|\nabla V(x^e(s)) g^v(x^e(s)) - \nabla V(x^e(s)) g^v(x)\| \\ &\qquad + \|\nabla V(x^e(s)) g^v(x) - \nabla V(x) g^v(x)\| \\ &\leq \quad \bar{K}\|x\|\,\|x^e(s) - x\| \end{aligned}$$

with $\bar{K} = K^2(K_1 + 1)$.

Hence, if $T < T_1$ then $x^e(s) \in B(0, 2R)$ for all $s \in [0, T]$ and, using (4.6) in (4.8), yields

$$V(x^e(T)) - V(x) \quad \leq \quad -\eta\,\|x\|^2\, T + \bar{K}\|x\|^2 \int_0^T (\exp(K\,s) - 1)\, ds \;\leq\; -\frac{\eta}{2}\|x\|^2 q(T)$$

where $q(T) \stackrel{\text{def}}{=} \left(2 + \frac{2\bar{K}}{\eta}\right) T - \frac{2\bar{K}}{\eta K}(\exp(K\,T) - 1)$. If $r(T) \stackrel{\text{def}}{=} q(T) - T$, then $r(0) = 0$ and $r'(0) \stackrel{\text{def}}{=} \frac{dr}{dT}\big|_{T=0} = 1$, so there exists a $T_{max} \leq T_1$ such that $r(T) \geq 0$ for all $T \in [0, T_{max}]$. Hence $q(T) \geq T$ for all $T \in [0, T_{max}]$ which proves (4.5). $\square$

In this context it is desirable to construct an open-loop control for system $\Sigma$ which guarantees an equivalent decrease in the Lyapunov function as stated by (4.5). This can be achieved by the construction of any $\bar{u} \in \mathcal{P}^m$ which ensures:

$$x(T, x, \bar{u}) \in \mathcal{R}_{\mathcal{G}}(T, x, U^e(x)) \tag{4.9}$$

The above is a control problem for which the terminal constraint set has no direct characterization. However, when (4.9) is re-stated on the Lie group $H$ it translates into a computationally feasible nonlinear programming problem which can be formulated in terms of the $\gamma$-coordinates. This is done as follows.

By virtue of the definitions in Section 4.2 the reachable set $\mathcal{R}_{\mathcal{G}}(T, x, U^e(x))$ is the orbit $G(T, U^e(x))\, x = \mathcal{R}_{\mathcal{G}}(T, x, U^e(x))$ where

$$G(T, U^e(x)) \stackrel{\text{def}}{=} \{\exp(T g^v) \mid v \in U^e(x)\} \subset G \tag{4.10}$$

Also

$$H(T, U^e(x)) \stackrel{\text{def}}{=} \{S^e(T, v) \mid v \in U^e(x)\} = (\phi_G^+)^{-1}\left(G(T, U^e(x))\right) \subset H \tag{4.11}$$

where $S^e(T, v)$ denotes the value of the solution to equation (2.9) at time $T$ and due to extended control $v$. When expressed in the global coordinate system, (2.10), each element of $H(T, U^e(x))$ has the representation:

$$S^e(T, v) = \prod_{i=0}^{r-1} \exp(\gamma_i^e(T, v^d)\psi_i) \tag{4.12}$$

where $\gamma^e(T, v^d) \stackrel{\text{def}}{=} [\gamma_0^e, \ldots, \gamma_{r-1}^e](T, v^d)$, is the value of the solution to equation (2.17) at time $T$ and due to control $v^d = [1\ v]^T$ with $v \in U^e(x)$. Since

$$x(T, x, \bar{u}) = \phi_G^+\left(\prod_{i=0}^{r-1} \exp(\gamma_i(T, \bar{u}^d)\psi_i)\right) x \tag{4.13}$$

where $\gamma(T,\bar{u}^d) \stackrel{\text{def}}{=} [\gamma_0,\ldots,\gamma_{r-1}](T,\bar{u}^d)$ is the value of the solution to equation (2.17) at time $T$ and due to control $\bar{u}^d = [1\ \bar{u}\ 0\ \ldots\ 0]^T \in \mathcal{P}^r$ with $\bar{u} \in \mathcal{P}^m$, then (4.9) holds if

$$\prod_{i=0}^{r-1} \exp(\gamma_i(T,\bar{u}^d)\psi_i) \in H(T,U^e(x)) \tag{4.14}$$

Due to the representation (4.12), it hence follows that (4.9) holds if

$$\gamma(T,\bar{u}^d) \in \mathcal{R}_\gamma(T,U^e(x)) \tag{4.15}$$

where

$$\mathcal{R}_\gamma(T,U^e(x)) \quad \stackrel{\text{def}}{=} \quad \{\gamma^e(T,v^d) \mid v \in U^e(x)\} \tag{4.16}$$

For any *constant* control $v^d = [1,v] \in \mathbb{R}^r$ equation (2.17) can be integrated symbolically to yield $\gamma^e(T,v^d) = \int_0^T \Gamma^{-1}(\gamma^e(\tau,v^d))d\tau\, v^d \stackrel{\text{def}}{=} M(T)\, v^d$. Since $\Gamma$ is nonsingular and triangular for any of its arguments $\gamma^e$, then $M(T)$ is also nonsingular and triangular for any integrated trajectory $\gamma^e(\tau,v^d)$, $\tau \in [0,T]$, with a constant $v^d$. Since one of the components of $v^d$ is equal to one (due to the presence of the drift vector field), an *analytic expression* can be derived for the inverse mapping $F : \mathbb{R}^r \times \mathbb{R} \to \mathbb{R}^{r-1}$ such that $v = F(\gamma^e(T,v^d),T)$.

It follows that

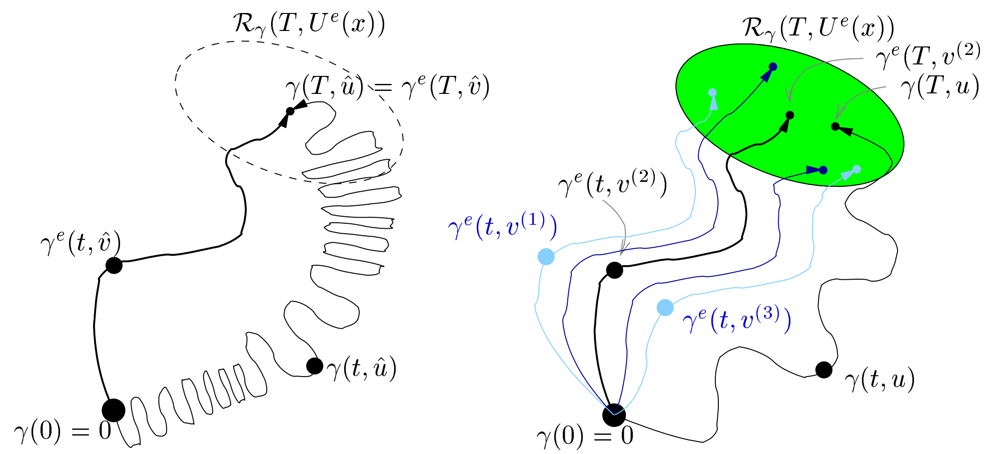
$$\mathcal{R}_\gamma(T,U^e(x)) \quad = \quad \{\gamma \in \mathbb{R}^r \mid F(\gamma,T) \in U^e(x)\} \tag{4.17}$$

which is given explicitly and permits to express (4.9) as a nonlinear programming problem with respect to the variables parametrizing the piece-wise constant control $\bar{u}$. Assuming that such a parametrization is given by $u_{(k)}$, $k = 1,\ldots,s$, $s \in \mathbb{N}$, so that

$$\bar{u}(\tau) \ \stackrel{\text{def}}{=}\ u_{(k)}, \quad \tau \in [t_k, t_k+\varepsilon), \quad \varepsilon s = T \text{ and } t_1 = 0,\ t_k = t_{k-1}+\varepsilon,\ k = 1,2,\ldots,s \tag{4.18}$$

the solution of (2.17), $\gamma(T,\bar{u}^d)$, is also parametrized by $u_{(k)}$, $k = 1,\ldots,s$.

Before stating the nonlinear programming problem corresponding to (4.9) is worth noting that (4.9) can be regarded as a *relaxed trajectory interception problem*, since the original system trajectory must reach at time $T$ *any* state in the set of states attainable at time $T$ by the extended system using controls $v \in U^e$, instead of *exactly matching* a given state of system $\Sigma^e$. By the above discussion, the relaxed TIP translates accordingly into a *relaxed logarithmic coordinate interception problem* (relaxed LCIP), shown in Fig. 4.2. The freedom to reach any $\gamma$-coordinate in $\mathcal{R}_\gamma(T, U^e)$ as depicted in Fig. 4.2, instead of requiring an exact end-point matching of the $\gamma$-coordinates such that $\gamma(T) = \gamma^e(T)$ as in Fig. 2.2, is what makes of this approach to be computationally feasible as compared to the *exact*, but more restrictive, LCIP in (2.22).




(a) The exact trajectory interception problem in the logarithmic coordinates.

(b) The logarithmic coordinate satisficing problem.

FIGURE 4.2. The "exact" and the "relaxed" trajectory interception problems in the logarithmic coordinates of flows are shown in figures (a) and (b), respectively. In the above diagrams, $\gamma(t, u)$ and $\gamma^e(t, v)$ denote the logarithmic coordinates of systems $\Sigma$ and $\Sigma^e$ at time $t$ and due to controls $u$ and $v$, respectively. In contrast to the exact trajectory interception problem, in which the logarithmic coordinates of both systems $\Sigma$ and $\Sigma^e$ must match *exactly* at time $T$, the satisficing problem simply requires that $\gamma(T, u)$ belongs to the reachable set $\mathcal{R}_\gamma(T, U^e(x)) = \{\gamma^e(T, v) \mid v \in U^e(x)\}$ in equation (4.16). It thus is conceivable that $\gamma(t, u)$ is *smoother* than $\gamma(t, \hat{u})$.

The nonlinear programming problem equivalent to (4.9) is now stated as the following *satisficing problem* (SP1) for Method 1:

**SP1:** For given constants $\eta > 0$, $T > 0$ and $M > 0$, and for $x \in B(0, R)$,

find feasible parameter vectors $u_{(k)}$, $k = 1, \ldots, s$, such that:

$$\gamma(T, \bar{u}^d) \in \mathcal{R}_\gamma(T, U^e(x)) \tag{4.19}$$

Concerning the selection of the constant $M$ and the existence of solutions to SP1, it is possible to show the following.

PROPOSITION 4.2. *Under assumptions H1–H3 and H7, for any neighbourhood of the origin $B(0,R)$ and any $\eta > 0$, there exists a constant $M(R,\eta) > 0$, such that a solution to SP1 exists for any $x \in B(0,R)$, and any control horizon $T > 0$, provided that $s \in \mathbb{N}$, the number of switches in the control sequence $\bar{u}$, is allowed to be large enough.*

PROOF. For any $\epsilon \in (0, \frac{\eta}{\zeta^2}]$ and any given $x \in B(0,R)$ let

$$z(x) \stackrel{\text{def}}{=} -\epsilon \nabla V^T(x) \tag{4.20}$$

Then, by virtue of hypothesis H7.b, $z(x)$ satisfies

$$\nabla V z(x) = -\epsilon \|\nabla V(x)\|^2 \leq -\epsilon\zeta^2 \|x\|^2 \leq -\eta \|x\|^2 \tag{4.21}$$

and is realizable as the right-hand side of the extended system (2.1), i.e. there exists an extended control $v$ such that

$$g_0(x) + \sum_{i=1}^{r-1} g_i(x) v_i \quad = \quad z(x) \tag{4.22}$$

It follows from (4.22) that for a given $x \in \mathbb{R}^n$:

$$v(x) \quad = \quad Q^\dagger(x)\left(z(x) - g_0(x)\right), \qquad v = [v_1 \; v_2 \; \ldots \; v_{r-1}]^T \tag{4.23}$$

where $Q^\dagger = Q^T \left(Q\, Q^T\right)^{-1}$ is the pseudo-inverse of the $n \times (r-1)$ matrix $Q(x) = [g_1(x) \; g_2(x) \; \ldots \; g_{r-1}(x)]$, which is guaranteed to exist for all $x \in \mathbb{R}^n$ because $\operatorname{rank}\big(Q(x)\big) = n$ by construction of the extended system (2.1). Moreover, $Q^\dagger$ is a smooth matrix function of $x$, thus there exists a constant $c(R) > 0$ such that

$$\|Q^\dagger(x)\| \leq c, \qquad \forall\, x \in B(0,R) \tag{4.24}$$

By Lipschitz continuity of $g_0$ and $\nabla V$, there exist constants $d(R) > 0$ and $K(R) > 0$ such that:

$$\begin{aligned} \|v\| &\leq \|Q^{\dagger}(x)\| \, \|z(x) - g_0(x)\| \\ &\leq \|Q^{\dagger}(x)\| \, (\|z(x)\| + \|g_0(x)\|) \\ &\leq c\left(\frac{\eta K}{\zeta^2}\|x\| + d\|x\|\right) \end{aligned} \tag{4.25}$$

and hence with $M = c\left(\frac{\eta K}{\zeta^2} + d\right)$, the extended control $v$ is in the set $U^e(x)$; it is only one of many controls which satisfy $v \in U^e(x)$. Let $\gamma^e(T)$ be the $\gamma$-coordinates of the flow at time $T$ of the extended system (2.1) with control $v$. Then $\gamma^e(T) \in \mathcal{R}_\gamma(T, U^e(x))$. By virtue of the controllability assumption H2, there exists an open-loop control $\bar{u} \in \mathcal{P}^m$ such that $\gamma^e(T, v^d) = \gamma(T, \bar{u}^d)$. Hence, $\gamma(T, \bar{u}^d) \in \mathcal{R}_\gamma(T, U^e(x))$ thus proving the existence of solutions to SP1 at any $x \in B(0, R)$. □

PROPOSITION 4.3. *Let $\bar{u}(x, \tau)$, $\tau \in [0, T]$, be a control generated by the solution $\bar{u}$ to SP1. There exists a $T_{max} > 0$ such that for all $T \in [0, T_{max}]$:*

$$V(x(T, x, \bar{u})) - V(x) < -\frac{\eta}{2}\|x\|^2 T \tag{4.26}$$

PROOF. If $\bar{u}$ solves SP1 then $\gamma(T, \bar{u}^d) \in \mathcal{R}_\gamma(T, U^e(x))$. Hence, there exists a control $v \in U^e(x)$ such that $x(T, x, \bar{u}) = x^e(T, x, v) \in \mathcal{R}_{\mathcal{G}}(T, x, U^e(x))$ which proves (4.26) by virtue of Proposition 4.1.

□

The results presented above underlie the construction of the stabilizing feedback presented in Section 4.4.

### 4.3.2. Method 2: Stabilizing Feedback Design using the CBH formula

A result essential to the construction of the stabilizing control is the Campbell-Baker-Hausdorff (CBH) formula. By virtue of the CBH formula, the composition of exponentials in (2.3) can be expressed in terms of a single exponential. This is seen as follows:

$$\begin{aligned} \exp(\varepsilon_1 f^{u_{(1)}}) \circ \cdots \circ \exp(\varepsilon_k f^{u_{(k)}}) &= \phi_G^+ \left(\exp(\varepsilon_1 \lambda_1) \circ \cdots \circ \exp(\varepsilon_k \lambda_k)\right) \\ &= \phi_G^+ \left(\exp(T\bar{\lambda})\right) \quad \text{for all } \varepsilon_i,\ i = 1, \ldots, k \end{aligned} \tag{4.27}$$

where $\phi_L^+(\lambda_i) = f^{u_{(i)}}$ for $i = 1, \ldots, k$, $T = \sum_{i=1}^k \varepsilon_i$, and where $\bar{\lambda} \in L(H)$ is a uniquely defined element which exists because the exponential map on $H$ is a global diffeomorphism of $L(H)$ onto $H$, so that the CBH formula holds globally on $H$, see [**162**, Thm. 3.6.1 and 3.6.2].

Let $\bar{u} \stackrel{\text{def}}{=} \{u_{(1)}, \ldots, u_{(s)}\}$ and $\bar{\varepsilon} \stackrel{\text{def}}{=} \{\varepsilon_1, \ldots, \varepsilon_s\}$, and let

$$\bar{f}(\cdot, \bar{u}, \bar{\varepsilon}) \stackrel{\text{def}}{=} \phi_L^+\left(\bar{\lambda}\right) \in L(\mathcal{F}) \tag{4.28}$$

Therefore,

$$\exp(\varepsilon_1 f^{u_{(1)}}) \circ \cdots \circ \exp(\varepsilon_s f^{u_{(s)}}) = \exp(T\bar{f}) \quad \text{for all } \bar{u} \in \mathcal{P}^m \text{ and } \bar{\varepsilon} \in \mathbb{R}^s \tag{4.29}$$

By hypothesis H3.a, $\bar{f}$ has the following finite expansion in terms of the vector fields in the definition of system $\Sigma^e$:

$$\bar{f}(x, \bar{u}, \bar{\varepsilon}) = g_0(x) + \sum_{i=1}^{r-1} c_i(\bar{u}, \bar{\varepsilon}) g_i(x) \quad \text{for all } x \in B(0, R) \tag{4.30}$$

The coefficients $c_i$ are nonlinear functions in the components of $(\bar{u}, \bar{\varepsilon})$, whose analytic expressions can be determined from the CBH formula, after collection of terms. For an arbitrary $x \in B(0, R)$, the components of $(\bar{u}, \bar{\varepsilon})$ are employed in the parametrization of the piece-wise constant control, $\bar{u}(x, \tau)$, as follows:

$$\bar{u}(x, \tau) \stackrel{\text{def}}{=} u_{(k)}, \quad \tau \in [t_k, t_k + \varepsilon_k), \quad \textstyle\sum_{i=1}^s \varepsilon_i = T, \quad k = 1, 2, \ldots, s \tag{4.31}$$

where $t_1 = 0$ and $t_k = t_{k-1} + \varepsilon_{k-1}$, $k = 2, \ldots, s$.

Since the state of the system $\Sigma$ at time $T$ resulting from the application of control $\bar{u}$ in equation (4.31) is given by $x(T, x, \bar{u}) = G_T\, x = \exp(T\bar{f}(x, \bar{u}, \bar{\varepsilon}))\, x$, it is possible to regard $x(T, x, \bar{u})$ as the solution to the differential equation $\dot{x} = \bar{f}(x, \bar{u}, \bar{\varepsilon})$ and therefore the standard Lyapunov argument towards establishing the stability of system $\Sigma$, which requires that for all $x \in B(0, R)$, $\dot{V}(x) = \nabla V \bar{f}(x, \bar{u}, \bar{\varepsilon}) = L_{\bar{f}} V(x) < 0$, gives an explicit condition for the selection of the parameters in $(\bar{u}, \bar{\varepsilon})$ and permits for the following formulation of a *satisficing problem* (SP2) with respect to the pair $(\bar{u}, \bar{\varepsilon})$, where the selection of the constants $M$ and $\eta$ will be specified later, and the control horizon $T$ is the same as the one used in hypothesis H2:

**SP2:** For given constants $\eta > 0$, $T > 0$ and $M > 0$, and for a given $x \in B(0,R)$ find a feasible pair $(\bar{u},\bar{\varepsilon}) \in \mathcal{P}^m \times \mathbb{R}^s$, such that for some $s < \infty$:

$$\nabla V \bar{f}(x,\bar{u},\bar{\varepsilon}) \leq -\eta\|x\|^2 \tag{4.32}$$

$$\|c(\bar{u},\bar{\varepsilon})\| \leq M\|x\| \tag{4.33}$$

where $\bar{f}$ and $c(\bar{u},\bar{\varepsilon}) \stackrel{\text{def}}{=} [c_1(\bar{u},\bar{\varepsilon}) \ \cdots \ c_{r-1}(\bar{u},\bar{\varepsilon})]^T$ are defined by (4.30), and $s$ is the number of switches in the control sequence $\bar{u} \in \mathcal{P}^m$.

REMARK 4.2. *A comment concerning the practical implementation of SP2 is worth at this point. Noting that $\bar{f}$ is written in terms of the elements of $\mathcal{G}$ (see hypothesis H3, Section 2.2, p.27), as the right-hand side of system $\Sigma^e$, but with the coefficients $c_i$ replacing the parameters $v_i$. It is convenient to solve the above satisficing problem as an optimization problem involving two stages. First, values for the* parameters *$v_i$ determining the orientation and magnitude of the extended system's vector field $g^v(x) \stackrel{\text{def}}{=} \sum_{i=0}^{r-1} g_i(x)v_i$, $v_0 = 1$, are found such that they solve the following constrained minimization problem for a given $x \in \mathbb{R}^n$:*

$$\min_{v\in\mathbb{R}^{r-1}} F(v) = \nabla V g^v(x)$$

$$\textit{subject to:} \begin{cases} F(v) & \leq -\eta\|x\|^2 \\ \left(\sum_{i=1}^{r-1}(v_i)^2\right)^{1/2} & \leq M\|x\| \end{cases}$$

*If the the coefficients $c_i$ in SP2 can be made equal to the parameters $v_i$ computed from the above minimization problem, then they would clearly be a candidate solution to SP2. A feasible solution to SP2 thus can finally be obtained by solving the following least squares minimization problem:*

$$\min_{(\bar{u},\bar{\varepsilon})} \sum_{i=1}^{r-1}[v_i - c_i(\bar{u},\bar{\varepsilon})]^2$$

*which **ideally** yields the values for the control pair $(\bar{u},\bar{\varepsilon})$ such that $c_i(\bar{u},\bar{\varepsilon}) = v_i$, $i = 1,\ldots,r-1$. In practice, though, the non-uniqueness of minimizers and the limitations of the existing optimization algorithms can make it difficult to find the global minimizer for the latter optimization problem.*

The following results warrant the existence of solutions to SP2 and their stabilizing properties.

PROPOSITION 4.4. *Under assumptions H1–H3 and H7, for any neighbourhood of the origin $B(0,R)$ and any $\eta > 0$, there exists a constant $M(R,\eta) > 0$, such that a solution to SP2 exists for any $x \in B(0,R)$, and any control horizon $T > 0$, provided that $s \in \mathbb{N}$, the number of switches in the control sequence $\bar{u}$, is allowed to be large enough.*

PROOF. For any $\epsilon \in (0, \frac{\eta}{\zeta^2}]$ and any given $x \in B(0,R)$ let

$$z(x) \stackrel{\text{def}}{=} -\epsilon \nabla V^T(x) \tag{4.34}$$

Then, by virtue of hypothesis H7.a, $z(x)$ satisfies

$$\nabla V z(x) = -\epsilon \|\nabla V(x)\|^2 \leq -\epsilon \zeta^2 \|x\|^2 \leq -\eta \|x\|^2 \tag{4.35}$$

Furthermore, $z(x)$ can be written in terms of the right-hand side of the extended system $\Sigma^e$ as

$$z(x) = g_0(x) + \sum_{i=1}^{r-1} g_i(x) v_i \stackrel{\text{def}}{=} g^v(x) \tag{4.36}$$

where $g_i(x)$ are vector fields defined as in (4.30) and $v \stackrel{\text{def}}{=} [v_1 \ v_2 \ \cdots \ v_{r-1}]^T \in \mathbb{R}^{r-1}$ is a vector of constant parameters.

It follows from (4.36) that for a given $x \in \mathbb{R}^n$:

$$v(x) \quad = \quad Q^\dagger(x) \left( z(x) - g_0(x) \right), \qquad v = [v_1 \ v_2 \ \ldots \ v_{r-1}]^T \tag{4.37}$$

where $Q^\dagger(x) = Q^T \left( Q \, Q^T \right)^{-1}$ is the pseudo-inverse of the $n \times (r-1)$ matrix $Q(x) = [g_1(x) \ g_2(x) \ \ldots \ g_{r-1}(x)]$, which is ensured to exist for all $x \in \mathbb{R}^n$ because rank $\big(Q(x)\big) = n$ by the assumption that $g_i$, $i = 1, \ldots, r-1$, contains the basis for $L_x(\mathcal{F})$. Moreover, $Q^\dagger$ is a smooth matrix function of $x$ so there exists a constant $c(R) > 0$ such that

$$\|Q^\dagger(x)\| \leq c, \qquad \forall\, x \in B(0, R) \tag{4.38}$$

By Lipschitz continuity of $g_0$ and $\nabla V$ there exist constants $d(R) > 0$ and $K(R) > 0$, respectively, such that:

$$\begin{aligned} \|v\| &\leq \|Q^\dagger(x)\|\,\|z(x) - g_0(x)\| && (4.39)\\ &\leq \|Q^\dagger(x)\|\,(\|z(x)\| + \|g_0(x)\|) && (4.40)\\ &\leq c\left(\frac{\eta K}{\zeta^2}\|x\| + d\|x\|\right) && (4.41) \end{aligned}$$

Let $M = c\left(\frac{\eta K}{\zeta^2} + d\right)$ be the constant employed in (4.33). Then, a solution to SP exists if there is a control pair $(\bar{u}, \bar{\varepsilon})$ such that $c_i(\bar{u}, \bar{\varepsilon}) = v_i$, $i = 1, \ldots, r-1$. To demonstrate the existence of a control pair $(\bar{u}, \bar{\varepsilon})$, consider the extended system:

$$\dot{y} = g^v(y), \qquad y(0) = x \tag{4.42}$$

with state $y \in \mathbb{R}^n$ and constant control $v$ defined as in (4.37), (note that $v$ is a function of $x$ but not of $y$).

The integration of (4.42) over $[0, T]$ yields

$$\begin{aligned} y(T) &= \exp(Tg^v)x \\ &= \phi_G^+(S^e(T))x \end{aligned} \tag{4.43}$$

where $S^e(\cdot)$ is the solution to the system (4.42) reformulated as a right-invariant system on the Lie group $H$. By virtue of global strong controllability of $\Sigma_H$ on $H$, as demonstrated in Proposition 2.1, there exists a control pair $(\bar{u}, \bar{\varepsilon}) \in \mathcal{P}^m \times \mathbb{R}^s$, which steers system $\Sigma_H$ from $e \in H$ to $S(T) = S^e(T)$ in time $T$, i.e.:

$$\exp(\varepsilon_1\lambda_1) \circ \cdots \circ \exp(\varepsilon_s\lambda_s) = \exp(T\bar{\lambda}) = S^e(T) \tag{4.44}$$

where $\lambda_i = (\phi_L^+)^{-1}(f^{u_{(i)}})$, for $i = 1, \ldots, s$, and $\bar{\lambda}$ results from the application of the CBH formula on $H$. It follows from (4.43) and (4.44) that

$$\exp(T\bar{\lambda}) = (\phi_G^+)^{-1}\exp(Tg^v) = \exp(T(\phi_L^+)^{-1}(g^v)) \tag{4.45}$$

Since the exponential map is a global diffeomorphism on $H$, it follows that $\bar{\lambda} = (\phi_L^+)^{-1}(g^v)$, i.e. $\phi_L^+(\bar{\lambda}) = \bar{f} = g^v$, with $\bar{f}$ as in (4.28). Due to the expansion (4.30), $c_i = v_i$ for all $i = 1, \ldots, r-1$, as required. $\square$

PROPOSITION 4.5. *Let $\bar{u}(x,\tau)$, $\tau \in [0,T]$, be a control generated by the solution pair $(\bar{u}, \bar{\varepsilon})$ to SP2. There exists a $T_{max} > 0$ such that for all $T \in [0, T_{max}]$:*

$$V(x(T,x,\bar{u})) - V(x) \leq -\frac{\eta}{2}\|x\|^2 T \tag{4.46}$$

PROOF. Since $V \in \mathcal{C}^2$, $\bar{f}$ is analytic and linear in $c_i(\bar{u},\bar{\varepsilon})$, $i = 1, \ldots, r-1$, and $f_0(0) = 0$, then $\nabla V$ and $\bar{f}$ are Lipschitz continuous on $B(0, 2R)$, uniformly with respect to $c(\bar{u},\bar{\varepsilon}) = [c_1(\bar{u},\bar{\varepsilon}) \ \cdots \ c_{r-1}(\bar{u},\bar{\varepsilon})]^T$ satisfying $\|c(\bar{u},\bar{\varepsilon})\| \leq M\|x\|$. Hence, there exists a $K > 0$ such that:

$$\left\|\bar{f}(y,\bar{u},\bar{\varepsilon}) - \bar{f}(x,\bar{u},\bar{\varepsilon})\right\| \leq K\|y - x\| \quad \text{and} \quad \|\nabla V(y) - \nabla V(x)\| \leq K\|y - x\| \tag{4.47}$$

for all $x \in B(0,R)$, $y \in B(0,2R)$, and for all control pairs $(\bar{u},\bar{\varepsilon})$ satisfying $\|c(\bar{u},\bar{\varepsilon})\| \leq M\|x\|$.

Let $x(t) \stackrel{\text{def}}{=} x(t,x,\bar{u})$, $t \geq 0$. First, it is shown that there exists a $T_1 > 0$ and a constant $K_1 > 0$ such that

$$\|x(s) - x\| \quad \leq \quad \|x\|\left(\exp(K\,s) - 1\right) \tag{4.48}$$

and

$$\|x(s)\| \leq K_1\|x\| \tag{4.49}$$

for all $s \in [0, T_1]$ such that $x(s) \in B(0, 2R)$.

To this end it suffices to notice that

$$\|x(s) - x\| \;\leq\; \int_0^s \|\bar{f}(x,\bar{u},\bar{\varepsilon})\|d\tau + \int_0^s \|\bar{f}(x(\tau),\bar{u},\bar{\varepsilon}) - \bar{f}(x,\bar{u},\bar{\varepsilon})\|d\tau \;\leq\; K\,\|x\|\,s + \int_0^s K\|x(\tau) - x\|d\tau$$

which, by the application of the Gronwall-Bellman lemma (see Appendix C.1, p. 244), yields inequality (4.48).

It is possible to see that if $T_1$ is chosen so that $(\exp(K\,T_1)-1) \le \frac{1}{2}$ then (4.48) holds for $s \in [0, T_1]$. By contradiction, suppose that there exists an $s_1 < T_1$ such that $\|x(s_1)\| = 2R$. It follows that $2R \le \|x\| + \|x(s_1) - x\| \le R + \|x\|\,(\exp(K\,s_1) - 1) \le \frac{3}{2}R$ which is false, and hence (4.48) is valid for $s \in [0, T_1]$. Inequality (4.49) follows from (4.48) since

$$\|x(s)\| \;\le\; \|x(s) - x\| + \|x\| \;\le\; \|x\|\exp(K\,s) \;\le\; K_1\|x\|$$

with $K_1 = \exp(KT_1)$.

Now,

$$\begin{aligned} V(x(T)) - V(x) \;&\le\; \nabla V(x)\bar f(x,\bar u,\bar\varepsilon)T + \int_0^T \left\|\nabla V(x(s))\bar f(x(s),\bar u,\bar\varepsilon) - \nabla V(x)\bar f(x,\bar u,\bar\varepsilon)\right\| ds \\ &\le\; -\eta\,\|x\|^2\,T + \bar K\|x\| \int_0^T \|x(s) - x\|\,ds \end{aligned} \tag{4.50}$$

since

$$\begin{aligned} \left\|\nabla V(x(s))\bar f(x(s),\bar u,\bar\varepsilon) - \nabla V(x)\bar f(x,\bar u,\bar\varepsilon)\right\| \;&\le\; \|\nabla V(x(s))\bar f(x(s),\bar u,\bar\varepsilon) - \nabla V(x(s))\bar f(x,\bar u,\bar\varepsilon)\| \\ &\quad + \|\nabla V(x(s))\bar f(x,\bar u,\bar\varepsilon) - \nabla V(x)\bar f(x,\bar u,\bar\varepsilon)\| \\ &\le\; \bar K\|x\|\,\|x(s) - x\| \end{aligned}$$

with $\bar K = K^2(1 + K_1)$.

Hence, if $T < T_1$ then $x(s) \in B(0, 2R)$ for all $s \in [0, T]$ and, using (4.48) in (4.50), yields

$$V(x(T)) - V(x) \;\;\le\;\; -\eta\,\|x\|^2\,T + \bar K\|x\|^2 \int_0^T (\exp(K\,s) - 1)\,ds \;\le\; -\frac{\eta}{2}\|x\|^2 q(T)$$

where $q(T) \overset{\text{def}}{=} \left(2 + \frac{2\,\bar K}{\eta}\right)T - \frac{2\,\bar K}{\eta\,K}(\exp(K\,T) - 1)$. If $r(T) \overset{\text{def}}{=} q(T) - T$, then $r(0) = 0$ and $r'(0) \overset{\text{def}}{=} \frac{dr}{dT}\big|_{T=0} = 1$, so there exists a $T_{max} \le T_1$ such that $r(T) \ge 0$ for all $T \in [0, T_{max}]$. Hence $q(T) \ge T$ for all $T \in [0, T_{max}]$ which proves (4.46). □

The results concerning the control $\bar{u} \in \mathcal{P}^m$ obtained as a solution to the satisficing problem SP1 or SP2 presented above in Section 4.3.1 and Section 4.3.2, respectively, serve now for the construction of the stabilizing feedback.

## 4.4. The Stabilizing Feedback and its Analysis

The stabilizing feedback control, $u^c(x,\tau)$, $\tau \geq 0$, $x \in B(0,R)$ for system $\Sigma$ is defined as a concatenation of solutions to SP1 or SP2, $\bar{u}(x(nT),\tau)$, $\tau \in [nT,(n+1)T]$, computed at discrete instants of time $nT$, $n \in \mathbb{Z}_+$:

$$u^c(x,\tau) \stackrel{\text{def}}{=} \bar{u}(x(n\,T),\tau) \text{ for all } \tau \in [nT,(n+1)T], \quad n \in \mathbb{Z}_+ \tag{4.51}$$

where $x(nT)$ is the state of the closed-loop system $\Sigma$ at time $nT$.

Remark 4.3.

- *The concatenated control $u^c(x,t)$ is a feedback control in the sense that a solution to SP1 or SP2 is computed at each $t = nT$, $n \in \mathbb{Z}_+$, and thus depends on $x(nT)$.*
- *An off-line construction of the feedback law could possibly be envisaged in that the satisficing problem could be solved on a finite collection of compact non-overlapping subsets $C_s$ covering $B(0,R)$. The objective function for SP1 should for this purpose be modified as follows:*

$$\gamma(T,\bar{u}^d) \in \bigcup_{x \in C_s} \mathcal{R}_\gamma(T,U^e(x)) \tag{4.52}$$

- *The computation of the analytic expression for: (a) the mapping $F$ defining the reachable set $\mathcal{R}_\gamma(T,U^e(x))$ in (4.17) of SP1, or (b) the coefficients $c_i$ defining $\bar{f}(x,\bar{u},\bar{\varepsilon})$ in (4.30) of SP2, can be facilitated by adequate supporting software for symbolic manipulation of Lie algebraic expressions. Such software has been developed in the form of a software package for Lie algebraic computations in Maple, see Chapter 6 or* [**175**]*, which can be used here for the construction of a basis for the controllability Lie algebra, simplification of arbitrary Lie bracket expressions, the derivation of the equation for the evolution of the*

*$\gamma$-coordinates, and the composition of flows using the Campbell-Baker-Hausdorff-Dynkin formula.*

The stabilizing property of the feedback (4.51) constructed by concatenation of the solutions to SP1 or SP2 can now be stated.

**Theorem 4.1.** *Let $T \in [0, T_{max}]$, where $T_{max}$ is specified in Proposition 4.1 (for Method 1) or 4.5 (for Method 2) and let the constant $M$ be selected as in Proposition 4.2 (for Method 1) or 4.4 (for Method 2). Suppose there exists a constant $C > 0$ such that the solutions to SP1 or SP2 are bounded as follows*

$$\|\bar{u}(x(nT), \tau)\| \leq C\|x(nT)\| \quad \text{for all } \tau \in [0, T], \; n \in \mathbb{Z}_+ \tag{4.53}$$

*Under these conditions, the concatenated control $u^c(x, \tau)$ given by (4.51) renders the closed-loop system $\Sigma$ uniformly asymptotically stable.*

Proof. Let $t_k = t_0 + kT$, $k \in \mathbb{N}$, and let $x(t)$ denote the state of the closed-loop system at time $t$ due to control (4.51). By Proposition 4.3 (for Method 1) or 4.5 (for Method 2), the state of system $\Sigma$ with control input (4.51) satisfies

$$V(x(t_{k+1})) - V(x(t_k)) \leq -\varphi(\|x(t_k)\|) \qquad \forall \; k \in \mathbb{Z}_+ \tag{4.54}$$

where $\varphi(\|x(t_k)\|) = \frac{\eta}{2}\|x(t_k)\|^2 T$.

By invoking the Gronwall-Bellman lemma with $\bar{u}$ satisfying (4.53) it is easy to show (see the proof of Proposition 4.1 for Method 1, or Proposition 4.5 for Method 2, in which the state of $\Sigma^e$ should be replaced by the state of $\Sigma$) that for any constants $R > 0$ and $T > 0$, there exist constants $r \in [0, R]$ and $K > 0$ such that

$$\|x(t_k + \tau)\| \leq \|x(t_k)\| \exp(K\tau) < R \tag{4.55}$$

for all $x(t_k) \in B(0, r)$, and for all $\tau \in [0, T]$.

To prove global uniform asymptotic stability, it is necessary to show that: (1) the equilibrium point of (1.1) is uniformly stable, and (2) that the trajectories $x(t)$, $t \geq 0$, converge to the origin uniformly with respect to time.

<u>(1) Uniform stability:</u>

Uniform stability is proved by showing that for all $R > 0$ there exists $\delta(R) > 0$ such that for $\|x(t_0)\| < \delta(R)$, the state remains in $B(0, R)$, i.e. $\|x(t)\| < R$ for all $\forall\ t, t_0$, with $t \geq t_0$. To this end, define $\delta(R) \stackrel{\text{def}}{=} \beta^{-1}(\alpha(r))$. By assumption H7.b, $\alpha(\delta) \leq \beta(\delta) = \alpha(r)$, so $\delta \leq r$. For all $x(t_0) \in B(0, \delta) \subset B(0, r)$, we further have that $\beta(\|x(t_0)\|) < \beta(\delta) = \alpha(r)$ because $\beta(\cdot)$ is strictly increasing. Therefore, by assumption H7.b: $V(x(t_0)) \leq \beta(\|x(t_0)\|) < \alpha(r)$, and due to (4.54), $V(x(t_1)) < V(x(t_0)) < \alpha(r)$, whenever $x(t_0) \neq 0$. Again, by assumption H7.b, $\alpha(\|x(t_1)\|) \leq V(x(t_1))$, which implies that $\alpha(\|x(t_1)\|) < \alpha(r)$, so $\|x(t_1)\| < r$. Now, suppose that $V(x(t_n)) < \alpha(r)$, for some integer $n$. If $x(t_n) \neq 0$ then, by virtue of the same argument as the one presented above, $V(x(t_{n+1})) < V(x(t_n)) < \alpha(r)$, and $\alpha(\|x(t_{n+1})\|) \leq V(x(t_{n+1})) < \alpha(r)$, so, again $\|x(t_{n+1})\| < r$. By induction, then $\|x(t_k)\| < r$ for all $k \in \mathbb{Z}_+$, and by direct application of (4.55), $\|x(t_k + \tau)\| \leq \|x(t_k)\| \exp(K\tau) < r \exp(K\tau) \leq R$ for all $\tau \in [0, T]$, $k \in \mathbb{Z}_+$. It hence follows that system $\Sigma$ with feedback law (4.51) is uniformly stable.

<u>(2) Global uniform convergence:</u>

Global uniform convergence, requires the existence of a function $\xi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $x \in \mathbb{R}^n$, $lim_{t\to\infty}\xi(x, t) = 0$ and such that $\|x(t)\| \leq \xi(x(t_0), t - t_0)$ for all $t_0$, $t \geq t_0$. The last inequality translates into the requirement that for all $R > 0$ there exists $\bar{T}(R, x_0) \geq 0$ such that $\|x(t)\| < R$, for all $t_0 > 0$ and for all $t \geq t_0 + \bar{T}$.

Let $R > 0$ be any given constant and let $\delta(R) > 0$ be such that $\|x(t_0)\| < \delta(R)$ implies that $\|x(t)\| < R$ for all $t_0 \geq 0$ and for all $t \geq t_0$. Such a $\delta$ exists by virtue of uniform stability of the closed-loop system. It remains to show that there exists an index $k^*(x(t_0), \delta) \in \mathbb{Z}_+$ such that

$$\|x(t_{k^*})\| < \delta \tag{4.56}$$

By contradiction, suppose that $\|x(t_k)\| \geq \delta$, for all $k \in \mathbb{Z}_+$. By virtue of (4.54), for any $k \in \mathbb{N}$:

$$\begin{aligned} V(x(t_1)) - V(x(t_0)) &\leq -\varphi(\|x(t_0)\|) \leq -\varphi(\delta) \\ &\vdots \\ V(x(t_k)) - V(x(t_{k-1})) &\leq -\varphi(\|x(t_{k-1})\|) \leq -\varphi(\delta) \\ V(x(t_{k+1})) - V(x(t_k)) &\leq -\varphi(\|x(t_k)\|) \leq -\varphi(\delta) \end{aligned} \tag{4.57}$$

Adding the above $(k+1)$ inequalities, yields

$$V(x(t_{k+1})) - V(x(t_0)) \leq -(k+1)\varphi(\delta) \tag{4.58}$$

which implies that

$$V(x(t_{k+1})) \leq V(x(t_0)) - (k+1)\varphi(\delta) \leq \beta(\|x(t_0)\|) - (k+1)\varphi(\delta) \quad \forall\ k \in \mathbb{Z}_+ \tag{4.59}$$

The above inequality directly indicates the existence of a finite index $\bar{k} \geq 0$ such that $V(x(t_{\bar{k}})) < 0$ which contradicts the fact that $V$ is positive definite. Hence, there exists a finite index $k^* \in \mathbb{N}$ such that (4.56) is valid. Clearly, the index $k^*$ depends only on the value of $x(t_0)$ and $R$, but is independent of the particular value of $t_0$.

By virtue of uniform stability, we now conclude that for all $t_0 \geq 0$ and for all $t > t_0 + k^*T$, $\|x(t)\| < R$, which proves global uniform convergence with $\bar{T}(x_0, R) \stackrel{\text{def}}{=} k^*T$. This completes the proof of global uniform asymptotic stability of the closed-loop system $\Sigma$ with control (4.51). □

Remark 4.4.

- *The assumption (4.53) is not restrictive as it can always be satisfied if the system is uniformly controllable in the following sense: for every constant control $v_i = c_i(\bar{u}, \bar{\varepsilon})$, $i = 1, \ldots, r-1$, there exists a $\bar{u} \in \mathcal{P}^m$ such that the state $x(T, x, \bar{u})$ of system $\Sigma$ is equal to the state $x^e(T, x, v)$ of system $\Sigma^e$ in (2.1) is equal to the state $x(T, x, \bar{u})$ of system $\Sigma$, i.e. $x(T, x, \bar{u}) = x^e(T, x, v)$ and such that $\|\bar{u}(\tau)\| \leq M_1\|v\|$, for all $\tau \in [0, T]$ and some $M_1 > 0$. The last requirement is not included as a design condition in SP for brevity of exposition and because it is easy to satisfy.*

- *For any given $R$ and $\eta$, the constant $M$ should be chosen using the estimates derived in the proof of Proposition 4.2 or Propostion 4.4.*
- *The above proof is similar to the proof of a general result found in* [**9**] *concerning a general criterion for asymptotic stability of nonlinear time-variant differential equations. The result presented here applies to time-invariant control systems of the form (1.1) while the result in* [**9**] *applies to time-varying uncontrolled systems.*

The next two sections present application examples of the proposed approaches. Although more efficient stabilization methods have been conceived in the literature for the particular systems considered in the examples, these methods explicitly exploit the structure and the form of the system equations. The purpose of these examples is merely to explain the approaches presented which apply to systems with drift, in their full generality.

## 4.5. Examples for Method 1

This section presents two examples that illustrate the design of stabilizing feedback controls using the approach described in Section 4.3.1. The first example considers the steering of a unicycle, while the challenging problem of stabilizing both the orientation and the angular velocities of an underactuated rigid body in space is treated in the second example.

### 4.5.1. Steering the Unicycle

The stabilization of the kinematic model of the unicycle is presented in this section. The simulation results obtained by solving the satisficing problem SP1 in a single or two stages, as explained in Remark 4.2, are also compared. The kinematic model of the unicycle is given by the following driftless system (see [**7**]):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \cos(x_3)u_1 \\ \sin(x_3)u_1 \\ u_2 \end{bmatrix} \tag{4.60}$$

where $(x_1, x_2)$ refer to the unicycle's position on the plane and $x_3$ is the orientation angle with respect to the $x_1$-axis as shown in Fig. 4.3. The inputs $u_1$ and $u_2$ are the driving speed and the rate of change in the orientation angle, respectively. The system is not nilpotent but can be nilpotenized by the following feedback [**44**]:

$$\begin{aligned} u_1 &= \frac{1}{\cos(x_3)} w_1 \\ u_2 &= \cos^2(x_3) w_2 \end{aligned} \tag{4.61}$$

which transforms (4.60) to

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ \tan(x_3) \\ 0 \end{bmatrix}}_{f_1(x)} w_1 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \cos^2(x_3) \end{bmatrix}}_{f_2(x)} w_2 \tag{4.62}$$

The latter model is nilpotent of order two with $[f_1, f_2] = [0 \ -1 \ 0]^T$ (all higher order brackets are zero). Letting $g_1 = f_1$, $g_2 = f_2$, $g_3 = [f_1, f_2]$, a basis $\{g_1, g_2, g_3\}$ can be constructed for $L(\mathcal{F})$ (which is verified to satisfy the LARC).

Let $a_i(x) \stackrel{\text{def}}{=} \nabla V g_i(x)$, $i = 1, 2, 3$, denote the coefficients that define the set $U^e(x)$ in (4.3), given here by:

$$U^e(x) = \left\{ v \in \mathbb{R}^3 \mid \nabla V g^v(x) = \sum_{i=1}^{3} a_i(x) v_i < -\eta \|x\|^2, \ \|v\| \leq M \|x\| \right\}$$

Choosing $V(x) = \frac{1}{2}\|x\|^2$ then

$$\begin{aligned} a_1(x) &= x_1 + x_2 \tan(x_3) \\ a_2(x) &= x_3 \cos^2(x_3) \\ a_3(x) &= -x_2 \end{aligned}$$

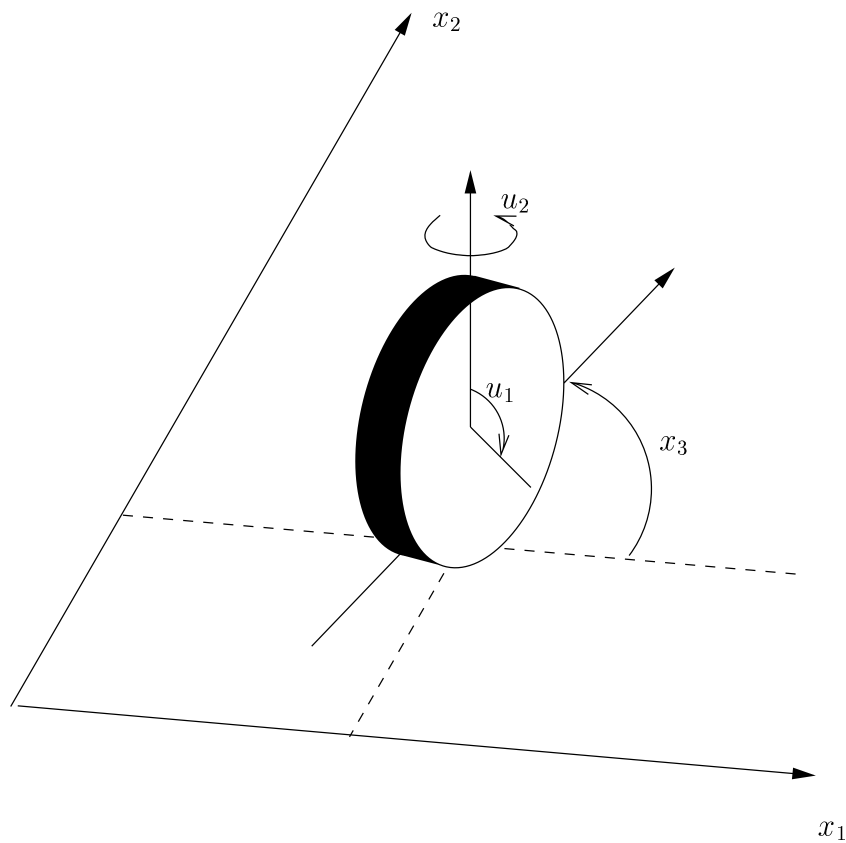



FIGURE 4.3. Unicycle or single-wheel drive cart modelled as a rolling disk.

The $\gamma$-coordinates equations for system (4.62) are easily calculated (following the steps explained in Chapter 6, p. 148) to be:

$$\begin{bmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\gamma_1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{4.63}$$

with $\gamma_1(0) = \gamma_2(0) = \gamma_3(0) = 0$. Integration of (4.63) for $t \in [0, T]$ yields

$$\gamma_1(T) = u_1 T \qquad \gamma_2(T) = u_2 T \qquad \gamma_3(T) = -u_1 u_2 \frac{T^2}{2} + u_3 T \tag{4.64}$$

Substituting the constant extended system controls $v_i$, $i = 1, 2, 3$, in (4.64) and solving for $v_i$ in terms of $\gamma_i$, $i = 1, 2, 3$, yields the map $F : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$. The components of the map $F(\gamma, T)$ are given by[1]:

$$
\begin{aligned}
v_1 &= F_1(\gamma^e, T) = \frac{\gamma_1^e(T)}{T} \\
v_2 &= F_2(\gamma^e, T) = \frac{\gamma_2^e(T)}{T} \\
v_3 &= F_3(\gamma^e, T) = \frac{1}{T}\left(\gamma_3^e(T) + \frac{\gamma_1^e(T)\gamma_2^e(T)}{2}\right)
\end{aligned}
$$

These expressions allow to re-state the set $U^e(x)$ in terms of the $\gamma$-coordinates.

Substituting $v_i = F_i(\gamma^e, T)$, $i = 1, 2, 3$, in (4.3) yields the following reachable set for the $\gamma$-coordinates of the extended system:

$$
\begin{aligned}
\mathcal{R}_\gamma(T, U^e(x)) \quad = \quad & \Bigg\{ \gamma \in \mathbb{R}^3 : \sum_{i=1}^{3} \nabla V g_i(x) F_i(\gamma, T) \; = \\
& a_1(x)\gamma_1^e(T) + a_2(x)\gamma_2^e(T) + a_3(x)\left(\gamma_3^e(T) + \frac{\gamma_1^e(T)\gamma_2^E(T)}{2}\right) < -\eta\|x\|^2, \\
& \|F(\gamma, T)\| \leq M\|x\| \Bigg\}
\end{aligned}
\tag{4.65}
$$

The last step involves finding a control sequence $\bar{u}$ such that the $\gamma$-coordinates at time $T$ for the original system, $\gamma(T, \bar{u})$, belong to the set $\mathcal{R}_\gamma(T, U^e(x))$. To this is end, consider a control sequence with $s = 4$ and equal time intervals $\varepsilon_i = \epsilon = T/4$. Integration of the $\gamma$-equations (4.63) with $u_3 = 0$ and the corresponding concatenated control, $u_1 = \bar{u}_1(x, t)$ and $u_2 = \bar{u}_2(x, t)$, over the period $T$, yields:

$$
\begin{aligned}
\gamma_1(T) &= (u_{1(1)} + u_{1(2)} + u_{1(3)} + u_{1(4)})\epsilon \\
\gamma_2(T) &= (u_{2(1)} + u_{2(2)} + u_{2(3)} + u_{2(4)})\epsilon \\
\gamma_3(T) &= -(u_{1(1)}u_{2(1)} + 2u_{1(1)}u_{2(2)} + u_{1(2)}u_{2(2)} + 2u_{2(3)}(u_{1(1)} + u_{1(2)}) + \\
&\quad u_{1(3)}u_{2(3)} + 2u_{2(4)}(u_{1(1)} + u_{1(2)} + u_{1(3)}) + u_{1(4)}u_{2(4)})\frac{\epsilon^2}{2}
\end{aligned}
\tag{4.66}
$$

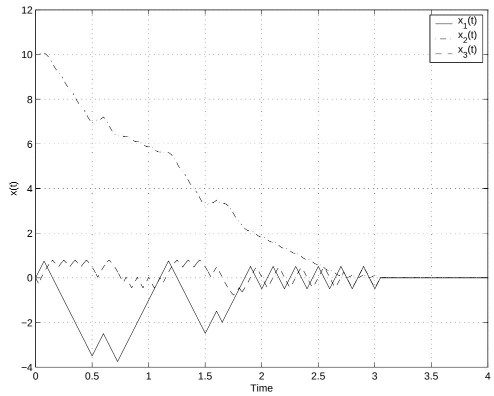
[1] In this example the map $F : (\gamma^e, T) \to F(\gamma^e, T)$ is from $\mathbb{R}^r \times \mathbb{R}$ to $\mathbb{R}^r$, instead of $\mathbb{R}^r \times \mathbb{R}$ to $\mathbb{R}^{r-1}$, because the system has no drift an its Lie algebra is of dimension $r = 3$.

Since the above $\gamma_i(T)$ must be in $\mathcal{R}_\gamma(T, U^e(x))$, they must satisfy (4.66). Hence, replacing each $\gamma_i^e$ in (4.66) by $\gamma_i(\bar{u}, \bar{\varepsilon})$, $i = 1, 2, 3$, the satisficing problem can be solved directly in terms of the parameters of the control pair $(\bar{u}, \bar{\varepsilon})$ sought.

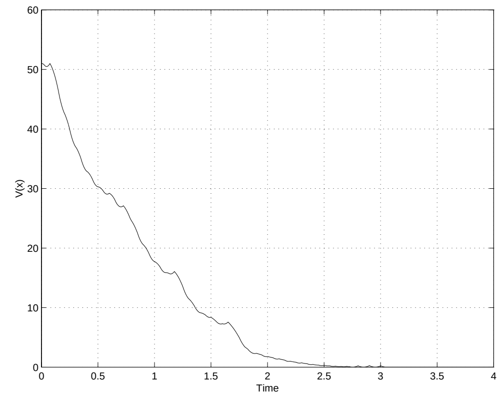
Figure 4.4 shows the stabilization of (4.60) using the control law obtained by solving the satisficing problem in one stage, i.e. by solving SP1 directly with respect to the control pair $(\bar{u}, \bar{\varepsilon})$. Figure 4.5 shows the simulation results corresponding to the solution of SP1 in two stages, i.e. by finding first values for the functions $F_i(\gamma, T)$, $i = 1, 2, 3$, that ensure greatest decrease in $V$, and then solving for the control pair $(\bar{u}, \bar{\varepsilon})$ which yields the closest values to those of $F_i(\gamma, T)$, $i = 1, 2, 3$.

The initial condition in both simulations is $x_0 = [0\ 10.1\ 0]^T$, which is in the bad set defined by $\dot{V}(x, u) = 0$ for all $u \in \mathbb{R}^2$, i.e. in the set where there are no control values that instantaneously decrease $V$. The period of the control sequence was chosen to be $T = 0.1$. For the single stage implementation of SP1, the constant $C$ bounding the magnitudes of the controls $u_{i(k)}$ was found to be $C = 10$, while for the two-stage approach the bound $M$ on the extended system controls yielding the best results (in terms of convergence speed) was $M = 26$.

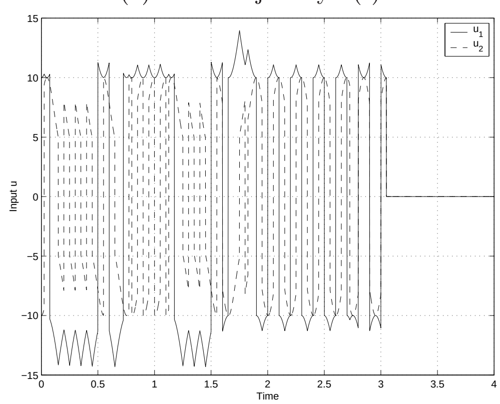
A comparison of the unicycle's trajectories in the plane resulting from each approach to the implementation of SP1 are shown in Fig. 4.6. From Fig. 4.6 and comparing Fig. 4.4 and Fig. 4.5 it is possible to conclude that the trajectory resulting from the two-stage solution of SP1 traverses a longer distance than the one obtained from the single stage implementation of SP1. However, the trajectory computed using the two-stage approach converges faster to the origin (compare the decrease of the Lyapunov functions) as the controls $\bar{u}$ that result from SP1 are of greater magnitude. As a concluding remark to this example, it is worth noting that the solutions to SP1 are not unique. This encourages the exploration of other approaches, for example, approaches in which SP1 is solved using continuous or smooth time-varying feedback controls.



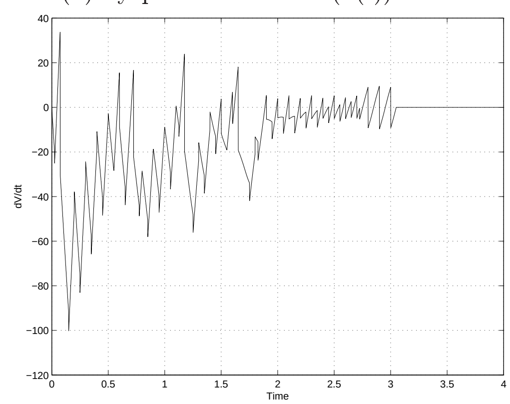

(a) State trajectory $x(t)$.




(b) Lyapunov function $V(x(t))$.




(c) Control inputs $u(t)$.



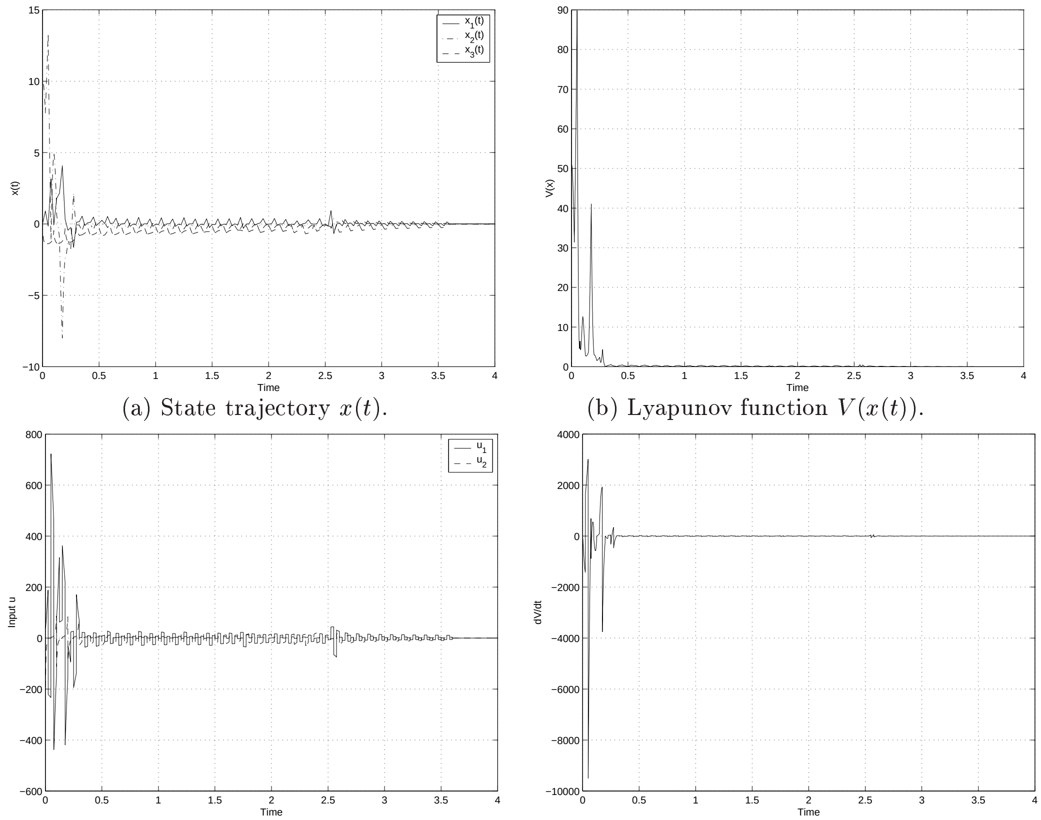

(d) Time derivative of $V(x(t))$.

FIGURE 4.4. Results for the stabilization of the unicycle obtained by solving the single-stage implementation of SP1.

### 4.5.2. Stabilization of the Attitude and Angular Velocities of an Underactuated Rigid Body in Space

The second stabilization approach is applied here to the model in $\mathbb{R}^6$ of the underactuated rigid body with two inputs given by equation (1.4) on p. 5. After the input transformation:

$$\tau_1 \; = \; J_1(u_1 - a_1\, x_5\, x_6) \qquad\qquad \tau_2 \; = \; J_2(u_2 - a_2\, x_4\, x_6) \tag{4.67}$$




(a) State trajectory $x(t)$.

(b) Lyapunov function $V(x(t))$.

(c) Control inputs $u(t)$.

(d) Time derivative of $V(x(t))$.

FIGURE 4.5. Results for the stabilization of the unicycle obtained by solving the two-stage implementation of SP1.

and with $a_3 = a = -0.5$, the model equation (1.4) becomes

$$\dot{x} = f_0(x) + f_1(x)u_1 + f_2(x)u_2 \tag{4.68}$$

$$\begin{aligned}
\text{where,}\quad f_0(x) \quad &= \quad (\sin(x_3)\,\sec(x_2)\,x_5 + \cos(x_3)\,\sec(x_2)\,x_6)\,\frac{\partial}{\partial x_1} \\
&\quad + (\cos(x_3)\,x_5 - \sin(x_3)\,x_6)\,\frac{\partial}{\partial x_2} \\
&\quad + (x_4 + \sin(x_3)\,\tan(x_2)\,x_5 + \cos(x_3)\,\tan(x_2)\,x_6)\,\frac{\partial}{\partial x_3} + a\,x_4 x_5\,\frac{\partial}{\partial x_6}, \\
f_1(x) \quad &= \quad \frac{\partial}{\partial x_4}, \quad f_2(x) = \frac{\partial}{\partial x_5}, \quad \text{and} \quad x = [x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6]^T
\end{aligned}$$

which is more convenient to analyze since several of its Lie brackets are zero.

The model used here is non-nilpotent, thus it does not lend itself directly to the application of the method presented. It was however selected to indicate to the reader that the approach can be

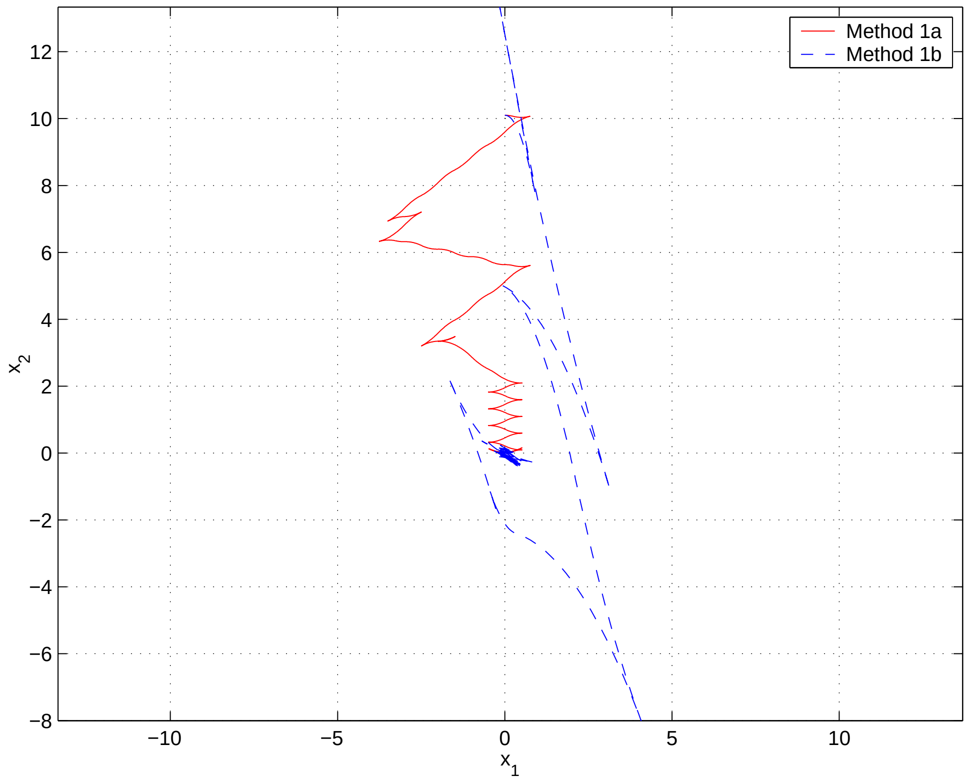



FIGURE 4.6. Comparison of the trajectories of the unicycle obtained by solving SP1 in one (Method 1a) and two stages (Method 1b).

extended to non-nilpotent systems, provided that the last are suitably "approximated" by nilpotent systems. At this stage, it is not our aim to introduce rigorous criteria for obtaining such approximations (see [**69**]), but rather to demonstrate that even a type of nilpotent truncation of the controllability Lie algebra of the original system can prove sufficient to implement the method. Obviously, any such approximation or truncation should necessarily preserve controllability of the system. It is further known, (see Theorem 2 in [**44**], or Theorem 5.1 on p. 111), that the steering error introduced while employing a truncated version of the controllability Lie algebra is a decreasing function of the distance between the initial and target points. It follows that the steering error can be controlled by selecting an adequately small time horizon $T$. Both the degree of nilpotency and the horizon $T$ can be selected on a trial and error basis by requesting periodic decrease in the Lyapunov function which is a directly verifiable criterion for the adequacy of the truncation.

In the above context, system (4.68) is assumed to be approximated by another system of a similar structure

$$\tilde{\Sigma}: \qquad \dot{x} = g_0(x) + g_1(x)u_1 + g_2(x)u_2$$

whose controllability Lie algebra, $L(g_0, g_1, g_2)$, corresponds to a nilpotent truncation of order four of $L(f_0, f_1, f_2)$. This is to say that $L(g_0, g_1, g_2)$ is nilpotent of order four and has

$$\{g_0,\, g_1,\, g_2,\, [g_0, g_1],\, [g_0, g_2],\, [g_1, [g_0, g_2]],\, [[g_0, g_1], [g_0, g_2]]\}$$

as its basis. Such truncation preserves the STLC property of the system, as is easily verified using Theorem 7.3 in [**39**] (see Thm. B.3 on p. of Appendix B).

The differential equations (2.17) for the evolution of the $\gamma$-coordinates for the approximating system are:

$$\begin{bmatrix} \dot{\gamma}_0 \\ \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \\ \dot{\gamma}_4 \\ \dot{\gamma}_5 \\ \dot{\gamma}_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\gamma_0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\gamma_0 & 0 & 1 & 0 & 0 \\ 0 & \gamma_0\gamma_2 & \gamma_0\gamma_1 & -\gamma_2 & -\gamma_1 & 1 & 0 \\ 0 & -a\gamma_0^2\gamma_2 & \gamma_0\gamma_3 - a\gamma_0^2\gamma_1 & a\gamma_0\gamma_2 & a\gamma_0\gamma_1 - \gamma_3 & -a\gamma_0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} \tag{4.69}$$

with $\gamma_i(0) = 0$, $i = 0, 1, \ldots, 6$. A detailed derivation of these equations is presented in Example 6.4.2 on p. 159 and makes extensive use of the package for Lie algebraic computations described in Chapter 6 and [**175**]. A quadratic Lyapunov function $V(x) = \frac{1}{2}\|x\|^2$ is chosen to define $U^e(x)$.

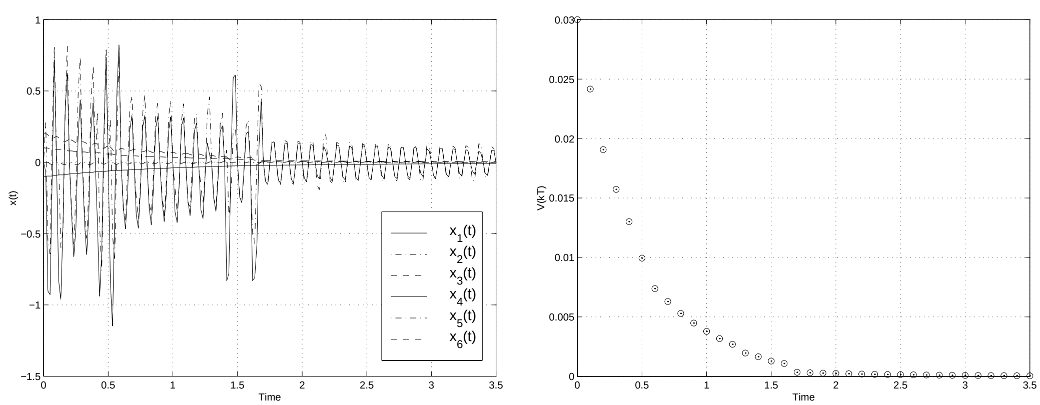
Integrating (4.69) with constant controls $v_i$ over the interval $[0, T]$ yields the required map $F$ : $\mathbb{R}^7 \times \mathbb{R} \to \mathbb{R}^6$

$$
\begin{aligned}
v_1 &= F_1(\gamma^e, T) = \frac{\gamma_1^e(T)}{T} \\
v_2 &= F_2(\gamma^e, T) = \frac{\gamma_2^e(T)}{T} \\
v_3 &= F_3(\gamma^e, T) = \frac{1}{T}\left(\gamma_3^e(T) + \frac{\gamma_0^e(T)\gamma_1^e(T)}{2}\right) \\
v_4 &= F_4(\gamma^e, T) = \frac{1}{T}\left(\gamma_4^e(T) + \frac{\gamma_0^e(T)\gamma_2^e(T)}{2}\right) \\
v_5 &= F_5(\gamma^e, T) = \frac{1}{T}\left(\gamma_5^e(T) + \frac{\gamma_1^e(T)\gamma_4^e(T)}{2} + \frac{\gamma_2^e(T)\gamma_3^e(T)}{2} - \frac{\gamma_0^e(T)\gamma_1^e(T)\gamma_2^e(T)}{6}\right) \\
v_6 &= F_6(\gamma^e, T) = \frac{1}{T}\left(\gamma_6^e(T) + \frac{a\,\gamma_0^e(T)\gamma_5^e(T)}{2} + \frac{\gamma_3^e(T)\gamma_4^e(T)}{2} + \frac{a\,\gamma_0^{e\,2}(T)\gamma_1^e(T)\gamma_2^e(T)}{12}\right. \\
&\qquad \left. - \frac{\gamma_0^e(T)\gamma_2^e(T)\gamma_3^e(T)}{12}(1+a) + \frac{\gamma_0^e(T)\gamma_1^e(T)\gamma_4^e(T)}{12}(1-a)\right)
\end{aligned} \tag{4.70}
$$

The above results in the following expression for the reachable set in the $\gamma$-coordinates:

$$
\mathcal{R}_\gamma(T, U^e(x)) = \left\{ \gamma \in \mathbb{R}^7 \mid \sum_{i=0}^{6} \nabla V g_i(x) F_i(\gamma, T) < -\eta \|x\|^2,\ \|F(\gamma, T)\| \leq M\|x\| \right\} \tag{4.71}
$$

The expressions for $\gamma(T, \bar{u}^d)$ in terms of the constant parameters defining $\bar{u}^d \in \mathcal{P}^r$ are easily obtained by symbolic integration and are subsequently employed to solve SP1. The values $s = 6$, $T = 0.1$, $\eta = 1$, $M = 10$, $R = 2$ and $C = 50$, were assumed in the solution of SP1. The simulation results are obtained using an initial condition $x_0 = [-0.1\ 0\ 0.2\ 0\ 0\ 0.1]^T$ and are shown in Figure 4.7.




(a) State trajectory $x(t)$.

(b) Lyapunov function $V(x(t))$ for $t = kT$, $k = 0, 1, \ldots, 35$.

FIGURE 4.7. Results for the attitude and angular velocities stabilization of the rigid body obtained by solving SP1.

## 4.6. Examples for Method 2

The design of stabilizing controls using the approach presented in Section 4.3.2 is illustrated by some examples of practical interest such as is the steering of the unicycle, the steering of a front-wheel drive car, and the rotational stabilization of an underactuated rigid-body in space.

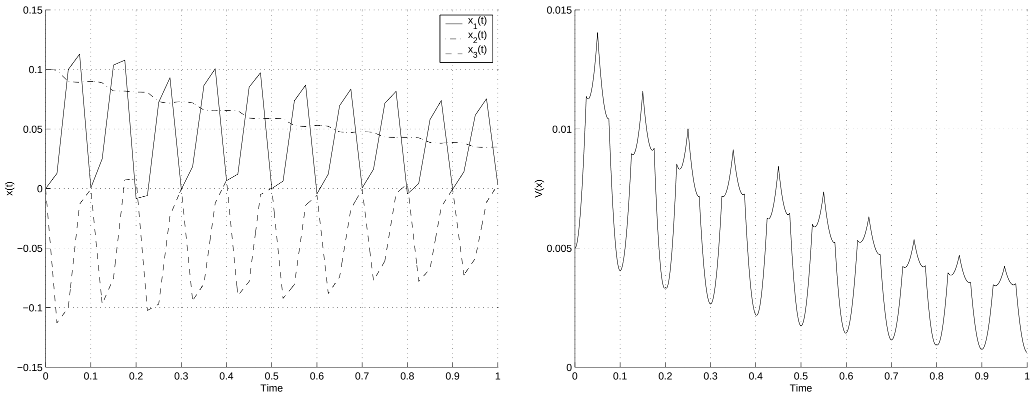
### 4.6.1. Steering the Unicycle

The model of the unicycle given in (4.60) is considered once again with the purpose of illustrating the synthesis of a feedback law by the approach proposed in Method 2.

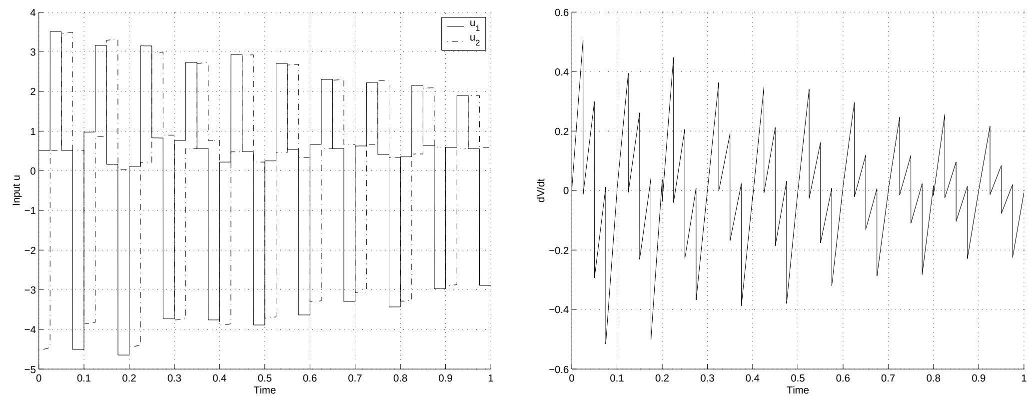
Using a control sequence with four intervals, i.e. $s = 4$, application of the CBH formula to the composition of flows (4.29) yields the following coefficients $c_i$ to the vector fields $g_i$, $i = 1, 2, 3$, of $\bar{f}$ in (4.30):

$$\begin{aligned}
c_1(\bar{u}, \bar{\varepsilon}) &= \sum_{k=1}^{4} \varepsilon_k u_{1(k)} \\
c_2(\bar{u}, \bar{\varepsilon}) &= \sum_{k=1}^{4} \varepsilon_k u_{2(k)} \\
c_3(\bar{u}, \bar{\varepsilon}) &= \frac{\varepsilon_1 \varepsilon_2}{2}(u_{1(2)} u_{2(1)} - u_{1(1)} u_{2(2)}) \\
&\quad + \frac{\varepsilon_3}{2}\big((\varepsilon_1 u_{2(1)} + \varepsilon_2 u_{2(2)}) u_{1(3)} - (\varepsilon_1 u_{1(1)} + \varepsilon_2 u_{1(2)}) u_{2(3)}\big) \\
&\quad + \frac{\varepsilon_4}{2}\big((\varepsilon_1 u_{2(1)} + \varepsilon_2 u_{2(2)} + \varepsilon_3 u_{2(3)}) u_{1(4)} - (\varepsilon_1 u_{1(1)} + \varepsilon_2 u_{1(2)} + \varepsilon_3 u_{1(3)}) u_{2(4)}\big)
\end{aligned}$$

The stabilization results obtained by solving SP2 with the above coefficients and $V(x) = \frac{1}{2}\|x\|^2$ are shown in Fig. 4.8. In order to clearly show the periodic decrease of $V$, the time interval for this simulation was selected to be $[0, 1]$. The complete simulation, however, does show the convergence of $x_2$ to zero in a larger time interval. The initial condition $x_0 = [0\ 0.1\ 0]^T$ is in the set $\{x \mid L_{f_i} V(x) = 0, i = 1, 2\}$ in which $\dot{V}(x, u) = 0$ independently of the input values. The time intervals are equal, i.e. $\varepsilon_i = \frac{T}{4}$, $i = 1, 2, 3, 4$, with $T = 0.1$. In this simulation, SP2 is solved in a single stage with $R = 10^4$ and $C = 0.1$. Notice that the constant $M$ is not actually needed in the one-step solution to SP2.




(a) State trajectory $x(t)$. (b) Lyapunov function $V(x(t))$.




(c) Control inputs $u(t)$. (d) Time derivative of $V(x(t))$.

FIGURE 4.8. Results for the stabilization of the unicycle obtained by solving SP2.

### 4.6.2. Steering the Front-Wheel Drive Car

The kinematic model of a front-wheel drive car, shown in Fig. 4.9, is given by (see [**6**]):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \cos(x_3)\cos(x_4)u_1 \\ \cos(x_3)\sin(x_4)u_1 \\ u_2 \\ \frac{1}{L}\sin(x_3)u_1 \end{bmatrix} \tag{4.72}$$

The first two components of the state vector, $x_1$ and $x_2$, represent the Cartesian position of the rear axle center. Component $x_3$ corresponds to the steering angle of the front wheels and $x_4$ to the angle between the main (longitudinal) axis of the car and the $x_1$-axis. The controls are the driving speed $u_1$ and the steering speed of the front wheels $u_2$.

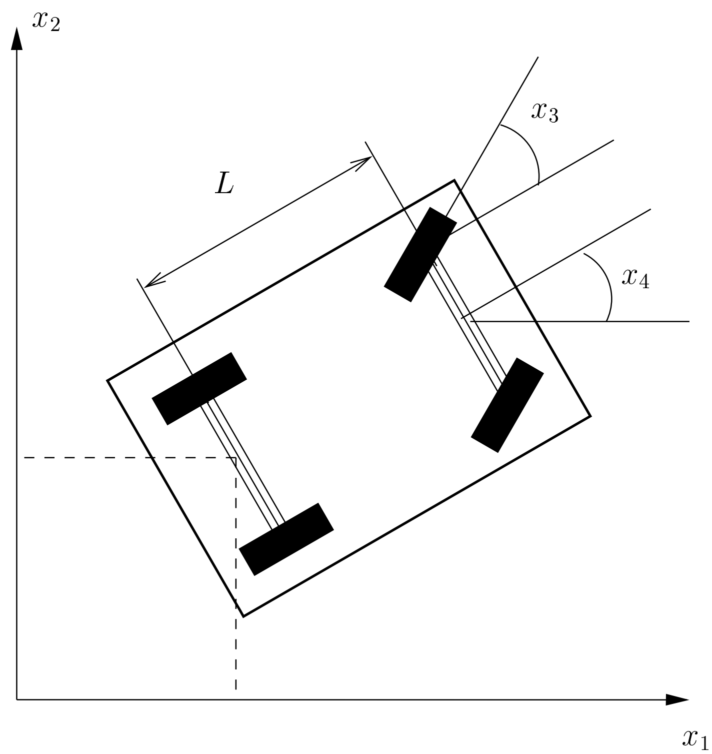



FIGURE 4.9. Front-wheel drive car.

System (4.72) can be nilpotenized by the following state feedback, [**44**],

$$\begin{aligned} u_1 &= \frac{1}{\cos(x_3)\cos(x_4)} w_1 \\ u_2 &= \cos^3(x_4)\cos^2(x_3) w_2 - \frac{3}{L}\frac{\sin(x_4)\sin^2(x_3)}{\cos^2(x_4)} w_1 \end{aligned} \tag{4.73}$$

in combination with the state transformation:

$$z_1 = x_1, \quad z_2 = \frac{\tan(x_3)}{\cos^3(x_4)}, \quad z_3 = \tan(x_4), \quad z_4 = x_2 \tag{4.74}$$

The nilpotenized system in $z$-coordinates can be rewritten as:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ z_2/L \\ z_3 \end{bmatrix}}_{f_1(z)} w_1 + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{f_2(z)} w_2 \tag{4.75}$$

A basis for the Lie algebra $L(\mathcal{F})$ is obtained by first constructing a Hall basis for the free Lie algebra $L(X_1, X_2)$ of two indeterminates $X_1$, $X_2$ (see Chapter 6) and then applying the canonical homomorphism $\nu : X_i \to f_i$ which assigns $X_i$ to $f_i$. The basis so constructed for $L(\mathcal{F})$ of the nilpotenized system (4.75) is defined by:

$$\begin{array}{rclrcl} g_1(z) & = & f_1(z) & g_3(z) & = & [f_1, f_2] = [0\ 0\ -1/L\ 0]^T \\ g_2(z) & = & f_2(z) & g_4(z) & = & [f_1, [f_1, f_2]] = [0\ 0\ 0\ 1/L]^T \end{array} \tag{4.76}$$

It can be verified that $\nu([X_2, [X_1, X_2]]) = [f_2, [f_1, f_2]] = 0$ and that all brackets of order higher than three also vanish. Hence, the Lie algebra $L(\mathcal{F})$ is nilpotent of order three with a basis of dimension $r = 4$.

The coefficients $c_i$ to the vector fields $g_i = 1, 2, 3, 4$, of $\bar{f}$ in (4.30) obtained by application of the CBH formula to the composition of flows (4.29) arising due to a control sequence with $s = 4$ are now given by:

$$c_1(\bar{u}, \bar{\varepsilon}) = \sum_{k=1}^{4} \varepsilon_k u_{1(k)} \tag{4.77}$$

$$c_2(\bar{u}, \bar{\varepsilon}) = \sum_{k=1}^{4} \varepsilon_k u_{2(k)} \tag{4.78}$$

$$c_3(\bar{u}, \bar{\varepsilon}) = a_4 + a_5 \tag{4.79}$$

$$c_4(\bar{u}, \bar{\varepsilon}) = \frac{\varepsilon_4}{2} a_4 u_{1(4)} + \frac{\varepsilon_4^2}{12} a_5 u_{1(4)} - \frac{\varepsilon_4}{12} a_1 a_5 + a_6 \tag{4.80}$$

where

$$\begin{aligned}
a_1 &= \sum_{k=1}^{3} \varepsilon_k u_{1(k)} \\
a_2 &= \sum_{k=1}^{3} \varepsilon_k u_{2(k)} \\
a_3 &= (\varepsilon_1 u_{2(1)} + \varepsilon_2 u_{2(2)}) u_{1(3)} - (\varepsilon_1 u_{1(1)} + \varepsilon_2 u_{1(2)}) u_{2(3)} \\
a_4 &= \frac{\varepsilon_3}{2} a_3 + \frac{\varepsilon_1 \varepsilon_2}{2} (u_{1(2)} u_{2(1)} - u_{2(2)} u_{1(1)}) \\
a_5 &= \frac{\varepsilon_4}{2} (a_2 u_{1(4)} - a_1 u_{2(4)}) \\
a_6 &= \frac{1}{12} \left( \left( \varepsilon_2^2 \varepsilon_1 u_{1(2)} - \varepsilon_1^2 \varepsilon_2 u_{1(1)} \right) \left( u_{1(2)} u_{2(1)} - u_{2(2)} u_{1(1)} \right) \right) + \\
&\quad \frac{\varepsilon_1 \varepsilon_2 \varepsilon_3}{4} \left( u_{1(2)} u_{2(1)} - u_{2(2)} u_{1(1)} \right) u_{1(3)} + \frac{\varepsilon_3^2}{12} a_3 u_{1(3)} - \frac{\varepsilon_3}{12} \left( \varepsilon_1 u_{1(1)} + \varepsilon_2 u_{1(2)} \right) a_3
\end{aligned}$$

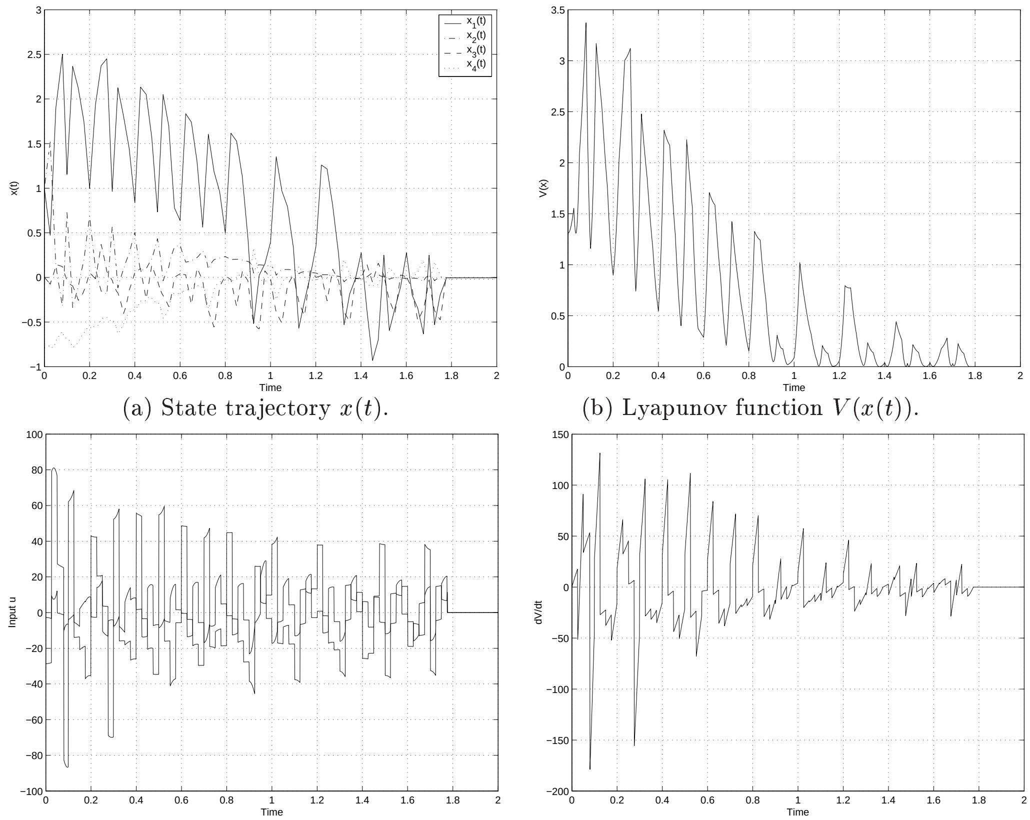
The derivation of the above expressions is facilitated by the software package described in Chapter 6.

The stabilization results obtained by solving SP2 with the above coefficients and $V(x) = \frac{1}{2}\|x\|^2$ are shown in Fig. 4.10. The initial condition $x_0 = [1\ 1\ 0\ -\frac{\pi}{4}]^T$ is in the set $\{x \mid L_{f_i} V(x) = 0, i = 1, 2\}$ in which $\dot{V}(x, u) = 0$ for all input values. The time intervals are equal, i.e. $\varepsilon_i = \frac{T}{4}$, $i = 1, 2, 3, 4$, with $T = 0.1$. In this simulation, SP2 is solved in a two stages with $R = 1.61$, $M = 0.1$ and $C = 100$. It is possible to see in Fig. 4.10 that in this case the concatenated control $u^c(x, t)$ steers the car to the origin in finite time.

### 4.6.3. Stabilization of the Angular Velocities of an Underactuated Rigid Body in Space

This example considers the rotational stabilization of an underactuated rigid-body in space. After a suitable feedback transformation, the model equations are given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ a_3 x_1 x_2 \end{bmatrix}}_{f_0(x)} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{f_1(x)} u_1 + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{f_2(x)} u_2 \tag{4.81}$$




(a) State trajectory $x(t)$.

(b) Lyapunov function $V(x(t))$.

(c) Control inputs $u(t)$.

(d) Time derivative of $V(x(t))$.

FIGURE 4.10. Results for the stabilization of the front-wheel drive car obtained by solving SP2.

with $a = -0.5$. The above model corresponds to the last three equations of system (1.4), describing the evolution of the angular velocities $x_i$, $i = 1, 2, 3$, of the rigid body about its principal axes. For details on the model derivation see [**198**].

The vector fields $f_0$, $f_1$ and $f_2$ generate a nilpotent Lie algebra of order three and dimension $r = 6$. A basis for the controllability Lie algebra $L(f_0, f_1, f_2)$ can be constructed by forming all Lie product combinations according to the Hall procedure for free Lie algebras, see [**6**], and by noting that the brackets $[f_1, f_2]$, $[f_0, [f_0, f_1]]$, $[f_0, [f_0, f_2]]$, $[f_1, [f_0, f_1]]$, $[f_2, [f_0, f_2]]$, $[f_1, [f_1, f_2]]$, $[f_2, [f_1, f_2]]$ and those of order four and higher are all zero. The resulting basis for $L(f_0, f_1, f_2)$ is thus given by the Lie algebra generators $g_i = f_i$, $i = 0, 1, 2$, and the Lie products $g_i$, $i = 3, 4, 5$, where:

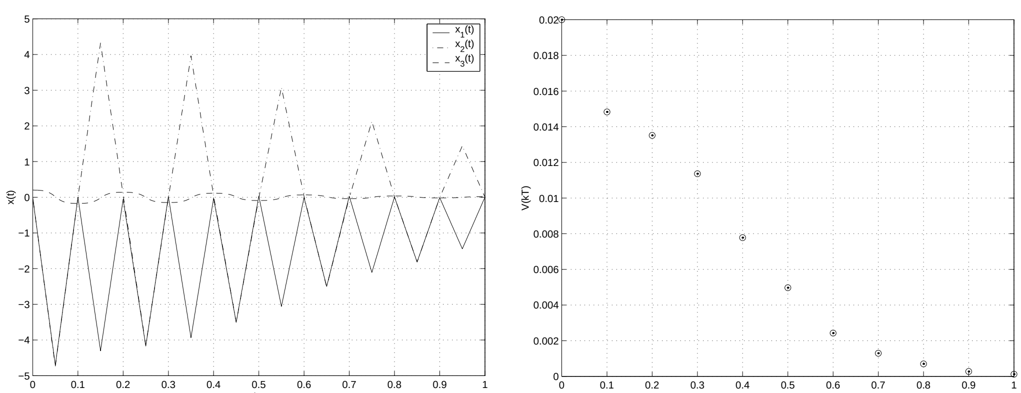
$$\begin{aligned} g_0(x) &= f_0(x) = [0\ 0\ a_3x_1x_2]^T & g_3(x) &= [f_0, f_1] = [0\ 0\ -a_3x_2]^T \\ g_1(x) &= f_1(x) = [1\ 0\ 0]^T & g_4(x) &= [f_0, f_2] = [0\ 0\ -a_3x_1]^T \\ g_2(x) &= f_2(x) = [0\ 1\ 0]^T & g_5(x) &= [f_1, [f_0, f_2]] = [0\ 0\ -a_3]^T \\ && g_6(x) &= [f_2, [f_0, f_1]] = g_5(x) \end{aligned} \tag{4.82}$$

In this example a control sequence of length $s = 2$ is considered. The corresponding composition of flows, (4.29), employing the CBH formula yields the following coefficients $c_i$ to the above $g_i$, $i = 0, \ldots, 5$, defining $\bar{f}$ as in (4.30):

$$\begin{aligned} c_0(\bar{u}, \bar{\varepsilon}) &= 1 & c_3(\bar{u}, \bar{\varepsilon}) &= \tfrac{\varepsilon_1\varepsilon_2}{2T}(u_{1(2)} - u_{1(1)}) \\ c_1(\bar{u}, \varepsilon) &= \tfrac{1}{T}\textstyle\sum_{k=1}^{2} \varepsilon_k u_{1(k)} & c_4(\bar{u}, \bar{\varepsilon}) &= \tfrac{\varepsilon_1\varepsilon_2}{2T}(u_{2(2)} - u_{2(1)}) \\ c_2(\bar{u}, \bar{\varepsilon}) &= \tfrac{1}{T}\textstyle\sum_{k=1}^{2} \varepsilon_k u_{2(k)} \end{aligned}$$

$$c_5(\bar{u}, \bar{\varepsilon}) = \tfrac{1}{12\,T}\left(\varepsilon_1^2\varepsilon_2(u_{1(1)}u_{2(2)} - u_{2(1)}u_{1(1)}) + \varepsilon_1\varepsilon_2^2(u_{2(1)}u_{1(2)} - u_{1(2)}u_{2(2)})\right)$$

The stabilization results obtained by solving SP2 with the above coefficients and $V(x) = \frac{1}{2}\|x\|^2$ are shown in Fig. 4.11. The initial condition $x_0 = [0\ 0\ 0.2]^T$ is a member of the set $\mathcal{S} \stackrel{\text{def}}{=} \{x \mid L_{f_0}V(x) \geq 0,\ L_{f_i}V(x) = 0, i = 1, 2\}$. It is easily seen that for any $x \in \mathcal{S}$, $\dot{V}(x) \geq 0$ independently of the values of the inputs. The time intervals are equal, i.e. $\varepsilon_1 = \varepsilon_2 = \frac{T}{2}$, and $T = 0.1$. The values $R = 0.2$, $M = 1$ and $C = 500$ are considered for the solution of SP2.




(a) State trajectory $x(t)$.

(b) Lyapunov function $V(x(t))$ for $t = kT$, $k = 0, 1, \ldots, 10$.

FIGURE 4.11. Results for the angular velocity stabilization of the rigid body obtained obtained by solving SP2.

# CHAPTER 5

## Stabilization of Bilinear Systems with Unstable Drift

This chapter presents two different methods for the stabilization of bilinear systems with unstable drift, [**76, 78**]. The first method relies on the solution of the trajectory interception problem (2.21) described in Section 2.5 of Chapter 2. This method is universal in the sense that it does not depend on the particular vector fields defining the dynamics of the system and does not require a Lyapunov function.

In the second approach, stabilization is carried out in two stages. In the first stage, the system is steered to a stable manifold which corresponds to a special selection of constant controls in the original bilinear system. This stage employs a control Lyapunov function in conjunction with a Lie algebraic control allowing the system to reach a stable manifold in finite time. The Lie algebraic control involves a solution to a non-linear programming problem whose formulation results from a direct application of the Campbell-Baker-Hausdorff formula for composition of flows as explained in Section 4.3.1 of Chapter 4. In the second stage, the system is made to "slide" through the ensemble of stable manifolds corresponding to the largest set of constant controls for which such stable manifolds exist.

Although the second method is not as general as the first approach, (because it requires the identification of the ensemble of stable manifolds and the definition of a control Lyapunov function as the "distance" to the stable manifold), its advantage lies in the fact that it is much simpler to implement. In fact, identifying a stable (linear) subspace is not difficult, and can be done even for high order systems using standard numerical procedures to calculate eigenvalues and eigenvectors.

## 5.1. Introduction

This chapter presents two approaches to the stabilization of homogeneous (in the state) bilinear systems of the form (1.5), re-stated below:

$$\Sigma_{BS}: \qquad \begin{aligned} \dot{x} &= A_0 x + \sum_{i=1}^{m} A_i x u_i \\ &= \underbrace{(A_0 + \sum_{i=1}^{m} A_i u_i)}_{A(u)}\, x \end{aligned} \tag{5.1}$$

Systems of this form are of practical relevance as they arise from the linearization of certain nonlinear systems with respect to the state only; see [**132, 120**]. The stabilization problem for bilinear systems has thus received much attention in the literature, see for example [**102, 104, 115, 120, 127, 125, 126, 76**] and [**132**] for a recent survey.

Several methods for stabilizing (5.1) start by finding a suitable Lyapunov function for the free system $\dot{x} = A_0 x$ (assuming that $A_0$ is stable), cf. [**100, 126**]. The proposed feedback controls are either linear, quadratic, or piece-wise constant, and usually result in slower than exponential asymptotic stability.

Stabilization of homogeneous bilinear systems in the plane has been fully analyzed. Bacciotti and Boieri, [**118**], using constant, linear, and quadratic feedbacks, and Chabour et al., [**120**], using feedbacks differentiable except at zero, have given complete classifications of the possibilities for stabilizability of $\dot{x} = A_0\, x + A_1\, x\, u$ on $\mathbb{R}^2 \setminus 0$. The methods of analysis in these papers again involve Lyapunov functions, center manifolds, and properties of plane curves.

For higher dimensional systems, however, relatively few methods for feedback stabilization are available. Although in [**127**], Wang gives a sufficient condition for stabilizability of systems in $\mathbb{R}^n$ by piece-wise constant controls, no general procedure for their construction is provided. There is a lack of constructive approaches to stabilization of higher order systems for which the matrix $A_0 + \sum_{i=1}^{m} A_i u_i$ is unstable for all choices of constants $u_i$.

Unlike many existing stabilization methods, the control laws proposed in this chapter do not require the drift term $A_0x$ to be critically stable, or stabilizable by constant controls. With this motivation, the following stabilization approaches are proposed:

- **Method 1:** The first proposed approach follows the idea, already suggested in [**132**], of stabilizing (5.1) by employing time-periodic feedback which brings into play the Lie brackets of the system matrices $A_0, A_1, \ldots, A_m$.

  The method relies on the Lie algebraic extension (2.1) of the original system (5.1). Under reasonable assumptions, a stabilizing feedback control is easy to construct for the extended system. The stabilizing time-invariant feedback control for the extended system is then combined with a periodic continuation of a specific solution to the open-loop, finite horizon control problem on the Lie group, described in Section 2.5, p. 35. The open-loop control problem is posed in terms of the $\gamma$-coordinates of flows and its purpose is to generate open-loop controls such that the trajectories of the extended system and the original system intersect after a finite time $T$, independent of their common initial condition. It is hence, a finite horizon interception problem for the flows parametrized by the $\gamma$-coordinates. While the time-invariant feedback for the extended system dictates the speed of convergence of the original system trajectory to the desired terminal point, the open-loop solution serves to ensure that the motion of the original system is on "average" that of the controlled extended system.

  The proposed approach demonstrates that synthesis of time-varying feedback stabilizers for bilinear systems is possible and can be viewed as a procedure of combining static feedback laws for a Lie algebraic extension of the system with a time-varying solution of an open-loop control problem on the associated Lie group.
- **Method 2:** The second stabilization approach merely requires the existence of a constant control which renders the resulting linear system to have at least one eigenvalue in the open left-half of the complex plane. By continuity, the latter induces the existence of a set of constant controls which yield a family of linear systems with stable manifolds.

The proposed control law comprises two phases: the *reaching phase* and the *sliding phase*. In the reaching phase, the state of the system is steered to a selected stable manifold employing a suitably designed control Lyapunov function in conjunction with a Lie algebraic control. The latter is necessary when there do not exist smooth controls that generate instantaneous velocities decreasing the Lyapunov function. The Lie algebraic control is constructed in terms of a sequence of constant controls which yield a decrease in the value of the Lyapunov function after a finite time $T$. The control sequence is based on the method proposed in Section 4.3.1 of Chapter 4 and is calculated as a solution to a non-linear programming problem whose formulation results from a direct application of the Campbell-Baker-Hausdorff formula for composition of flows. Once the set of stable manifolds is reached, the control is switched to its sliding phase whose task is to confine the motion of the closed-loop system to the latter set, making it invariant under limited external disturbances.

The contributions of this chapter are summarized as follows:

- Two novel Lie algebraic approaches to the synthesis of stabilizing feedback control for homogeneous bilinear systems with an unstable drift are presented.
- The methods are relatively general and apply to systems in which the drift cannot be stabilized by any constant control. Feedback stabilization of such systems has not yet found a general solution (except for systems which evolve in the plane).
- Sufficient conditions for the existence of the proposed control laws are given, including conditions under which the constructed feedback control of the second approach renders the stable manifold globally attractive and attainable in finite time.
- Examples are given to demonstrate the effectiveness of the proposed stabilizing feedback laws.

## 5.2. Problem Definition and Basic Assumptions

**Problem Definition.** *The objective is to construct time-varying feedback controls* $u_i(x,t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$, $i = 1, \ldots, m$, *in the class of piece-wise continuous functions in* $x$, *such that system (5.1) is Lyapunov asymptotically stable.*

For simplicity of notation, system $\Sigma_{BS}$ in (5.1) will simply be referred to as "system $\Sigma$" or just "$\Sigma$". Similarly, the Lie algebraic extension (2.1) corresponding to the bilinear system (5.1) will be denoted by $\Sigma^e$. Let $\mathcal{A} \stackrel{\text{def}}{=} \{A_0, A_1, \ldots, A_m\}$ be the set of matrices describing system (5.1), and denote by $L(\mathcal{A})$ the *matrix Lie algebra* of $(n \times n)$-dimensional square and real matrices generated by $\mathcal{A}$ with the Lie product of two matrices $A$ and $B$ defined as the matrix commutator $[A, B] = BA - AB$.

For the construction to be valid, the basic assumptions H1–H2 introduced in Chapter 2 are now replaced by the following hypotheses:

- H1.BS The vector fields $f_i(x) = A_i x$ $i = 0, 1, \ldots, m$, are real linearly independent vector fields that generate a Lie algebra $L(\mathcal{F})$, of dimension is $\dim L(\mathcal{F}) = r \geq n + 1$.
- H2.BS The system $\Sigma$ is strongly controllable, i.e. for any $T > 0$ and any two points $x_0, x_f \in \mathbb{R}^n$, $x_f$ is reachable from $x_0$ by some control $u \in \mathcal{P}^m$ of $\Sigma$ in time not exceeding $T$; i.e. there exists a control $u \in \mathcal{P}^m$ and a time $t \leq T$ such that $x(t, x_0, u) = x_f$.

REMARK 5.1. *Note that, unlike hypothesis H1 of Chapter 2, the above hypothesis H1.BS does not assume that the Lie algebra of vector fields* $L(\mathcal{F})$ *is nilpotent. Also analyticity and completeness are not assumed, because these properties are obviously satisfied by the linear vector fields in* $\mathcal{F}$.

*The Lie algebra* $L(\mathcal{F})$ *of a bilinear system is only finite dimensional, and in general, it is not nilpotent even if the vector fields* $f_i(x) = A_i x$, $i = 0, \ldots, m$, *defining the system's dynamics are linear. The finite dimensionality of* $L(\mathcal{F})$ *follows from the fact that the structure of the Lie algebra* $L(\mathcal{F})$ *is defined by the structure of the matrix Lie algebra* $L(\mathcal{A})$. *Specifically,* $\dim L(\mathcal{F}) \leq n^2$, *because the dimension of* $L(\mathcal{A})$ *is at most* $n^2$ *(the dimension of the space of* $n \times n$ *matrices).*

*Furthermore, there do not exist general transformations, either by state feedback or coordinate changes, that render the system nilpotent while preserving its bilinear form. The only general transformations that preserve the system's bilinearity are linear coordinate transformations of the form*

*$z = Tx$. However, the Lie algebra structure remains unchanged by this transformation, i.e. the Lie algebras are* equivalent *as their elements are related by the similarity transformation $\hat{X} = TXT^{-1}$, where $X$ and $\hat{X}$ denote elements of the Lie algebra of the original system (5.1) and the transformed one (using the coordinate change $z = Tx$):*

$$\Sigma^* : \qquad \dot{z} = TA_0T^{-1}z + \sum_{i=1}^{m} TA_iT^{-1}zu_i = \hat{A}_0 z + \sum_{i=1}^{m} \hat{A}_i z u_i \tag{5.2}$$

*It can easily be verified that the Lie brackets of (5.1) and (5.2) satisfy:*

$$[\hat{A}_i, \hat{A}_j] = [TA_iT^{-1}, TA_jT^{-1}] = T[A_i, A_j]T^{-1} \tag{5.3}$$

*and hence, the Lie algebra structures of the matrix Lie algebras $L(\mathcal{A})$ and $L(\hat{\mathcal{A}})$ of systems (5.1) and (5.2), respectively, are the same.*

*These facts suggest that even though the apparent linear aspect, bilinear systems should be treated within the class of general more complex non-nilpotent nonlinear systems. The lack of nilpotency significantly complicates the stabilization of such systems. The latter should be clear from the fact that neither the CBH formula nor the $\gamma$-coordinate equations (2.17) can easily be determined exactly, as both involve infinite series in the Lie products generated by $\mathcal{F}$. This in turn means that previous approaches that steer the system exactly between any two points $p$ and $q$ can now only steer it approximately. Note, however, that the exponential CBH formula (A.12), (see Appendix A, p. 219), is absolutely and uniformly convergent for all $t \in \mathbb{R}$, (see* [**12**]*), since for any two matrices $X$, $Y$:*

$$\begin{aligned} \|e^{Xt}Ye^{-Xt}\| &= \|Y + [Xt, Y] + \frac{1}{2!}[Xt, [Xt, Y]] + \frac{1}{3!}[Xt, [Xt, [Xt, Y]]] + \ldots\| \\ &\leq \|Y\| \left(1 + 2\|X\|t + \frac{1}{2!}(2\|X\|t)^2 + \frac{1}{3!}(2\|X\|t)^3 + \ldots\right) \\ &\leq \|Y\| \exp(2\|X\|t) \end{aligned} \tag{5.4}$$

*as the norm of the $n{+}1$-th term in the series (5.4) is not greater than $2\|X\|^n\|Y\|/n! \leq 2^n\|X\|^n\|Y\|/n!$.*

Once again, it will be useful to consider the Lie algebraic extension $\Sigma^e$ of the original bilinear system $\Sigma$ in (5.1), given here by:

$$\Sigma^e : \qquad \dot{x} = A_0 x + \sum_{i=1}^{r-1} A_i x v_i \stackrel{\text{def}}{=} g^v(x) \tag{5.5}$$

where the additional matrices $A_i$, $i = m+1, \ldots, r-1$, are Lie products of the original matrices in $\mathcal{A}$ such that:

H3.BS.a. The vector fields in $\mathcal{G} \stackrel{\text{def}}{=} \{g_0, \ldots, g_{r-1}\}$, $g_i(x) = A_i x$, $i = 0, \ldots, r-1$, define a basis for $L(\mathcal{F})$.

H3.BS.b. The set of vectors $g_i(x) = A_i x$, $i = 1, \ldots, r-1$, of the extended system $\Sigma^e$ contains a basis for $L_x(\mathcal{F})$, $x \in \mathbb{R}^n \setminus 0$.

H3.BS.c. The set of matrices $\{A_0, A_1, \ldots, A_{r-1}\}$ is a basis for the matrix Lie algebra $L(\mathcal{A})$ such that the extended system corresponds to a strongly controllable nilpotent approximation of system $\Sigma$.

Remark 5.2.

*Unlike assumption H3.b of Chapter 2, the above hypothesis H3.BS.b does not require here that for any $x$ in a sufficiently large neighbourhood of the origin $B(0, R)$, the set of vector fields $\mathcal{G}(x) \stackrel{\text{def}}{=} \{g_0(x), \ldots, g_{r-1}(x)\}$ contains a basis for $L_x(\mathcal{F})$. This is because the vector fields $g_i(x) = A_i x$, $i = 0, \ldots, r-1$, are linear, and hence, any basis of $L_x(\mathcal{F})$ for $x \in B(0, R) \setminus 0$ is also a basis for all $x \in \mathbb{R}^n \setminus 0$.*

The fact that $L(\mathcal{F})$ is in general non-nilpotent implies that the TIP, or equivalently LCIP, can only be solved approximately. This is because the infinite series involved in the derivation of the logarithmic coordinates equation are now replaced by finite series. In these circumstances, the computation of an approximate solution to LCIP does not differ from the one previously discussed in Chapter 2, but the trajectories of system $\Sigma$ and $\Sigma^e$ are no longer guaranteed to intercept periodically every $T$ units of time, and hence, it is not obvious that the stabilizing properties of the constructed time-varying feedback law are preserved. It is thus necessary to ensure that the error made in steering the system while neglecting higher order brackets can be made sufficiently small. An important result which delivers an estimated bound on the error was obtained by Lafferriere in [**44**]. This result, adequately interpreted in the context of the TIP, is essential to the stability analysis of system $\Sigma$ and is restated here as follows.

**Theorem 5.1** (Laferriere, [**44**]). *Suppose that the Lie algebra of vector fields $L(\mathcal{F})$ is not nilpotent but that the TIP is solved as an LCIP on a nilpotent Lie group $G_k$ of order $k < \infty$, corresponding to a nilpotent approximation of $L(\mathcal{F})$. Let $\hat{u}$ denote the approximate solution of the LCIP, and let $t \to x(t, x_0)$ and $t \to x^e(t, x_0)$ denote the integral curves, through $x_0 = x(t_0)$ at time $t_0$, of the original system $\Sigma$ with control $\hat{u}$ and the extended system $\Sigma^e$ with control $\hat{v}$, respectively. Further, let $x_T \stackrel{\text{def}}{=} x(T, x_0)$ and $x_T^e \stackrel{\text{def}}{=} x^e(T, x_0)$. Finally, let $\mathcal{R}$ be a bounded region in $\mathbb{R}^n$. Under these conditions, there exists a function $F : [0, \infty) \to [0, \infty]$, which is finite and bounded near zero, such that if $x_0, x_T^e \in \mathcal{R}$, $t_0 \geq 0$, then*

$$\|x_T^e - x_T\| \leq F(\|x_T^e - x_0\|)\|x_T^e - x_0\|^{1+\frac{1}{k}} \tag{5.6}$$

It follows from Theorem 5.1 that the steering error defined as the difference between trajectories of the original system and the trajectories of the approximating system (constructed in terms of a truncated version of the original controllability Lie algebra; see example in Section 5.4, p. 120) is a decreasing function of the distance between the initial and target points. This error can be controlled by selecting the degree of nilpotency and the time horizon $T$. Hence, it will simply be assumed, in the sequel, that $L(\mathcal{F})$ is a nilpotent Lie algebra. The approaches to the design of the stabilizing feedback control are presented in the next two sections.

## 5.3. Method 1: Time-Varying Stabilizing Feedback Design Based on the TIP

This section presents a method for construction of time varying stabilizing feedback control for homogeneous bilinear systems. The method is universal in the sense that it is independent of the vector fields determining the motion of the system, and does not require a Lyapunov function. The proposed feedback law is constructed in terms of a repeated continuation of the specific solution to an open-loop control problem on the associated Lie group. The latter is posed as the trajectory interception problem (TIP) in the logarithmic coordinates of flows (2.22) described in Section 2.5,

p. 35. The method thus requires first the stabilization of the Lie algebraic extension $\Sigma^e$ of the original bilinear system $\Sigma$, which is described in the next section. The subsequent section discusses the solution of the open-loop control problem and the stabilizing properties of the resulting time-varying feedback law.

### 5.3.1. Stabilization of the Lie Algebraic Extension of the Original System

A stabilizing feedback for system $\Sigma^e$ is first defined in the form of (2.18) as

$$v(x) \stackrel{\text{def}}{=} [v_1(x) \ \cdots \ v_r(x)]^T = Q(x)^\dagger (A_d x - A_0 x) \tag{5.7}$$

where $Q(x) \stackrel{\text{def}}{=} [A_1 x \ \cdots A_r x]$, $Q(x)^\dagger$ denotes the Moore-Penrose right pseudo-inverse of the state dependent matrix $Q$, i.e. $Q(x)Q(x)^\dagger = I$, which under the assumption H3.BS is guaranteed to exist for all $x \in \mathbb{R}^n \setminus 0$, see (2.18) on p. 36, and where $A_d$ is some asymptotically stable "reference system" matrix, i.e. $\text{eig}(A_d) \in \mathbb{C}_-^\circ$; here $\text{eig}(A_d)$ denotes the set of eigenvalues of $A_d$. It follows that the trajectories of the extended system satisfy

$$\dot{x} = A_0 x + Q(x)v(x) = A_d x \tag{5.8}$$

so that the extended system is stable, as desired.

Although the feedback given in (5.7) is exponentially stabilizing for the extended system, it is not stabilizing for the original system unless the action of the $r-1$ feedback controls in (5.7) is somehow *translated* into a corresponding action of the $m < r-1$ controls of (5.1). To facilitate such a construction, the trajectory interception problem of Section 2.5, p. 35, is solved once and evaluated repeatedly every period $T$ using a *discretized* version $\hat{v}$ of the controls $v(x)$. The feedback controls for the extended system $\hat{v}$ are "updated" only at discrete moments of time $nT$, $n \in \mathbb{Z}_+$, in terms of the values of $v(x(nT))$. More precisely, on the $n+1$-th interval $[nT, (n+1)T]$, $n \in \mathbb{Z}_+$, the discretized control $\hat{v}(n)$ is defined as the constant control:

$$\hat{v}(n) \stackrel{\text{def}}{=} v(x(nT)), \qquad t \in [nT, (n+1)T), \ n \in \mathbb{Z}_+ \tag{5.9}$$

Hence, $\hat{v}(n)$ is obtained from $v$ by the "sample and hold" operation. The corresponding extended system with discretized controls is thus given by:

$$\dot{x} = A_0 x + \sum_{i=1}^{r-1} A_i \, x \, \hat{v}_i(n) \tag{5.10}$$

It should be pointed out that such a discretization is only needed if the extended controls are not constant, and is introduced in order to ensure that the feedback controlled extended system (5.10) has the same Lie algebraic structure as the original system (5.1) within each time interval $[nT, (n+1)T)$.

Intuitively, it is clear that sufficiently fine discretization of the extended controls (reflected by a sufficiently small $T$) will preserve their stabilizing properties; the latter is confirmed by the following result.

Proposition 5.1. *Suppose that hypotheses H1.BS–H3.BS are valid, so that the controlled extended system given by (5.8) is globally exponentially stable. Under these conditions, for any region $B(0,R) \subset \mathbb{R}^n$ there exists a constant $T > 0$, $T \in [0, T_{max}]$, such that the extended system (5.10) with discretized controls (5.9) is asymptotically stable with region of attraction $B(0,R)$.*

Proof. First, notice that the extended system (5.5) with control (5.7) satisfies equation (5.8), and therefore, by asymptotic stability of the linear system $\dot{x} = A_d x$ there exists[1] positive matrices $P > 0, Q > 0$ and a Lyapunov function $V(x) \stackrel{\text{def}}{=} x^T P x$, such that $\dot{V} = x^T(A_d^T P + P A_d)x \leq -x^T Q x < -\lambda_{min}(Q)\|x\|^2$ for all $x \neq 0$, where $\lambda_{min}(Q)$ denotes the smallest eigenvalue of $Q$. Clearly, the chosen Lyapunov function satisfies hypothesis H7 in Chapter 4 and with $\eta \stackrel{\text{def}}{=} -\lambda_{min}(Q)$

$$\dot{V}(x) = \nabla V g^v(x) = \nabla V A_d x < -\eta \|x\|^2, \quad \text{for all } x \in \mathbb{R}^n \setminus 0 \tag{5.11}$$

Since (5.11) is valid for all $x$ along the trajectory of (5.1), it is also true for $x = x(nT)$. Furthermore, by the definition (5.7) and the fact that the matrix $Q^\dagger$ is a smooth matrix function of $x$, there exists constants $K_1(R)$ and $K_2(R)$ such that

$$\|Q^\dagger(x)\| \leq K_1, \qquad \forall \; x \in B(0,R) \tag{5.12}$$

[1]This is a well known converse of Lyapunov's Theorem, see for example [**4**, Thm. 3.12, p. 149].

and

$$\begin{aligned} \|v\| &\leq \|Q^{\dagger}(x)\| \, \|A_d x - A_0 x\| \\ &\leq K_1 \, K_2 \, \|x\| \end{aligned} \tag{5.13}$$

and hence, with $M = K_1 \, K_2$, $\|\hat{v}\| \leq M\|x\|$ and $\hat{v}(n) = v(x(nT)) \in U^e(x(nT))$, (see equation (4.3), p. 69), for some constant $M(R) > 0$ and all $x \in B(0, R)$. By Proposition 4.1, the latter implies that there exists $T > 0$ in $[0, T_{max}]$ such that $V(x^e(T, x, \hat{v})) - V(x) < -\frac{\eta}{2}\|x\|^2 T$. Final application of Theorem 4.1 to the extended system with discretized controls (instead of the original system) guarantees its asymptotic stability. □

### 5.3.2. The Open-Loop Control Problem on the Lie Group

The next step is to generate open-loop controls $u_i$, $i = 1, \ldots, m$, for the original system such that its trajectories and the trajectories of the discretized extended system (5.10) intersect periodically with the given frequency of discretization $1/T$. To this end, the open-loop control problem discussed in Section 2.5, p. 35, is solved yielding controls $\hat{u}(\hat{v}, t)$ such that the trajectory $x(t, x, \hat{u})$ of the original system, starting from $x = x(0) \neq 0$ at time $t = 0$, intersects the trajectory $x^e(t, x, \hat{v})$ of the *discretized* extended system (5.10), at time $T$, i.e. such that $x(T, x, \hat{u}) = x^e(T, x, \hat{v})$. The reformulation of the TIP as an LCIP, presented in Section 2.5 for a rather general class of systems, is described next with reference to the specific class of bilinear systems.

Re-stating systems $\Sigma$ and $\Sigma^e$ as system on the Lie group $H$ is essential for the design and analysis of the proposed feedback law. This reformulation is possible due to the results described in Chapter 2, specifically Theorem 2.1 by Palais and the result by Wei-Norman [**149**]. However, a less rigorous approach is adopted in the present exposition[2], not only for simplicity of notation, but rather to make more tangible the application of the ideas presented in Chapter 2. The rigour is sacrificed here in the sense that the relation between the Lie groups $H$ and $G$ established by the homomorphism $\phi_G^+$, and the corresponding Lie algebra homomorphism $\phi_L^+$, are "ignored". Thus with some abuse of notation the solution $x(t, x, u) = \phi_G^+(S(t, u))x$ to system $\Sigma$ is simply written as $x(t, x, u) = S(t, u)x$

[2]This informal exposition is similar to that found in [**6**] or [**180**].

and $e^A \in G$, $A \in \mathcal{A}$ are used in place of $e^\psi \in H$, $\psi \in L(H)$, i.e. $(\phi_L^+)^{-1}(A) = \psi$ is simply identified with its image such that "$A = \psi$".

Since any solution trajectory of $\Sigma$, $x(t,x,u) = S(t,u)x$ starting from $x \in \mathbb{R}^n$, clearly must satisfy its differential equation, a representation of $\Sigma$ in terms of $S(t,u)$ as a right-invariant system on the Lie group $H$ can be obtained by substituting $x(t,x,u) = S(t,u)x$ in (5.1) yielding the following differential equation, "on the Lie group $H$" , for $S(t,u)$:

$$\begin{aligned} \dot{S}(t,\hat{u}) &= \left[A_0 + \sum_{i=1}^{m} A_i \hat{u}_i(\hat{v},t)\right] S(t,\hat{u}) \\ S(0) &= I \end{aligned}$$

An analogous equation also holds for the extended system $\Sigma^e$, which together with the above equation allows to reformulate the TIP as the following flow interception problem (FIP):

**FIP:** Consider the two formal initial value problems:

$$S1: \quad \begin{cases} \dot{S}^e(t) = \left[A_0 + \sum_{i=1}^{r-1} A_i \hat{v}_i\right] S^e(t) \\ S^e(0) = I \end{cases} \tag{5.14}$$

$$S2: \quad \begin{cases} \dot{S}(t) = \left[A_0 + \sum_{i=1}^{m} A_i \hat{u}_i(\hat{v},t)\right] S(t) \\ S(0) = I \end{cases} \tag{5.15}$$

For a fixed value of the time horizon $T \leq T_{max}$, find control functions $\hat{u}_i(\hat{v},t)$, $i = 1,\ldots,m$, $t \in [nT,(n+1)T)$, $n \in \mathbb{Z}_+$, in the class of functions which are continuous in $\hat{v}$ and piece-wise continuous in $t$, such that for *any constant control vector* $\hat{v} \in \mathbb{R}^{r-1}$ the above flows (of the extended and original systems, respectively) intersect at time $T$, thus satisfying:

$$S(T,\hat{u}) = S^e(T,\hat{v}) \tag{5.16}$$

Employing the powerful formalism of Wei-Norman [**149**] is now essential as it enables to find a solution of the TIP while abstracting from the actual form of the matrices $A_0,\ldots,A_m$, any particular values of the initial condition $x$, as well as the values of the extended system control $\hat{v}$. The result of Wei-Norman implies that the solution to both initial value problems $S1$ and $S2$ has the same general form (see Section 2.4, p. 31, and also Chapter 6, p. 148 for additional details concerning this result in the general setting of free Lie algebras of indeterminates):

$$S^e(t) = \prod_{i=0}^{r-1} \exp(\gamma_i^e(t) A_i) \tag{5.17}$$

$$S(t) = \prod_{i=0}^{r-1} \exp(\gamma_i(t) A_i) \tag{5.18}$$

where the matrix exponentials are defined in the usual way:

$$\exp(\gamma(t)A) \stackrel{\text{def}}{=} I + \gamma(t)A + \frac{\gamma(t)^2}{2!}A^2 + \dots \tag{5.19}$$

and where $\{A_0, \dots, A_{r-1}\}$ is a basis for the algebra $L(\mathcal{A})$. The functions $\gamma_i^e$ and $\gamma_i$, $i = 0, \dots, r-1$, will be dependent on the control values $\hat{v}_i$, $i = 1, \dots, r$, and $\hat{u}_i$, $i = 1, \dots, m$, respectively. The representations (5.17)–(5.18) are generally only local (valid for sufficiently small times $t$). The latter can be shown to be global if the algebra has special properties (is solvable), or else in the case of real $2 \times 2$ systems.

The above equations (5.17)–(5.18) allow to obtain the $\gamma$-coordinates equations (2.17) and reformulate the FIP as an LCIP. The procedure to obtain (2.17) is explained in detail in Appendix A, p. 222. In the context of the class of systems considered here, it requires the substitution of expression (5.17) into equation (5.14) which it satisfies, and the application of a form of Campbell-Baker-Hausdorff, namely the first expression in (5.4) before application of the norm[3], to rearrange the equation in such a way as to be able to equate the coefficients which correspond to the same basis elements $A_0, \dots, A_{r-1}$ on its both sides. The resulting set of ordinary differential equations (2.17) has in general a matrix $\Gamma$ which is invertible only in a neighborhood of $\gamma^e = 0$. However, if $\Gamma^{-1}$ exists for all values of $\gamma^e$ then the representation (5.17) is global and the functions $\gamma^e$ can be solved explicitly by solving the corresponding equation in (2.17) with $v = [1\ \hat{v}]$. A similar solution procedure can, of course, be applied to the flow equation for the original system (5.18), which yields the final statement of the TIP, now with respect to the logarithmic coordinates of the corresponding flows:

[3]This formula is referred to as *exponential formula* in [**6**], and is also included on p. 219 of Appendix A.

**LCIP:** Consider the two formal "control systems" of equation (2.17):

$$CS1: \quad \dot{\gamma}^e(t) \;=\; \Gamma\left(\gamma^e(t)\right)^{-1} v^d, \quad \gamma^e(0) = 0 \tag{5.20}$$

$$CS2: \quad \dot{\gamma}(t) \;=\; \Gamma\left(\gamma(t)\right)^{-1} u^d, \quad \gamma(0) \;= 0 \tag{5.21}$$

with $u^d = [1 \ \hat{u}]$ and $v^d = [1 \ \hat{v}]$. For a fixed value of the time horizon $T \leq T_{max}$, find control functions $\hat{u}_i(\hat{v}, t)$, $i = 1, \ldots, m$, $t \in [nT, (n+1)T)$, $n \in \mathbb{Z}_+$, in the class of functions which are continuous in $\hat{v}$ and piece-wise continuous in $t$, such that for *any constant control vector* $\hat{v} \in \mathbb{R}^{r-1}$:

$$\gamma(T, \hat{u}) = \gamma^e(T, \hat{v}) \tag{5.22}$$

where $\gamma(T, \hat{u})$ and $\gamma^e(T, \hat{v})$ denote the $\gamma$-coordinates at time $T$ corresponding to systems $\Sigma_H$ and $\Sigma_H^e$ with controls $\hat{u} \in \mathbb{R}^m$ and $\hat{v} \in \mathbb{R}^{r-1}$, respectively.

Both FIP and LCIP are clearly independent of the initial condition $x(0) = x$, but the control functions $\hat{u}(\hat{v}, t)$ must be found in terms of the parameter $\hat{v}$, the value of the discretized extended controls.

The existence of solutions to LCIP is not obvious, and if it exists it is often not unique. Solutions to LCIP will however always exist under assumption H1.BS, since it implies that the motion in the direction of any Lie bracket $A_i$, $i = m+1, \ldots, r-1$, can be realized by switching controls in the original system.

Using Proposition 5.1, it is now possible to show the following stabilization result for the closed-loop system (5.1).

**Theorem 5.2.** *Under the hypotheses H1.BS–H3.BS, and with $\hat{v}_i(n)$, $i = 1, \ldots, r-1$, defined in (5.9) (evaluated at periodically at states $x(nT)$, $n \in \mathbb{Z}_+$, of the original system), for any region $B(0, R)$, there exists a constant $T > 0$ in $[0, T_{max}]$ such that the time-varying feedback controls $u_i(x, t)$, $i = 1, \ldots, m$, defined as the periodic continuation of the solutions to LCIP:*

$$u_i(x, t) = \hat{u}_i(\hat{v}(n)), \quad \textit{for } t \in [nT, (n+1)T), \ n \in \mathbb{Z}_+, \ i = 1, \ldots, m, \tag{5.23}$$

*are asymptotically stabilizing for system (5.1), with region of attraction* $B(0, R)$.

PROOF. Proceeding similarly as in [**75**] and propositions 4.1 and 4.3 of Chapter 2, let $x_T \stackrel{\text{def}}{=} x(T, x, \hat{u})$ denote the state of system $\Sigma$ in (5.1) reached from $x$ at time $T$ by application of the time-varying feedback control $\hat{u}$ (which approximately solves the TIP), and let $x^e_T \stackrel{\text{def}}{=} x^e(T, x, \hat{v})$ denote the state of system $\Sigma^e$ in (5.5) reached from $x$ at time $T$ by application of the discretized control $\hat{v}$.

By Propostion 5.1 the extended system is Lyapunov stable with Lyapunov function $V(x) = x^T P x$, and therefore there exists a closed ball $\bar{B}(0, R)$, $R > 0$, which contains all the trajectories $x^e_t \stackrel{\text{def}}{=} x^e(t, x, \hat{v})$, $t \geq 0$ for all $(t, x) \in \mathbb{R}_+ \times B(0, R)$. Additionally, there exists $\eta > 0$ and $T > 0$ such that for any $(t, x) \in \mathbb{R}_+ \times B(0, R)$:

$$V(x^e_T) - V(x) \leq -\frac{\eta}{2}\|x\|^2 T \tag{5.24}$$

with extended system controls $\hat{v}$ satisfying $\|\hat{v}\| \leq M\|x\|$.

Let $K_A$ be a common bound for $g_i(x) = A_i x$, $i = 0, 1, \ldots, r-1$, on $\bar{B}(0, R)$. Since for all $s \in [0, T]$, $x^e_s \in B(0, R)$, there exists a constant $K_1$ such that $\|x^e_s\| \leq K_1\|x\| \leq R$ and the following estimates are immediate:

$$\begin{aligned} \|x^e_T - x\| &\leq \int_{t_0}^{t_0+T} \|A_0 x^e_s\| + \sum_{i=1}^{r-1} \|A_i x^e_s\|\|\hat{v}\| ds \\ &\leq K_1\|A_0\|\|x\|T + K_A\, M\|x\|T \\ &\leq cT\|x\| \end{aligned} \tag{5.25}$$

where $c = K_1\|A_0\| + K_A\, M$, and

$$\|x^e_T\| \leq \|x^e_T - x\| + \|x\| \leq (1 + cT)\|x\| \tag{5.26}$$

By (5.24),

$$V(x^e_T) \leq V(x) - \frac{\eta}{2}\|x\|^2 T \tag{5.27}$$

and letting $\Delta x_T = x_T - x_T^e$, the last inequality clearly implies that

$$\begin{aligned} V(x_T) &= V(x_T^e) + (V(x_T) - V(x_T^e)) \\ &\leq V(x) - \frac{\eta}{2}\|x\|^2 T + (V(x_T) - V(x_T^e)) \\ &\leq V(x) - \frac{\eta}{2}\|x\|^2 T + (x_T^e + \Delta x_T)^T P (x_T^e + \Delta x_T) - (x_T^e)^T P x_T^e \\ &\leq V(x) - \frac{\eta}{2}\|x\|^2 T + 2(\Delta x_T)^T P x_T^e + (\Delta x_T)^T P \Delta x_T \\ &\leq V(x) - \frac{\eta}{2}\|x\|^2 T + \|P\| \left(2\|x_T^e\|\,\|\Delta x_T\| + \|\Delta x_T\|^2\right) \end{aligned} \tag{5.28}$$

Let $\delta \in (0,1)$ be such that the function $\xi \to F(\xi)$ of Theorem 5.1 is bounded by some constant $M_F$ for $\xi \in [0,\delta]$. Since $\bar{B}(0,R)$ is bounded, there exists a $T_1$ such that

$$\max\{cT_1, T_1^{\frac{1}{2}}\}\|x\| \leq \delta, \qquad \text{for all } x \in \bar{B}(0,R) \tag{5.29}$$

Then, by virtue of the definition of $\Delta x_T$, equation (5.25) and the result of Theorem 5.1:

$$\begin{aligned} \|\Delta x_T\| &\leq M_F \|x_T^e - x\|^{1+\frac{1}{k}} \\ &\leq M_F c^{1+\frac{1}{k}} T^{1+\frac{1}{2k}} \|x\| \left(T^{\frac{1}{2}}\|x\|\right)^{\frac{1}{k}} \\ &\leq M_F c^{1+\frac{1}{k}} T^{1+\frac{1}{2k}} \|x\| \end{aligned} \tag{5.30}$$

for all $T \leq T_1$, and all $x \in \bar{B}(0,R)$, as $\delta < 1$. Using (5.26) and (5.30), the "error" on the right-hand side of (5.28) can be bounded as follows:

$$\begin{aligned} &\|P\| \left(2\|x_T^e\|\,\|\Delta x_T\| + \|\Delta x_T\|^2\right) \\ &\leq \|P\| \left(2(1+cT) M_F c^{1+\frac{1}{k}} T^{1+\frac{1}{2k}} \|x\|^2 + M_F^2 c^{2+\frac{2}{k}} T^{2+\frac{1}{k}} \|x\|^2\right) \end{aligned} \tag{5.31}$$

for all $T \leq T_1$ and $(t_0, x) \in \mathbb{R}_+ \times \bar{B}(0,R)$.

Finally, by (5.28) and (5.31),

$$V(x_T) - V(x) \leq -q(T)\|x\|^2 \tag{5.32}$$

where

$$q(T) = \frac{\eta}{2}T - \|P\| \left(2(1+cT_1)M_F c^{1+\frac{1}{k}} T^{1+\frac{1}{2k}} + M_F^2 c^{2+\frac{2}{k}} T^{2+\frac{1}{k}}\right) \tag{5.33}$$

Noting that $q(0) = 0$ and that $\frac{dq}{dT}(0) = \frac{\eta}{2} > 0$, implies that there exists $T \leq T_1$ such that $q(T) > 0$ is increasing for all $T \in [0, T_1)$. Hence, by inequality (5.32) and Theorem 4.1 on page 84 of Chapter 4, the original system with time-varying control $\hat{u}$ (that approximately solves the TIP with period $T \leq T_1$), is also asymptotically stable in $B(0, R)$. □

Remark 5.3. *The estimation of the order of truncation $k$ and the constant $R$ is a difficult problem since both parameters depend on the structure and the dimension of the Lie algebra.*

## 5.4. Example Using the TIP Approach

For simplicity, the system to be stabilized is defined on the plane:

$$\dot{x} = A_0 x + A_1 x u \tag{5.34}$$

with $x \stackrel{\text{def}}{=} [x_1 \; x_2]^T$ and the matrices $A_0$, $A_1$ given by:

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & 4 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \tag{5.35}$$

It is worth noticing that the above system has the property that there exists no constant control $u$ for which the system matrix $A_0 + A_1 u$ becomes stable (in terms of $u$, the eigenvalues of this matrix are $\lambda_{\{1,2\}} = 2 \pm \sqrt{3-u}$), hence stabilization of (5.34) is non-trivial.

The Lie algebra $L(A_0, A_1)$ is of dimension four as shown by the following Lie bracket multiplication table in which the Lie product of any two matrices $A$ and $B$ is calculated as the matrix commutator $[A, B] \stackrel{\text{def}}{=} BA - AB$.

| | $A_0$ | $A_1$ | $A_2$ | $A_3$ |
|---|---|---|---|---|
| $A_0$ | 0 | $A_2$ | $A_3$ | $12A_3$ |
| $A_1$ | | 0 | $-2A_1$ | $-2A_2$ |
| $A_2$ | | | 0 | $24A_1 - 2A_3$ |
| $A_3$ | | | | 0 |

In the above table the following shorthand notation is used:

$$A_2 \stackrel{\text{def}}{=} [A_0, A_1] \tag{5.36}$$

$$A_3 \stackrel{\text{def}}{=} [A_0, [A_0, A_1]] \tag{5.37}$$

To facilitate further the derivation of the equations for the logarithmic coordinates of flows, the series expansions of the exponentials will be truncated at Lie brackets of order one whenever employing the following Campbell-Baker-Hausdorff formula (see Lemma A.1, on p. 219 of Appendix A):

$$\begin{aligned} \exp(X)Y\exp(-X) &= Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \ldots \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\mathrm{ad}_X^k Y \stackrel{\text{def}}{=} \exp(\mathrm{ad}_X)Y \end{aligned} \tag{5.38}$$

for any $X = g(t)A$ and $Y = h(t)B$, $g, h : \mathbb{R} \to \mathbb{R}$, $A, B \in \mathbb{R}^{n\times n}$, where by the bilinear nature of the Lie product $[g(t)A, h(t)B] = g(t)h(t)[A, B]$, and where the operation "ad" can be defined recursively as follows (see (A.9), p. 215):

$$\begin{aligned} \mathrm{ad}_X^k Y &\stackrel{\text{def}}{=} (\mathrm{ad}_X^{k-1})\mathrm{ad}_X Y \\ \mathrm{ad}_X Y &\stackrel{\text{def}}{=} [X,Y] \\ \mathrm{ad}_X^0 Y &\stackrel{\text{def}}{=} Y \end{aligned} \tag{5.39}$$

The latter amounts to stating that all the higher order Lie brackets of $A_0$ and $A_1$ can be assumed to be equal to zero. Considering this simplifying assumption, the extended system for (5.34) involves only the first order Lie bracket and is given by:

$$\dot{x} = A_0 x + A_1 x v_1 + [A_0, A_1] x v_2 \tag{5.40}$$

This simplification is possible due to the fact that

$$\operatorname{span} L(A_1 x, [A_0, A_1]x) = \mathbb{R}^2 \tag{5.41}$$

for all $x \in \mathcal{S} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^2 \mid x_1 \neq 0\}$. Thus the matrix $Q(x) \stackrel{\text{def}}{=} \big[A_1 x \;\; [A_0, A_1]x\big]$ is invertible on $\mathcal{S}$. The singularity of $Q(x)$ on the complement of $\mathcal{S}$, $\mathcal{S}^C$, does not incur problems as, in this case, the un-forced system escapes $\mathcal{S}^C$.

The extended system control can thus be evaluated as

$$v(x) \quad = \quad Q(x)^{-1}[A_d x - A_0 x] \tag{5.42}$$

where $v(x) \stackrel{\text{def}}{=} [v_1(x) \;\; v_2(x)]^T$, and where $A_d$ is a suitably chosen matrix such that the reference system $\dot{x} = A_d x$ is asymptotically stable.

The FIP now requires finding a control $\hat{u}(\hat{v}, t)$ such that the flows $S^e(t)$ and $S(t)$ , respectively satisfying:

$$\dot{S}^e(t) \quad = \quad (A_0 + A_1 \hat{v}_1 + A_2 \hat{v}_2) S^e(t), \qquad S^e(0) \;=\; I \tag{5.43}$$

$$\dot{S}(t) \quad = \quad (A_0 + A_1 \hat{u}(\hat{v}, t)) S(t), \qquad S(0) \;=\; I \tag{5.44}$$

intersect at $T$.

To this end, it remains to derive the equations describing the evolution of the corresponding logarithmic coordinates and to solve the associated LCIP. Assuming that the solution to (5.43) is of the form $S^e(t) = \exp(\gamma_0(t)A_0)\exp(\gamma_1(t)A_1)\exp(\gamma_2(t)A_2)$, its time derivative is calculated as:

$$\dot{S}^e(t) \quad = \quad [\dot{\gamma}_0 A_0 + \dot{\gamma}_1 \exp(\gamma_0 \mathrm{ad}_{A_0}) A_1 + \dot{\gamma}_2 \exp(\gamma_0 \mathrm{ad}_{A_0}) \exp(\gamma_1 \mathrm{ad} A_1) A_2]\, S^e(t) \tag{5.45}$$

Using the Campbell-Baker-Hausdorff formula (5.38) (with higher order brackets taken to be zero) yields:

$$\exp(\gamma_0 \mathrm{ad}_{A_0}) A_1 \quad = \quad A_1 + \gamma_0 A_2 \tag{5.46}$$

$$\exp(\gamma_0 \mathrm{ad}_{A_0}) \exp(\gamma_1 \mathrm{ad}_{A_1}) A_2 \quad = \quad A_2 \tag{5.47}$$

Substituting (5.45)–(5.47) into (5.43) and equating coefficients of $A_0, A_1$ and $A_2$ gives the control system CS1 of LCIP:

$$CS1: \begin{cases} \dot{\gamma}_0^e &= 1 \\ \dot{\gamma}_1^e &= \hat{v}_1 \\ \dot{\gamma}_2^e &= -\gamma_0^e \hat{v}_1 + \hat{v}_2 \end{cases} \tag{5.48}$$

Similarly the control system CS2 is

$$CS2: \begin{cases} \dot{\gamma}_0 &= 1 \\ \dot{\gamma}_1 &= \hat{u} \\ \dot{\gamma}_2 &= -\gamma_0 \hat{u} \end{cases} \tag{5.49}$$

It can be verified that one possible solution of the LCIP is

$$\hat{u}(\hat{v}, t) = \hat{v}_1 + \frac{2\pi \hat{v}_2}{T} \sin(\frac{2\pi}{T} t) \tag{5.50}$$

defined for $t \in [0, T]$. In terms of the continuous extended feedback controls the final stabilizing control law is thus

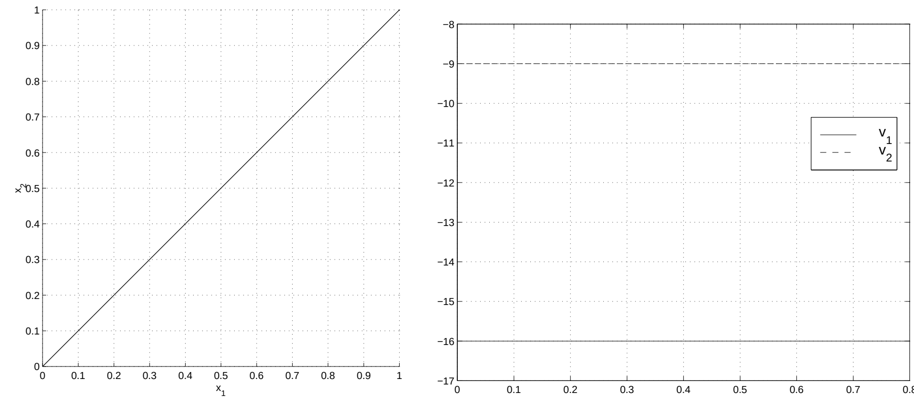
$$\hat{u}(v(x), t) = v_1(x) + \frac{2\pi v_2(x)}{T} \sin(\frac{2\pi}{T} t) \tag{5.51}$$

which is now defined for $t \in [0, \infty)$, due to the periodic continuation of the sine.

One set of simulation results is presented and corresponds to a reference system in which $A_d = -\alpha I$, with gain $\alpha = 8$. The period used in the solution of LCIP was $T = 0.01$. Fig. 5.1 (a)–(b) show the extended system trajectory and the corresponding extended controls, respectively. Fig. 5.2 (a)–(b) show the original system trajectory (in the phase plane) and the respective stabilizing control $u(x, t) = \hat{u}(v(x), t)$, in which the extended controls $v_i(x)$, $i = 1, 2$, have been updated every $T/10$. Finally, Fig. 5.3 displays the controlled system state variables versus time.

## 5.5. Method 2: Stabilization to the Stable Manifold Approach

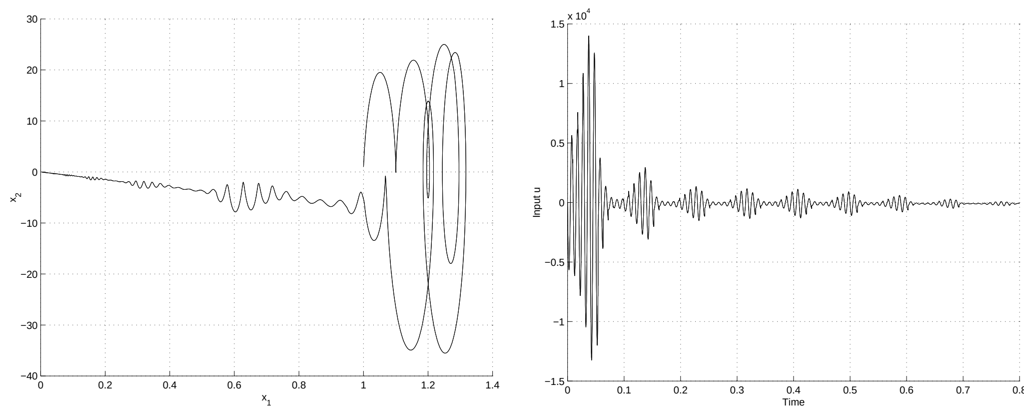
The underlying idea of the feedback synthesis is simple and draws on variable structure control approach. The feedback control comprises two stages: the reaching phase and the sliding phase.




(a) Trajectory of the extended system in the phase plane.

(b) Extended system controls $v_1(x)$, $v_2(x)$.

FIGURE 5.1. Stabilization of the Lie algebraic extension of the original bilinear system (5.34).




(a) Trajectory of the original system in the phase plane.

(b) Stabilizing control $u(x,t) = \hat{u}(\hat{v}(x),t)$.

FIGURE 5.2. Stabilization of the original bilinear system (5.34) using the trajectory interception approach.

The task of the reaching phase control is to steer the system to a stable subspace in finite time, while the task of the sliding phase control is to keep the system's state evolving in the set of stable subspaces in the presence of limited external disturbances.

For simplicity of exposition, the approach will be explained with reference to bilinear systems which, through the use of constant controls, yield linear systems with stable manifolds of co-dimension one with respect to the state space. The approach can be generalized to systems with stable manifolds of any co-dimension, as is later shown in terms of an example. Thus, in addition to assumptions H1.BS–H3.BS, the following assumption is needed:

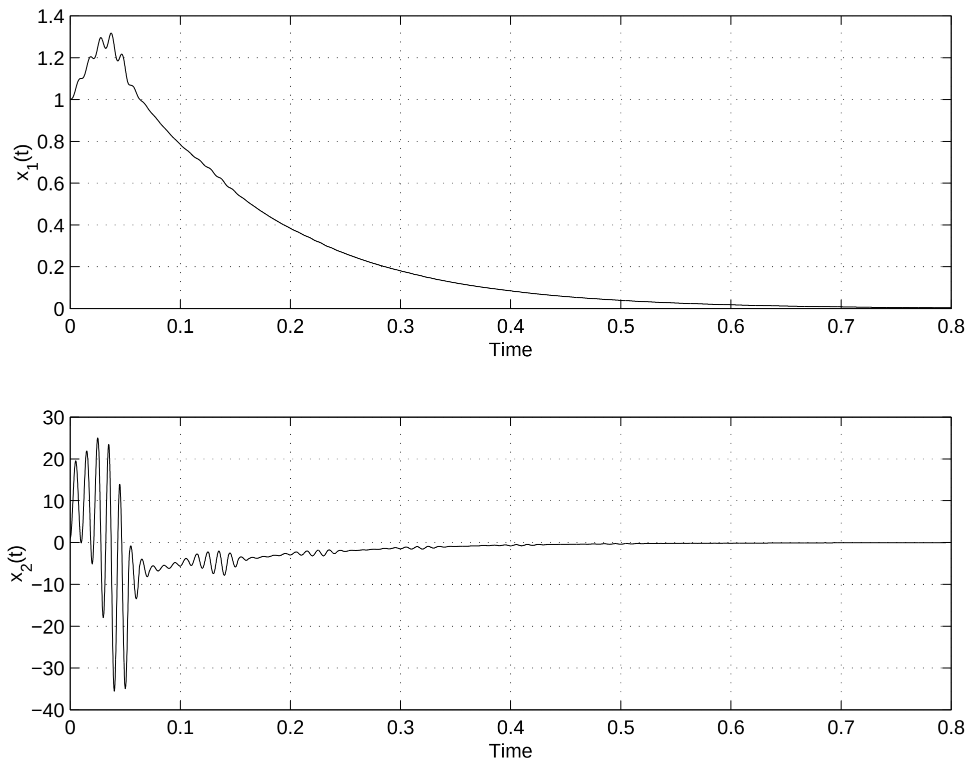



FIGURE 5.3. Original system state variables $x_1$, $x_2$ versus time.

H4.BS. There exists constant controls $u^* \stackrel{\text{def}}{=} [u_1^*, \ldots, u_m^*]^T$ such that the corresponding linear system with system matrix $A(u^*) = A_0 + \sum_{i=1}^m A_i u_i^*$ has $n-1$ stable eigenvalues $\lambda_i(u^*)$, $i = 1, \ldots, n-1$, i.e. $\lambda_i(u^*) \in \mathbb{C}_-^\circ \stackrel{\text{def}}{=} \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$, $i = 1, \ldots, n-1$.

REMARK 5.4. *Assumption H4.BS implies that there does not exist constant controls such that* $A(u) = A_0 + \sum_{i=1}^m A_i u_i$ *is stable.*

Under assumption H4.BS, let $\lambda_i(u^*) = a_i + \mathbf{i}\, b_i$, $i = 1, \ldots, n-1$, and let $w_i = s_i + \mathbf{i}\, v_i$, $i = 1, \ldots, n-1$, be the generalized eigenvectors of the matrix $A(u^*)$. Then the stable subspace corresponding to the constant control $u^*$ can be defined as:

$$E^s(u^*) = \text{span}\ \big\{ s_i, v_i \mid a_i < 0, i \in \{1, 2, \ldots, n\} \big\} \tag{5.52}$$

It will also be useful to define the collection of stable subspaces

$$\mathcal{S} \stackrel{\text{def}}{=} \bigcup_{u \in \mathcal{U}} E^s(u) \tag{5.53}$$

where $\mathcal{U} \stackrel{\text{def}}{=} B(u^*, \delta)$ is ball $B(u^*, \delta) \subset \mathbb{R}^m$ of center $u^*$ and radius $\delta$ for some $\delta > 0$ such that assumption H4.BS is satisfied.

### 5.5.1. The Reaching Phase Control

Let $n(u^*)$, $\|n(u^*)\| = 1$, be the normal vector to the stable subspace $E^s(u^*)$ defined by (5.52). Drawing on the ideas presented in [**93**], a generalized control Lyapunov function is defined by the signed distance to the stable subspace:

$$V(x) \stackrel{\text{def}}{=} \zeta n^T x \tag{5.54}$$

Along the trajectories of the system (5.1) the time derivative of $V(x)$ is given by

$$\dot{V}(x) \quad \stackrel{\text{def}}{=} \quad \frac{dV}{dt}(x) \quad = \quad \frac{\partial V}{\partial x} f^u(x) \quad = \quad a(x) + b(x)u$$

where

$$\begin{array}{rclcrcl} a(x) & \stackrel{\text{def}}{=} & \nabla V A_0 x, & \qquad & b_i(x) & \stackrel{\text{def}}{=} & \nabla V A_i x, \ i = 1, \ldots, m \\ b(x) & \stackrel{\text{def}}{=} & [b_1(x) \ \cdots \ b_m(x)], & & u & \stackrel{\text{def}}{=} & [u_1, \ldots, u_m]^T \end{array}$$

It is clear that the feedback control of the form (2.20) defined here as

$$u^r(x) \quad \stackrel{\text{def}}{=} \quad \frac{-a(x) - \eta \,\text{sign}\{V(x)\}}{\|b(x)\|^2} b(x)^T \tag{5.55}$$

with $\eta > 0$ is well defined and bounded for all $x \notin N_\epsilon(E^b)$, where $N_\epsilon(E^b)$ denotes a suitably chosen $\epsilon$-neighbourhood of the set

$$\begin{aligned} E^b \quad & \stackrel{\text{def}}{=} \quad \{x \in \mathbb{R}^n \mid b(x) = 0\} \\ & = \quad \{x \in \mathbb{R}^n \mid n^T A_i x = 0, \ i = 1, \ldots, m\} \end{aligned} \tag{5.56}$$

so that $N_\epsilon(E^b) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid \|b(x)\| < \epsilon\}$. Moreover, for all $x \notin N_\epsilon(E^b)$

$$\dot{V}(x) \quad \stackrel{\text{def}}{=} \quad a(x) + b(x)u^r(x) = -\eta \operatorname{sign}\{V(x)\} \tag{5.57}$$

which implies that the distance between the current state of the system using the control $u^r$ and the stable subspace $E^s(u^*)$ is decreasing at a constant rate $\eta$ as long as the closed-loop trajectory evolves in the complement of $N_\epsilon(E^b)$.

Once the system trajectory enters $N_\epsilon(E^b)$, a different control needs to be constructed as for $x \in N_\epsilon(E^b)$ there may not exist any control value $u$ which renders $\dot{V}(x) < 0$ guaranteeing monotonic decrease in $V$. In this event, further (non-monotonic) decrease in the generalized Lyapunov function is achieved employing the stabilization approach presented in Section 4.3.2, p. 76, for the synthesis of discontinuous time-varying controls. For clarity of exposition, this approach is explained here in the context of bilinear systems as follows.

Consider a sequence of constant inputs $\bar{u} \stackrel{\text{def}}{=} \{u_{(1)}, u_{(2)}, \ldots, u_{(s)}\}$, where each $u_{(i)}$ is applied to (5.1) for a time $\varepsilon_i$, $i = 1, \ldots, s$, in the set $\bar{\varepsilon} \stackrel{\text{def}}{=} \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s\}$, such that $T = \sum_{i=1}^{s} \varepsilon_i$. The state of (5.1), $x(T, x, \bar{u})$, resulting from the application of $(\bar{u}, \bar{\varepsilon})$ to this system with initial condition $x$ at time $t = 0$ is given as the composition of flows:

$$x(T, x, \bar{u}) = \exp(\varepsilon_s A(u_{(s)})) \circ \cdots \circ \exp(\varepsilon_2 A(u_{(2)})) \circ \exp(\varepsilon_1 A(u_{(1)}))x \tag{5.58}$$

where $\exp(\varepsilon A(u))x_0$ denotes the solution of $\dot{x} = A(u)x$ through $x_0$ at $t = 0$ evaluated at $t = \varepsilon$; i.e $\exp(\varepsilon A(u))x$ denotes the flow of $A(u)x$.

As in Section 4.3.2 (see equation (4.29)), by virtue of the CBH formula, for sufficiently small times $\bar{\varepsilon}$, the composition of flows (5.58) can be expressed in the form of a single flow; i.e. there exists a matrix $\bar{A}(\bar{u}, \bar{\varepsilon})$ such that

$$x(T, x, \bar{u}) = \exp(T\bar{A}(\bar{u}, \bar{\varepsilon}))x \text{ for all } \bar{u} \in \mathcal{P}^m \text{ and } \bar{\varepsilon} \in \mathbb{R}^s \tag{5.59}$$

By assumption H3.BS.a, the matrix $\bar{A}(\bar{u},\bar{\varepsilon})$ has the following finite expression in terms of the matrices in the definition of system $\Sigma^e_{BS}$:

$$\bar{A}(\bar{u},\bar{\varepsilon}) = A_0 + \sum_{i=1}^{r-1} c_i(\bar{u},\bar{\varepsilon})A_i \tag{5.60}$$

where the scalar coefficients $c_i$ are nonlinear functions in the components of $(\bar{u},\bar{\varepsilon})$, whose analytic expressions can be determined from the CBH formula after collection of terms involving the same basis elements, see Example 2 in Section 5.6.

Equation (5.59) can be regarded as the solution to a system $\dot{x} = \bar{f}(x,\bar{u},\bar{\varepsilon}) \stackrel{\text{def}}{=} A(\bar{u},\bar{\varepsilon})x$. In this sense, equation (4.30) is equivalent to (5.60), as $\bar{f}$ is given in the case of bilinear systems simply by $\bar{f} = \bar{A}x$. Now, since $\bar{f}$ is spanned by the elements of $\mathcal{G}$ (as implied by $\bar{f} = \bar{A}x$), motions of system $\Sigma$ along $\bar{f}$ can be made to lie in any direction of $\mathbb{R}^n$. To achieve motions in an adequate direction such that the Lyapunov function $V(x)$ is decreased, the SP2 in (4.32)–(4.33) is solved to yield a control pair $(\bar{u},\bar{\varepsilon})$.

For a given $x \in N_\epsilon(E^b)$, the satisficing problem SP2 is stated here as follows:

**SP-BS:** For given constants $\eta > 0$, $T > 0$ and $M > 0$, and for a given $x \in B(0,R)$, find a feasible pair $(\bar{u},\bar{\varepsilon}) \in \mathcal{P}^m \times \mathbb{R}^s$, such that for some $s < \infty$:

$$\begin{aligned} \operatorname{sign}\{V(x)\} n^T \bar{A}(\bar{u},\bar{\varepsilon})x &\leq -\eta\|x\| && (5.61) \\ \|c(\bar{u},\bar{\varepsilon})\| &\leq M\|x\| && (5.62) \end{aligned}$$

where $\bar{A}$ and $c(\bar{u},\bar{\varepsilon}) \stackrel{\text{def}}{=} [c_1(\bar{u},\bar{\varepsilon}) \ \cdots \ c_{r-1}(\bar{u},\bar{\varepsilon})]^T$ are defined by (5.60), and $s$ is the number of switches in the control sequence $\bar{u} \in \mathcal{P}^m$.

The following propositions ensure the existence of solutions to the above satisficing problem and their stabilizing properties.

Proposition 5.2. *Under assumptions H1.BS–H4.BS, for any neighbourhood of the origin $B(0,R)$ and any $\eta > 0$, there exists a constant $M(R,\eta) > 0$, such that a solution to SP-BS exists for any $x \in B(0,R)$, and any control horizon $T > 0$, provided that $s \in \mathbb{N}$, the number of switches in the control sequence $\bar{u}$, is allowed to be large enough.*

PROOF. For any $\epsilon \in (0, \frac{\eta}{\zeta^2}]$ and any given $x \in B(0,R)$ let

$$z(x) \stackrel{\text{def}}{=} -\epsilon \nabla V^T(x) \|x\| \tag{5.63}$$

Then,

$$\nabla V z(x) = -\epsilon \|\nabla V(x)\|^2 \|x\| \leq -\epsilon \zeta^2 \|x\| \leq -\eta \|x\| \tag{5.64}$$

since by (5.54), $\|\nabla V(x)\| = \zeta$ for all $x \notin E^b$.

Furthermore, $z(x)$ can be written in terms of the right-hand side of the extended system $\Sigma^e$ as

$$z(x) = g_0(x) + \sum_{i=1}^{r-1} g_i(x) v_i \stackrel{\text{def}}{=} g^v(x) \tag{5.65}$$

where $g_i(x) = A_i x$, $A_i \in \mathbb{R}^{n \times b}$ are matrices defined as in (5.60), and $v \stackrel{\text{def}}{=} [v_1 \ v_2 \ \cdots v_{r-1}]^T \in \mathbb{R}^{r-1}$ is a vector of constant parameters.

It follows from (5.65) that for a given $x \in \mathbb{R}^n$:

$$v(x) \quad = \quad Q^\dagger(x) \left( z(x) - g_0(x) \right), \qquad v = [v_1 \ v_2 \ \ldots \ v_{r-1}]^T \tag{5.66}$$

where $Q^\dagger(x) = Q^T \left( Q \, Q^T \right)^{-1}$ is the pseudo-inverse of the $n \times (r-1)$ matrix $Q(x) = [g_1(x) \ g_2(x) \ \ldots \ g_{r-1}(x)]$, which is ensured to exist for all $x \in \mathbb{R}^n$ because $\text{rank}\big(Q(x)\big) = n$ by the assumption that $g_i$, $i = 1, \ldots, r-1$, contains the basis for $L_x(\mathcal{F})$. Moreover, $Q^\dagger$ is a smooth matrix function of $x$ so there exists a constant $c(R) > 0$ such that

$$\|Q^\dagger(x)\| \leq c, \qquad \forall \ x \in B(0,R) \tag{5.67}$$

By Lipschitz continuity of $g_0 = A_0 x$ there exists a constant $d(R) > 0$, such that:

$$\begin{aligned} \|v\| \quad &\leq \quad \|Q^\dagger(x)\| \, \|z(x) - g_0(x)\| && (5.68) \\ &\leq \quad \|Q^\dagger(x)\| \, \left( \|z(x)\| + \|g_0(x)\| \right) && (5.69) \\ &\leq \quad c \left( \frac{\eta}{\zeta} \|x\| + d\|x\| \right) && (5.70) \end{aligned}$$

Let $M = c\left(\frac{\eta}{\zeta} + d\right)$ be the constant employed in (5.62). Then, a solution to SP-BS exists if there is a control pair $(\bar{u}, \bar{\varepsilon})$ such that $c_i(\bar{u}, \bar{\varepsilon}) = v_i$, $i = 1, \dots, r-1$. To demonstrate the existence of a control pair $(\bar{u}, \bar{\varepsilon})$, consider the extended system:

$$\Sigma^e : \qquad \dot{y} = g^v(y), \qquad y(0) = x \tag{5.71}$$

with state $y \in \mathbb{R}^n$ and constant control $v$ defined as in (5.66), (note that $v$ is a function of $x$ but not of $y$).

The integration of (5.71) over $[0, T]$ yields

$$\begin{aligned} y(T) &= \exp(Tg^v)x \\ &= \phi_G^+(S^e(T))x \end{aligned} \tag{5.72}$$

where $S^e(\cdot)$ is the solution to the system (5.71) reformulated as right-invariant system on the Lie group $H$. By virtue of global strong controllability of $\Sigma_H$ on $H$, as shown in Proposition 2.1, there exists a control pair $(\bar{u}, \bar{\varepsilon}) \in \mathcal{P}^m \times \mathbb{R}^s$, which steers system $\Sigma_H$ from $e \in H$ to $S(T) = S^e(T)$ in time $T$, i.e.:

$$\exp(\varepsilon_1 \lambda_1) \circ \cdots \circ \exp(\varepsilon_s \lambda_s) = \exp(T\bar{\lambda}) = S^e(T) \tag{5.73}$$

where $\lambda_i = (\phi_L^+)^{-1}(f^{u_{(i)}})$, for $i = 1, \dots, s$, and $\bar{\lambda}$ resulting from the application of the CBH formula on $H$. It follows from (5.72) and (5.73) that

$$\exp(T\bar{\lambda}) = (\phi_G^+)^{-1} \exp(Tg^v) = \exp(T(\phi_L^+)^{-1}(g^v)) \tag{5.74}$$

Since the exponential map is a global diffeomorphism on $H$ it follows that $\bar{\lambda} = (\phi_L^+)^{-1}(g^v)$, i.e. $\phi_L^+(\bar{\lambda}) = \bar{A}x = g^v$, with $\bar{f} = \bar{A}x$ and $\bar{f}$ as in (4.28). Due to the expansion (5.60), $c_i = v_i$ for all $i = 1, \dots, r-1$, as required. $\square$

PROPOSITION 5.3. *Let $\bar{u}(x,\tau)$, $\tau \in [0,T]$, be a control generated by the solution pair $(\bar{u},\bar{\varepsilon})$ to SP-BS. There exists a $T_{max} > 0$ such that for all $T \in [0, T_{max}]$:*

$$V(x(T,x,\bar{u})) - V(x) \leq -\frac{\eta}{2}\|x\|T \tag{5.75}$$

PROOF. Since $V(x) = \zeta n^T x$, for all $x \notin E^s$, and $\bar{f} = \bar{A}x$ is analytic and linear in $c_i$, $i = 1,\ldots,r-1$, and $f_0(0) = 0$, then $\nabla V = \zeta n^T$ and $\bar{f}$ is Lipschitz continuous on $B(0,2R)$, uniformly with respect to $c(\bar{u},\bar{\varepsilon}) = [c_1(\bar{u},\bar{\varepsilon}) \ \cdots \ c_{r-1}(\bar{u},\bar{\varepsilon})]^T$ satisfying $\|c(\bar{u},\bar{\varepsilon})\| \leq M\|x\|$. Hence, there exists a $K > 0$ such that:

$$\left\|\bar{f}(y,\bar{u},\bar{\varepsilon}) - \bar{f}(x,\bar{u},\bar{\varepsilon})\right\| \leq K\|y-x\| \tag{5.76}$$

for all $x \in B(0,R)$, $y \in B(0,2R)$, and for all control pairs satisfying $\|c(\bar{u},\bar{\varepsilon})\| \leq M\|x\|$.

Let $x(t) \stackrel{\text{def}}{=} x(t,x,\bar{u})$, $t \geq 0$. First, it is shown that there exists a $T_1 > 0$ and a constant $K_1 > 0$ such that

$$\|x(s) - x\| \quad \leq \quad \|x\|\left(\exp(K\,s) - 1\right) \tag{5.77}$$

and

$$\|x(s)\| \leq K_1\|x\| \tag{5.78}$$

for all $s \in [0,T_1]$ such that $x(s) \in B(0,2R)$.

To this end it suffices to notice that

$$\|x(s)-x\| \leq \int_0^s \|\bar{f}(x,\bar{u},\bar{\varepsilon})\|d\tau + \int_0^s \|\bar{f}(x(\tau),\bar{u},\bar{\varepsilon}) - \bar{f}(x,\bar{u},\bar{\varepsilon})\|d\tau \leq K\,\|x\|\,s + \int_0^s K\|x(\tau)-x\|d\tau$$

which, by the application of the Gronwall-Bellman lemma (see Appendix C.1, p. 244), yields inequality (5.77).

It is possible to see that if $T_1$ is chosen so that $(\exp(K\,T_1) - 1) \leq \frac{1}{2}$ then (5.77) holds for $s \in [0,T_1]$. By contradiction, suppose that there exists an $s_1 < T_1$ such that $\|x(s_1)\| = 2R$. It follows that $2R \leq \|x\| + \|x(s_1) - x\| \leq R + \|x\|\left(\exp(K\,s_1) - 1\right) \leq \frac{3}{2}R$ which is false, and hence (5.77) is valid for $s \in [0, T_1]$. Inequality (5.78) follows from (5.77) since

$$\|x(s)\| \le \|x(s) - x\| + \|x\| \le \|x\| \exp(K\,s) \le K_1 \|x\|$$

with $K_1 = \exp(KT_1)$.

Now,

$$\begin{aligned} V(x(T)) - V(x) &\le \nabla V(x) \bar{f}(x, \bar{u}, \bar{\varepsilon}) T + \int_0^T \left\| \nabla V(x(s)) \bar{f}(x(s), \bar{u}, \bar{\varepsilon}) - \nabla V(x) \bar{f}(x, \bar{u}, \bar{\varepsilon}) \right\| ds \\ &\le -\eta \|x\| T + K\,\zeta \int_0^T \|x(s) - x\|\, ds \end{aligned} \tag{5.79}$$

since $\nabla V(x) = \zeta n^T$ for any $x \notin E^s$ and

$$\begin{aligned} \left\| \nabla V(x(s)) \bar{f}(x(s), \bar{u}, \bar{\varepsilon}) - \nabla V(x) \bar{f}(x, \bar{u}, \bar{\varepsilon}) \right\| &\le \|\nabla V(x(s))\| \, \|\bar{f}(x(s), \bar{u}, \bar{\varepsilon}) - \bar{f}(x, \bar{u}, \bar{\varepsilon})\| \\ &\le K\,\zeta \|x(s) - x\| \end{aligned}$$

Hence, if $T < T_1$ then $x(s) \in B(0, 2R)$ for all $s \in [0, T]$ and, using (5.77) in (5.79), yields

$$V(x(T)) - V(x) \le -\eta \|x\| T + K\,\zeta \|x\| \int_0^T (\exp(K\,s) - 1)\, ds \le -\frac{\eta}{2} \|x\| q(T)$$

where $q(T) \stackrel{\text{def}}{=} \left(2 + \frac{2\,K\,\zeta}{\eta}\right) T - \frac{2\,\zeta}{\eta} (\exp(K\,T) - 1)$. If $r(T) \stackrel{\text{def}}{=} q(T) - T$, then $r(0) = 0$ and $r'(0) = 1$, so there exists a $T_{max} \le T_1$ such that $r(T) \ge 0$ for all $T \in [0, T_{max}]$. Hence $q(T) \ge T$ for all $T \in [0, T_{max}]$ which proves (5.75). $\square$

The concatenated control $u^c(x, \tau)$ is defined in terms of the solution pair $(\bar{u}, \bar{\varepsilon})$ to SP-BS computed at discrete instants of time $nT$, $n \in \mathbb{Z}_+$ as:

$$\begin{aligned} u^c(x, \tau) &\stackrel{\text{def}}{=} \bar{u}(x(n\,T), \tau) \text{ for all } \tau \in [nT, (n+1)T], \quad n \in \mathbb{Z}_+ \\ &= u_{(k)}, \text{ for } t \in [nT + \sum_{i=1}^{k-1} \varepsilon_i, nT + \sum_{i=1}^{k} \varepsilon_i], \quad k = 1, \ldots, s, \quad n \in \mathbb{Z}_+ \end{aligned} \tag{5.80}$$

where $x(nT)$ is the state of the closed-loop system $\Sigma$ at time $nT$.

The control $u^c(x,\tau)$ now serves in the definition of the reaching phase control, whose stabilizing property is established established by the following theorem.

**Theorem 5.3.** *Under hypotheses H1.BS–H4.BS, the reaching phase control defined by*

$$u^R(x,t) \stackrel{\text{def}}{=} \begin{cases} u^r(x) & x \notin N_\epsilon(E^b) \\ u^c(x,t) & x \in N_\epsilon(E^b) \end{cases} \tag{5.81}$$

*brings the system (5.1) to the set $N_\epsilon(E^s(u^*))$ in finite time $T^*$, where $N_\epsilon(E^s(u^*)) \stackrel{\text{def}}{=} \{x : \|x-y\| \leq \epsilon,$ for all $y \in E^s(u^*)\}$ is an arbitrary small $\epsilon$-neighbourhood of the stable subspace $E^s(u^*)$.*

Proof. If $x \notin E^b$, the reaching phase control $u^r(x)$, in (5.55), is such that $\dot{V}(x) = -\eta < 0$ if $V(x) > 0$ and $x \notin E^s(u^*)$, or $\dot{V}(x) = \eta > 0$ if $V(x) < 0$ and $x \notin E^s(u^*)$. Also, $\dot{V}(x) = 0$ if $V(x) = 0$ and $x \in E^s(u^*)$. It is then possible to conclude that $V(x(t))$ decreases in magnitude for all $x \notin E^s(u^*) \cup E^b$. Moreover, if $x(t) \notin E^b$ for all $t \geq t_0$ the set $E^s(u^*)$ may be reached in finite time $T^* = \frac{1}{\eta}|V(x_0)|$, i.e. $V(x(t_0+T^*)) = 0$, since $V(x(t)) = V(x(t_0)) - \eta \int_{t_0}^{t} \text{sign}\{V(x(s))\}ds = V(x(t_0)) - \eta\text{sign}\{V(x(t_0))\}(t-t_0)$ for all $t < t_0 + T^*$.

If $x(t) \in N_\epsilon(E^b)$ for some $t \geq t_0$ then the application of the time-varying concatenated control $u^c(x,t)$ is necessary. By Proposition 5.3 the state of system $\Sigma$ in (5.1) with control the concatenated control (5.80) satisfies $V(x(t_{k+1})) - V(x(t_k)) \leq -\frac{\eta}{2}\|x(t_k)\|T$, for all $k \in \mathbb{Z}_+$ with $t_k = kT$ and hence, application of Theorem 4.1, now with $\gamma(\|x(t_k)\|) = \frac{\eta}{2}\|x(t_k)\|T$, ensures the asymptotic convergence of the trajectories of the closed-loop system to the stable manifold $E^s(u^*)$, i.e. for any $\epsilon > 0$ there exits a time $T^* > t_0$ such that $x(t,x,u^R) \in N_\epsilon(E^s(u^*))$ for all $t \geq t_0 + T^*$. □

### 5.5.2. The Sliding Phase Control

Once the closed-loop system using the control $u^R$ reaches the the neighbourhood $N_\epsilon(E^s(u^*)) \cap \mathcal{S}$ of stable subspaces $E^s(u^*)$ it is then logical to switch the control to a constant value $u \in \mathcal{U} = B(u^*, \delta)$ such that $E^s(u) \ni x$ for $x \in \mathcal{S}$, and which, in the absence of any external disturbances, keeps the system's state evolving in $\mathcal{S}$ for all future times. The latter results in asymptotic stabilization since $\mathcal{S}$ is stable. As the assumption about the absence of external disturbances is not practical, a more realistic (feedback) version of the sliding control must take account of any possible deviations of the system's state from the desirable stable subspace $E^s(u^*)$. To this end, it is worth pointing out the following consequence of continuity of the eigenvalues as functions of the control $u \in \mathcal{U}$.

PROPOSITION 5.4. *For a system (5.1) satisfying H1.BS and H2.BS, let $u^* \in \mathbb{R}^m$ be a control which satisfies assumption H4.BS. Then there exists a $\delta > 0$ such that assumption H4.BS holds for all $u \in \mathcal{U} = B(u^*, \delta)$. The mapping $u \to n(u)$, with $n(u) \in \mathbb{R}^n$, $\|n(u)\| = 1$, the normal vector to $E^s(u)$, is continuous on $\mathcal{U}$. Consequently, the point to set mapping $u \to E^s(u)$ is continuous on $\mathcal{U}$.*

PROOF. First, note that for a fixed control $u$ the right hand side of equation (5.1) corresponds to a linear system with constant matrix $A(u)$. It is well-known that solutions to the corresponding characteristic equation, defined by $P(s,u) = \det(sI - A(u)) = 0$, are continuous with respect to the parameters $u$.

Also notice that the controllability assumptions H1.BS–H2.BS imply that no invariant subspace $Q$ exists such that solution trajectories starting from $x_0 \in Q$ remain in $Q$ for all $t \geq 0$ and any $u \in \mathcal{P}^m$. This means that there does not exist an invariant linear subspace and an associated set of spanning eigenvectors that remain constant for any choice of $u$. The latter implies by continuity of solutions of $P(s,u) = 0$ with respect to $u$ that for a given $u^*$ satisfying H4.BS, there exists a $\Delta u$ such that the solution to $P(s, u^* + \Delta u) = 0$ yields $n-1$ eigenvalues $\lambda_i(u^* + \Delta u) \in \mathbb{C}_-^\circ$, $i = 1, \ldots, n-1$, in some neighborhood of $\lambda(u^*)$. The existence of $\delta$ such that H4.BS is satisfied for all $u \in B(u^*, \delta)$ simply follows by letting $\delta = \inf_{\Delta u \in \{v | \mathrm{Re}(\lambda(u^*+v))=0\}} \|\Delta u\|$.

The continuity of solutions to $P(s,u) = 0$ also implies the continuity of the eigenvectors $w(u)$ that solve $[sI - A(u)]w(u) = 0$ for a given $u \in \mathbb{R}^m$, and hence that of the point to set mapping $u \to E^s(u)$ on $\mathcal{U}$. $\square$

The sliding phase control is now defined on the set $\mathcal{S}$ as:

$$u^S(x) \stackrel{\text{def}}{=} u \in \mathbb{R}^m \quad \text{such that } E^s(u) \ni x \tag{5.82}$$

where, additionally, $u^S(x(t))$ is required to be continuous along any trajectory of the system $t \to x(t) \in \mathcal{S}$. It is now possible to prove the following stabilization result.

**Theorem 5.4.** *Suppose all assumptions H1.BS–H4.BS are satisfied. Let $T$ be such that the reaching control defined by (5.81) steers the system from any initial point $x_0 \in \mathbb{R}^n$ to $N_\epsilon(E^s(u^*))$ in finite time, and let $u^{TIP} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ be the asymptotically stabilizing control (5.23) obtained using the approach of Section 5.3. Under these conditions, the combined reaching and sliding phase controls, together with $u^{TIP}$:*

$$u(x,t) \stackrel{\text{def}}{=} \begin{cases} u^R(x,t) & x \notin N_\epsilon(E^s(u^*)) \\ u^S(x) & x \in N_\epsilon(E^s(u^*)) \text{ and } x \in \mathcal{S} \\ u^{TIP}(x,t) & x \in N_\epsilon(E^s(u^*)) \text{ and } x \notin \mathcal{S} \end{cases} \tag{5.83}$$

*provide an asymptotically stabilizing feedback control for system (5.1).*

Proof. The asymptotic stability of system (5.1) with control $u(x,t)$ given by (5.83) directly follows from Theorem 5.3, which guarantees that the reaching control $u^R(x,t)$ steers the system to $N_\epsilon(E^s(u^*))$ in some finite time $T^*$, and the fact that once $x(t) \in N_\epsilon(E^s(u^*)) \cap \mathcal{S}$, $t \geq T^*$, the sliding control $u^S(x)$, whose continuity along trajectories $t \to x(t) \in \mathcal{S}$ is ensured by Proposition 5.4, keeps trajectories of the system confined to the set $\mathcal{S}$ of stable subspaces $E^s(u)$, $u \in \mathcal{U}$. If $x \in N_\epsilon(E^s(u^*)) \setminus \{N_\epsilon(E^s(u^*)) \cap \mathcal{S}\}$, i.e. if $x \in N_\epsilon(E^s(u^*))$ but $x \notin \mathcal{S}$, then $u^{TIP}$ asymptotically stabilizes the system by Theorem 5.2. □

Remark 5.5. *The above feedback control provides an alternative approach to the asymptotic stabilization of system (5.1) which is in most situations much simpler than that provided by the control* $u^{TIP}$ alone *(like in the method proposed in Section 5.3) since $u^{TIP}$ is usually not required as shown by the examples presented next.*

## 5.6. Examples for the Stabilization to the Stable Manifold Approach

### 5.6.1. Example 1: System on $\mathbb{R}^3$ with $\dim E^s(u^*) = 2$.

The feedback control constructed is applied to a single-input bilinear system on $\mathbb{R}^3$, with the following matrices:

$$A_0 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A_1 = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \tag{5.84}$$

It can easily be verified that:

- The drift term $A_0 x$ is unstable as all the eigenvalues of $A_0$ have positive real parts.
- The system matrix $A(u) = A_0 + A_1 u$ has at most two eigenvalues with negative real part for all $u \in \mathbb{R}$.
- The two stable eigenvalues of $A(u)$ occur for $u < 0$ and are complex conjugate.
- The Lie algebra generated by $A_0 x$ and $A_1 x$ satisfies the LARC condition.

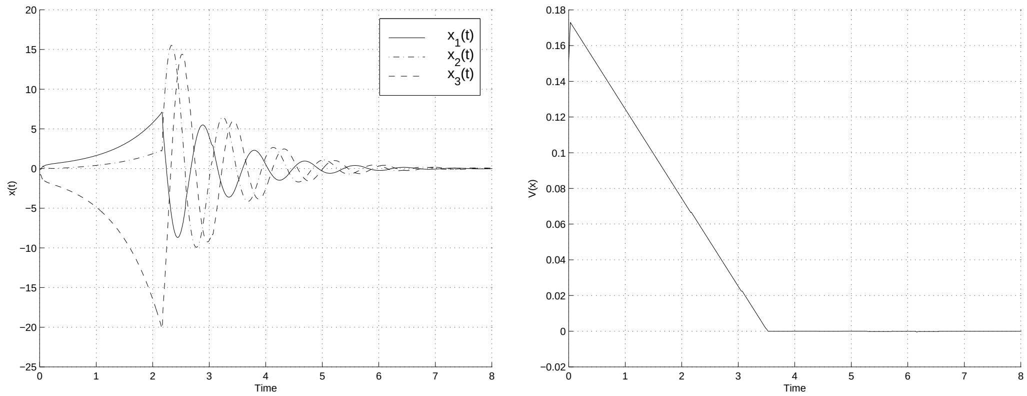
The trajectories of the closed-loop system are shown in Fig. 5.4 (a) for an initial state

$$x_0 = [-0.0939\ 0.6460\ \text{-}0.7575]^T$$

in $E^b$. The trajectory $x(t)$ never enters the set $E^b$ again, thus the control (5.55) is enough two reach $E^s$. A value of $u^* = -6$ was adopted for the design of the feedback $u^r(x)$, in which case

$$n(u^*) = [-0.8860\ \ -0.3047\ \ -0.3496]^T$$

In this simulation the constant rate of decrease $\eta$ was chosen to be 0.05.




(a) Closed-loop system trajectory $x(t)$. (b) Lyapunov function $V(x(t))$.

FIGURE 5.4. Stabilization of system (5.84) to a stable plane, with initial condition $x_0 \in E^b$.

### 5.6.2. Example 2: System on $\mathbb{R}^3$ with $\dim E^s(u^*) = 1$.

Consider a single-input bilinear system on $\mathbb{R}^3$ with matrices:

$$A_0 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \tag{5.85}$$

In this example $A(u) = A_0 + A_1 u$ has only one stable eigenvalue regardless of the choice of $u \in \mathbb{R}$. The set $\mathcal{S}$ is then a collection of one dimensional subspaces $E^s$, where:

$$E^s = \{x \in \mathbb{R}^n \mid x = s\, d(u), s \in \mathbb{R}\} \tag{5.86}$$

Here $d(u)$ denotes the eigenvector corresponding to the stable eigenvalue $\lambda(u)$. It is now convenient to consider a control Lyapunov function $V$, which is defined as the distance between the current state $x$ and its orthogonal projection on $E^s$, given by:

$$V(x) = x^T x - \frac{(d^T x)^2}{d^T d} \tag{5.87}$$

The reaching phase control $u^r(x)$ is hence defined as:

$$u^r(x) \stackrel{\text{def}}{=} \frac{-a(x) - K}{\|b(x)\|^2} b(x)^T \tag{5.88}$$

whenever $x \notin N_\epsilon(E^b)$, with $K > 0$ and

$$E^b \quad = \quad \{x \in \mathbb{R}^n \mid x^T(d^T dA_1 - dd^T A_1)x = 0\} \tag{5.89}$$

It is easy to check that $E^s \subset E^b$, thus in the process of reaching $E^s$ it is inevitable for the state of the controlled system to enter the set $N_\epsilon(E^b)$. This necessitates the construction of the time-varying component $u^c(x,t)$ of the reaching control $u^R(x,t)$.

Let $X_0 = A_0 x$ and $X_1 = A_1 x$. The system satisfies the LARC, since

$$\operatorname{span} L\{X_0, X_1, [X_0, X_1], [X_1, [X_0, X_1]]\}(x) = \mathbb{R}^3$$

The Lie algebra of this system is non-nilpotent, however it is finite dimensional and can be approximated by the following fourth order P. Hall basis:

$$\begin{array}{lllllllll}
B_1 & = & X_0 & B_2 & = & X_1 & B_3 & = & [X_0, X_1] \\
B_4 & = & [X_0, [X_0, X_1]] & B_5 & = & [X_1, [X_0, X_1]] & B_6 & = & [X_0, [X_0, [X_0, X_1]]] \\
B_7 & = & [X_1, [X_0, [X_0, X_1]]] & B_8 & = & [X_1, [X_1, [X_0, X_1]]] & & &
\end{array}$$

The stable subspace $E^s(u^*)$ is spanned by $d(u^*) = [0\ 0\ 1]^T$, which corresponds to the direction of the stable eigenvector generated with $u^* = 0$ and for which

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2) \tag{5.90}$$

The reaching phase feedback (5.81) may now be written as:

$$u^R(x,t) = \begin{cases} \frac{-(x_1^2+x_2^2)-K}{(x_1+x_2)x_3} & \text{for } x \notin N_\epsilon(E^b) \\ \bar{u}(x(t^*), T) & \text{for } x \notin N_\epsilon(E^b) \end{cases} \tag{5.91}$$

where $\mathcal{N}_\epsilon E^b = \{x \mid (x_1 + x_2)x_3 < \epsilon; \epsilon = 0.01\}$.

The computation of the values the control sequence $\bar{u}$ requires of the expressions for the coefficients $c_i(\bar{u}, \bar{\varepsilon})$ that define the vector field $\bar{f}(x, \bar{u}, \bar{\varepsilon}) = \bar{A}(\bar{u}, \bar{\varepsilon})x$ of equation (5.60), resulting from the composition of flows in equation (5.58). Considering a control sequence $\bar{u}$ with $s = 4$, application of the

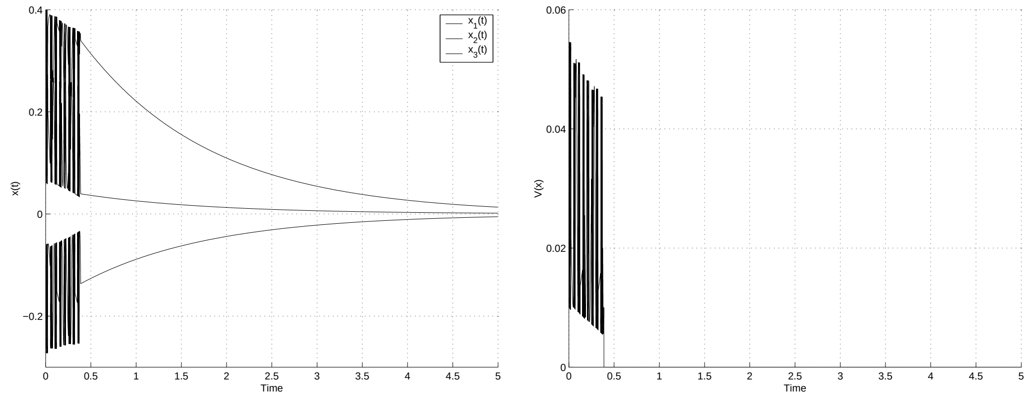
CBH formula yields:

$$
\begin{aligned}
c_1(\bar{u},\bar{\varepsilon}) &= T \\
c_2(\bar{u},\bar{\varepsilon}) &= (u_{(1)} + u_{(2)} + u_{(3)} + u_{(4)})\epsilon \\
c_3(\bar{u},\bar{\varepsilon}) &= (-3u_{(1)} - u_{(2)} + u_{(3)} + 3u_{(4)})\frac{\epsilon^2}{2} \\
c_4(\bar{u},\bar{\varepsilon}) &= (u_{(1)} - u_{(2)} - u_{(3)} + u_{(4)})\frac{\epsilon^3}{2} \\
c_5(\bar{u},\bar{\varepsilon}) &= (-3u_{(1)}^2 - 2u_{(1)}u_{(2)} + 4u_{(1)}u_{(3)} - 3u_{(2)}^2 - 2u_{(3)}u_{(2)} \\
&\quad -3u_{(3)}^2 + 10u_{(4)}u_{(1)} + 4u_{(4)}u_{(2)} - 2u_{(4)}u_{(3)} - 3u_{(4)}^2)\frac{\epsilon^3}{12} \\
c_6(\bar{u},\bar{\varepsilon}) &= (5u_{(1)} + 7u_{(2)} - 3u_{(3)} - 9u_{(4)})\frac{\epsilon^4}{24} \\
c_7(\bar{u},\bar{\varepsilon}) &= (5u_{(1)}^2 + 4u_{(1)}u_{(2)} + u_{(3)}^2 - 4u_{(4)}u_{(3)} + 4u_{(4)}u_{(2)} - 9u_{(4)}^2 - 4u_{(1)}u_{(3)} + 3u_{(2)}^2)\frac{\epsilon^4}{24} \\
c_8(\bar{u},\bar{\varepsilon}) &= (-5u_{(4)}^2u_{(1)} - 3u_{(4)}^2u_{(2)} - u_{(4)}^2u_{(3)} + 5u_{(4)}u_{(1)}^2 \\
&\quad +4u_{(4)}u_{(1)}u_{(2)} - 4u_{(4)}u_{(1)}u_{(3)} + 3u_{(4)}u_{(2)}^2 + u_{(4)}u_{(3)}^2)\frac{\epsilon^4}{24}
\end{aligned}
$$

for equal time intervals $\varepsilon_k = \epsilon = T/4$, $k = 1, 2, 3, 4$, and the basis (5.90).

With the above $c_i$ the control $\bar{u}$ is computed by solving (5.61)–(5.62) employing nonlinear least squares procedures, as explained in Remark 4.2.

The stabilization of (5.85), by means of (5.91) and the sliding phase control with $u^* = 0$ is shown in Fig. 5.5. The initial condition, $x_0 = [-0.1\ 0.1\ 0.4]^T$, lies in $E^b$. The control sequence period is selected to be $T = 1$. The simulation results show that the trajectory $x(t)$ stays in $N_\epsilon(E^b)$ for almost the first 0.5 seconds until it enters $\mathcal{S}$.




(a) Closed-loop system trajectory $x(t)$.

(b) Lyapunov function $V(x(t))$.

FIGURE 5.5. Stabilization of system (5.85) to a stable line, with initial condition $x_0 \in E^b$.

# CHAPTER 6

## A Software Package for Symbolic Lie Algebraic Computations

This chapter presents a computer algebra package that facilitates Lie algebraic symbolic computations required in the solution of a variety of problems, such as the solution of right-invariant differential equations evolving on Lie groups. Lie theory is a powerful tool, helpful in the analysis and design of modern nonlinear control laws, nonlinear filters, and the study of particle dynamics. The practical application of Lie theory often results in highly complex symbolic expressions that are difficult to handle efficiently without the aid of a computer software tool. The aim of the package is to facilitate and encourage further research relying on Lie algebraic computations. In view of the assistance the package might provide, its capabilities and applications are presented in [**175**].

### 6.1. Introduction

The purpose of this chapter is to describe some of the capabilities of a software package for symbolic calculations frequently encountered in the application of Lie algebra theory. The package, here referred to as the *Lie Tools Package* (LTP), is implemented in Maple and can be employed for computations involving Lie algebras of arbitrary type as it is constructed using a free Lie algebra of indeterminates as its base. The results obtained with the help of the package can subsequently be projected onto the specific Lie algebra arising in the concrete application of interest by the use of an adequately constructed Lie algebra homomorphism.

The theory of Lie algebras and groups was originally conceived by Sophus Lie as a tool for the solution of differential equations and has since then become a discipline in its own right. Lie theory brings together the mathematical disciplines of algebra and geometry to produce results relying on group-theoretic and differential geometric developments. Such results have proved essential in the study of kinematical symmetries in both classical and quantum mechanics [**164, 166**], the construction of nonlinear filters [**163, 165**], the analysis of dynamical systems, and the design of feedback control laws for nonlinear systems [**6, 2, 180**]. The use of Lie theory in the study of the symmetries of differential equations is described in [**168**] from a practical perspective. The application of Lie theory to the analysis and control of robotic systems is found in [**180, 167, 6**] and references therein. For a comprehensive review of other applications of the Lie theory, the reader is referred to the books by R. Gilmore and J. G. F. Belinfante [**155, 151**], which are intended to serve an audience of physicists and engineers. Important basic references in Lie algebras and group theory are the books by J.-P. Serre [**160**] and V.S. Varadarajan [**162**]. Despite the attention which the Lie theory has received in a variety of fields, it has been limited mostly because of the complexity of the symbolic calculations, and thus, the use of Lie theory for practical purposes has often been avoided, or has not yet received proper attention.

The importance of developing software tools that facilitate the tedious symbolic computations arising in applications of Lie theory has already been recognized by several authors, see for example [**167**, p. 60] and [**168**, p. v., pp. 333–335]. Schwarz in his article [**173**], acknowledges that Lie theory, as applied to identification of symmetries of differential equations, has previously failed to receive adequate attention. It is also argued that significant progress in this and other areas of mathematics has resulted from the development of software tools to assist in problems involving intensive algebraic computations.

The development of LTP was motivated by the lack of software capable of handling completely general symbolic Lie algebraic expressions and the fact that Lie algebraic computations are often prohibitively difficult to perform by hand. Developing software tools, such as the LTP, can prove essential to encourage further research based on Lie theoretical results [1].

[1] With this motivation in mind, the Lie Tools Package is freely available at
`http://www.cim.mcgill.ca/~migueltt/ltp/ltp.html`

The LTP package has unique capabilities which are not provided by other software such as, for example, the *liealg* package developed by Yuly Billig and Matthias Mazzag[2], which is conceived to perform specific calculations involving Kac-Moody and Virasoro algebras, and their representations. Specifically, the LTP package is capable of simplifying completely general Lie algebraic expressions with *symbolic coefficients* and is designed to serve as an efficient aid in the solution of right-invariant differential equations evolving on Lie groups. It can also deliver the Lie series which results in the application of the Campbell-Baker-Hausdorff formula; the latter containing Lie products up to any pre-specified order. The only previous computer-aided attempt to calculate the Campbell-Baker-Hausdorff formula is reported in [**159**]. However, the procedure employed did not take account of the dependencies between Lie products arising from the antisymmetry and Jacobi identities, thus the result is of limited significance.

None of the mature computer algebra systems (CAS), such as Axiom (former Scratchpad II by R. D. Jenks and D. Yun, IBM Watson Laboratories), Derive (D. R. Stoutemyer), Macsyma (Math Lab Group, MIT), Maple (B. Char, Waterloo Maple, Inc.), Mathematica (Wolfram Research, Inc.) or Reduce (A. C. Hearn), provide toolboxes with the functionality of LTP. For surveys and comparative reviews of the different CAS, the reader is referred to [**169, 197**], the references in [**168**], and the information on symbolic computation available through Internet sites, such as the comprehensive Computer Algebra Information Network (CAIN)[3] or the Symbolic Mathematical Computation Information Center[4]. Furthermore, existing programs such as [**171, 172, 173**] or Maple's *liesymm package* [**191**], (see also references to specific software in the surveys [**169, 170, 168**]), are very specialized and only focus on the computation of Lie symmetries.

Among its several capabilities, LTP greatly automates and simplifies the following computations:

- Construction of ordered bases for free Lie algebras of indeterminates (Hall bases).
- Simplification of arbitrary Lie algebraic expressions possibly involving symbolic scalar coefficients.

[2]The *liealg* package has been developed in the School of Mathematics and Statistics, Carleton University, and is available at `http://mathstat.carleton.ca/~billig/maple/`

[3]`http://www.mupad.de/CAIN/`

[4]`http://www.symbolicnet.org/`

- Composition of exponential mappings involving indeterminates by the use of Dynkin's expression for the Campbell-Baker-Hausdorff (CBH) formula.
- Construction of the Wei-Norman equations of logarithmic coordinates of flows on nilpotent Lie groups.

A complete description of the package can be found in [**174**], including details of the algorithms and their implementation. LTP has proved to be an essential tool in the development of the stabilization methods proposed in the previous chapters.

For clarity of exposition, a brief reminder of basic concepts of the theory of Lie groups and Lie algebras is introduced in the next section, and their practical relevance to applications is briefly discussed in Section 6.3. Section 6.4 presents some general examples illustrating the capabilities of the main functions provided by LTP. A practical example on the derivation of the Wei-Norman equations for an underactuated rigid body in space is also presented. These equations are important in the design of feedback control laws, as shown in Example 4.5.2 on p. 92, and are practically impossible to compute by hand. The example thus demonstrates the usefulness and functionality of the package. The chapter concludes with a summary of current work aiming at further extensions of the package.

## 6.2. Preliminary Notions and LTP Formalism

Let $\{X_1, \ldots, X_m\}$ denote a set of indeterminates. For brevity of notation, let $\bar{X}_m = (X_1, \ldots, X_m)$. Let $A(\bar{X}_m)$ denote the free associative algebra (over $\mathbb{R}$) of noncommutative polynomials in the indeterminates $X_1, X_2, \ldots, X_m$. Members of $A(\bar{X}_m)$ are known to have the form of infinite sums $\sum_I a_I X_I$, where the summation is over all possible multi-indices $I = (i_1, \ldots, i_m)$, with $i_j \in \{0, \ldots, m\}$, for $j = 1, \ldots, k$, and where the coefficients $a_I$ are real numbers, all of which vanish except for a finite number of them. Here $X_I = X_{i_1} \cdots X_{i_k}$, and $X_{I=\emptyset} = 1$, where, in general, $X_i X_j \neq X_j X_i$ as implied by noncommutativity.

Let a Lie product $[X_i, X_j]$ of two indeterminates be defined as the noncommutative polynomial $X_j X_i - X_i X_j$. The free associative Lie algebra of indeterminates $X_1, \ldots, X_m$ is the subspace $L(\bar{X}_m) \subseteq A(\bar{X}_m)$ consisting of those elements $S \in A(\bar{X}_m)$ which are linear combinations of

$X_1, \ldots, X_m$ or arbitrary nested Lie products involving $X_1, \ldots, X_m$. In other words, $L(\bar{X}_m)$ is a Lie algebra spanned by formal Lie products of $X_1, \ldots, X_m$. The elements of $L(\bar{X}_m)$ are referred to as *Lie polynomials.*

Further, let $\hat{L}(\bar{X}_m)$ denote the Lie algebra of Lie series in $X_1, \ldots, X_m$. The elements of $\hat{L}(\bar{X}_m)$ are formal series of the type $\sum_{i=1}^{\infty} a_i S_i$, where $a_i$ are real coefficients and $S_i \in L(\bar{X}_m)$. Clearly, any element $Z \in \hat{L}(\bar{X}_m)$ can be written as a formal infinite series $\sum_I a_I X_I$ in the indeterminates $X_1, \ldots, X_m$, in which $X_I$ is some monomial in $X_1, \ldots, X_m$ and $a_{I=\emptyset} = 0$.

For any element in $Z \in \hat{L}(\bar{X}_m)$ the formal power series

$$e^Z = \sum_{k=0}^{\infty} \frac{1}{k!} Z^k \tag{6.1}$$

is well defined because $1 \notin \hat{L}(\bar{X}_m)$. Here, $Z^k$ are infinite series in the indeterminates $X_1, \ldots, X_m$ obtained by the natural multiplication rule for the component monomials of $Z$, $X_I X_J = X_{I*J}$, where $I * J$ is the concatenation of the components of the multi-indices $I$ and $J$. The set $\hat{G}(\bar{X}_m) = \{e^Z : Z \in \hat{L}(\bar{X}_m)\}$ is called the *set of exponential Lie series* in the indeterminates $X_1, \ldots, X_m$.

### 6.2.1. Philip Hall Basis

Note that, due to the antisymmetry property and the Jacobi identity of the Lie product, not all the elements of a Lie algebra $L(\bar{X}_m)$ are linearly independent. A procedure to construct a basis for any Lie algebra of indeterminates, while taking into account the dependencies imposed by the antisymmetry and the Jacobi identities, which was conceived by Philip Hall, involves listing all the generators $X_1, \ldots, X_m$ and selecting some of their Lie products according to the rules given below, see [**6, 160, 152**].

**Definition 6.1. - Hall basis (HB).** *Let $B$ denote the basis for $L(\bar{X}_m)$, and let $B_i$ be the $i$-th element in this basis. Let the* length *(order) of a Lie product $G$, $l(G)$, be defined as the number of indeterminates in the expansion of $G$, also given recursively by:*

$$\begin{aligned} l(X_i) &= 1 \qquad\qquad i = 1, \ldots, m \\ l([G, H]) &= l(G) + l(H) \end{aligned}$$

*where $G$ and $H$ are Lie products.*

*Then a* Hall basis *is an ordered set of Lie products $\{B_i\}$ such that:*

(i) $X_i \in B,\ i = 1, \ldots, m$

(ii) *If $l(B_i) < l(B_j)$ then $B_i < B_j$*

(iii) *$[B_i, B_j] \in B$ if and only if*

(a) *$B_i, B_j \in B$ and $B_i < B_j$ and*

(b) *either $B_j = X_k$ for some $k$ or $B_j = [B_p, B_q]$ with $B_p, B_q \in B$ and $B_p \leq B_i$.*

The proof that a Hall basis indeed constitutes a basis for the Lie algebra $L(\bar{X}_m)$ is found in [**156, 160**].

Let $L_k(\bar{X}_m) \subset L(\bar{X}_m)$ denote the *free nilpotent Lie algebra of order $k$* defined by assuming that all the Lie products of degree $k+1$ or greater are equal to zero. The above procedure can still be employed to construct bases for $L_k(\bar{X}_m)$ simply by forming all the Lie products that satisfy the above properties and whose length does not exceed $k$.

By the result of Campbell, Baker, and Hausdorff, known as the **CBH** formula, it follows that $\hat{G}(\bar{X}_m)$ is closed under multiplication, and is in fact a group, as it can be verified that $e^Z e^{-Z} = 1$, for any $Z \in \hat{G}(\bar{X}_m)$. Moreover, the map $\exp : \hat{L}(\bar{X}_m) \to \hat{G}(\bar{X}_m)$ is a bijection from $\hat{L}(\bar{X}_m)$ onto $\hat{G}(\bar{X}_m)$. It follows that for any $Z_1, Z_2 \in \hat{L}(\bar{X}_m)$ we can compute a unique $Z_3 \in L(\bar{X}_m)$ such that

$$e^{Z_1} e^{Z_2} = e^{Z_3} \tag{6.2}$$

The way to compute $Z_3$ is also delivered by the CBH formula which, in Dynkin's form, is given by [**160, 147**]:

$$\begin{aligned} Z_3 &= \sum_{m=1}^{\infty} \sum \frac{(-1)^{m-1} (ad_{Z_2})^{q_m} (ad_{Z_1})^{p_m} \cdots (ad_{Z_2})^{q_1} (ad_{Z_1})^{p_1}}{m \left(\sum_{i=1}^{m} (p_i + q_i)\right) \prod_{i=1}^{m} (p_i! q_i!))} \\ &= Z_1 + Z_2 + \frac{1}{2}[Z_1, Z_2] + \frac{1}{12}([[Z_1, Z_2], Z_2] - [[Z_1, Z_2], Z_1]) \\ &\quad -\frac{1}{48}([Z_2, [Z_1, [Z_1, Z_2]]] + [Z_1, [Z_2, [Z_1, Z_2]]]) + \ldots \end{aligned} \tag{6.3}$$

where the inner sum is performed over all $m$-tuples of pairs of nonnegative integers $(p_i, q_i)$ such that $p_i + q_i > 0$. In (6.3), $(ad_X)Y \overset{def}{=} [X, Y]$ and $(ad_X) \overset{def}{=} X$.

It is worth noticing that the group $\hat{G}(\bar{X}_m)$ is not a Lie group because it is infinite dimensional.

As the package is primarily a tool for the analysis of dynamical systems, it will be applied in the context of groups of transformations acting on the underlying manifold on which the system evolves, see [**162**]. Since such groups of transformations are Lie groups, it is helpful to define $G_k(\bar{X}_m)$, a nilpotent version of $\hat{G}(\bar{X}_m)$:

$$G_k(\bar{X}_m) \overset{def}{=} \{e^Z : Z \in L_k(\bar{X}_m)\} \tag{6.4}$$

The group $G_k(\bar{X}_m)$ is now a Lie group with Lie algebra $L_k(\bar{X}_m)$, see [**162**]. For a systematic development it is assumed here that all groups of transformations act from the right on the underlying manifolds $M$. With this notation, for $x \in M$, the expression $xe^Z$ denotes the value of a group action $e^Z$ at a point $x \in M$, [**160**, p. LG 4.11] or [**162**, p. 74].

One of the many applications of the LTP package involves the solution of differential equations defined on Lie groups. As will be explained later, the trajectories of these equations relate (through a Lie group homomorphism) to trajectories evolving on $G_k(\bar{X}_m)$. It is hence mandatory that $G_k(\bar{X}_m)$ is equipped with a convenient coordinate system. Such a coordinate system can be constructed in terms of a Hall basis and has the advantage of being global (consisting of a single chart) since $G_k(\bar{X}_m)$ is nilpotent, see [**148**]. In full rigour, if $\{B_1, B_2, \ldots, B_r\}$ is the $r$-dimensional Hall basis for a given nilpotent Lie algebra $L_k(\bar{X}_m)$, then any element $P$ in the Lie group $G_k(\bar{X}_m)$ has the following unique representation, [**44**]:

$$P = e^{\gamma_1 B_1} e^{\gamma_2 B_2} \cdots e^{\gamma_r B_r} \tag{6.5}$$

The map $P \to (\gamma_1, \gamma_2, \ldots, \gamma_r)$ establishes a global diffeomorphism between $G_k(\bar{X}_m)$ and $\mathbb{R}^r$ and is thus a global coordinate chart on $G_k(\bar{X}_m)$. This coordinate system is a Lie-Cartan coordinate system of the second kind [**6**], and as in previous chapters, its coordinates will simply be referred to by the name of $\gamma$-coordinates.

### 6.2.2. Wei-Norman Equation

Equation (6.5) can be viewed as a way to represent an arbitrary group action as a composition of elementary group actions defined in terms of the elements of the Hall basis of the Lie algebra associated with the group. This fact has been exploited by Wei and Norman in the solution of right-invariant parametric differential equations evolving on $G_k(\bar{X}_m)$:

$$\begin{aligned} \dot{S}(t) &= \left(\sum_{i=1}^{m} X_i u_i(t)\right) S(t) \\ S(0) &= I \in G_k(\bar{X}_m) \end{aligned} \tag{6.6}$$

where $m < \infty$ (finite), $X_i$ are indeterminate operators independent of $t$ that generate $L_k(\bar{X}_m)$ under the commutator product $[X_i, X_j] = X_j X_i - X_i X_j$, and $u_i$ are scalar functions of $t$. Here, as $S(0) \in G_k(\bar{X}_m)$, $S(t)$ evolves on $G_k(\bar{X}_m)$.

Therefore, the solution to (6.6) is given by the product of exponentials:

$$S(t) = e^{\gamma_1(t)B_1} e^{\gamma_2(t)B_2} \cdots e^{\gamma_r(t)B_r} = \prod_{i=1}^{r} e^{\gamma_i(t)B_i} \tag{6.7}$$

where $\{B_1, B_2, \ldots, B_r\}$ is the Hall basis for the Lie algebra $L_k(\bar{X}_m)$, and the $\gamma_i$ are scalar functions of time, see [**6, 149**]. Without the loss of generality, it may be assumed that $B_i = X_i$, for $i = 1, \ldots, m$.

The $\gamma$-coordinates are shown to satisfy a set of nonlinear differential equations as is implied by the following derivation, see also [**149**] or Appendix A, p. 222.

Differentiating (6.7) yields,

$$\dot{S}(t) = \frac{dS(t)}{dt} = \sum_{i=1}^{r} \dot{\gamma}_i(t) \prod_{j=1}^{i-1} e^{\gamma_j B_j} B_i \prod_{j=i}^{r} e^{\gamma_j B_j} \tag{6.8}$$

Multiplying both sides of (6.8) by $S(t)^{-1}$ from the right and using the *exponential formula* (see [**162**, p. 40]):

$$\begin{aligned} (e^X)Y(e^{-X}) &= Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \ldots \\ &= \sum_{k=0} \frac{1}{k!}(ad_X^k)Y \\ &= (e^{ad_X})Y \end{aligned} \tag{6.9}$$

yields,

$$\dot{S}(t)S^{-1}(t) = \sum_{i=1}^{r} \dot{\gamma}_i(t) \prod_{j=1}^{i-1} e^{\gamma_j ad_{B_j}} B_i \tag{6.10a}$$

$$= \sum_{i=1}^{r} B_i u_i(t) \tag{6.10b}$$

with $u_i(t) = 0$ for $i = m+1, \ldots, r$, as $S(t)$ satisfies (6.6).

Equating the coefficients on both sides of the last equality gives:

$$\underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} \xi_{11}(\gamma) & \cdots & \xi_{1r}(\gamma) \\ \vdots & \ddots & \vdots \\ \\ \xi_{r1}(\gamma) & \cdots & \xi_{rr}(\gamma) \end{bmatrix}}_{\Gamma(\gamma)} \underbrace{\begin{bmatrix} \dot{\gamma}_1(t) \\ \dot{\gamma}_2(t) \\ \vdots \\ \dot{\gamma}_r(t) \end{bmatrix}}_{\dot{\gamma}}, \quad \gamma(0) = 0 \tag{6.11}$$

where the $\xi_{ij}(\gamma)$ are analytic functions of the $\gamma_i$'s. Clearly, $\gamma(0) = 0$ since $S(0) = I$.

It is worth noting that there exists a chain of ideals $0 \subseteq \mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \cdots \subseteq \mathcal{I}_r = L_k(\bar{X}_m)$ where each $\mathcal{I}_n$ is exactly of dimension $n$. The order of the elements in the Hall basis $\{B_1, \ldots, B_r\}$ is such that is the ideal $\mathcal{I}_n$ is generated by $\{B_n, \ldots, B_r\}$, which implies that the multiplication table for $L_k(\bar{X}_m)$ satisfies:

$$[B_i, B_j] = \sum_{n=i}^{r} c_n^{ij} B_n, \qquad \text{for } i > j \tag{6.12}$$

It can be shown, see [**149**], that such a multiplication table implies that $\Gamma(\gamma)$ is lower triangular and invertible for all $t$. Hence, (6.11) yields the system of differential equations for the computation of the $\gamma$-coordinates in explicit form:

$$\dot{\gamma}(t) = \Gamma^{-1}(\gamma)u(t), \qquad \gamma(0) = 0 \tag{6.13}$$

Equation (6.13) will be referred to as the *Wei-Norman equation*. Its solution delivers $S(t)$ of (6.7) which solves (6.6).

## 6.3. Relevance of LTP to Applications of Lie Algebras and Groups

To show the practical relevance of the concepts introduced in the previous section and further motivate the development of the LTP, a few applications are discussed next. A rigorous exposition of the examples presented and the associated assumptions can be found in [**2, 165, 6**] and references therein.

### 6.3.1. Trajectory Planning and Control

A wide class of nonlinear control systems can be described by an ordinary differential equation which is affine in the control inputs:

$$\dot{x} = f_0(x)u_0 + f_1(x)u_1 + \ldots + f_m(x)u_m = f(x, u)) \tag{6.14}$$

where $x \in \mathbb{R}^n$ is the state of the system, $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 0, 1, \ldots, m$, are smooth vector fields defined on $\mathbb{R}^n$, and $u_i \in \mathbb{R}$, $i = 0, 1, \ldots, m$, are scalar control inputs usually restricted to the class of piece-wise continuous functions in $t$. The control problem for (6.14) becomes challenging if $m < n$. Equation (6.14) can be thought to represent both driftless systems, and systems with drift (if $u_0 = 1$). Practical examples can be found in [**6, 180, 167**] and include robotic manipulators, mobile robots, underwater vehicles, and rigid bodies in space.

With reference to systems described by (6.14), the theory of Lie algebras and groups is known to be helpful in the following:

- Establishing the controllability properties of the system.
- Developing control laws that stabilize the system to a given equilibrium point, or ensure tracking of a desired reference trajectory.

Chow's Theorem delivers a conclusive result for the determination of complete controllability for driftless system (6.14). Chow's result involves the verification of the *Lie algebra rank condition* (LARC), see [**6**]:

$$\dim \bar{\mathcal{G}}(x) = n \ \forall x \in \mathbb{R}^n \tag{6.15}$$

where $\bar{\mathcal{G}}(x) \stackrel{\text{def}}{=} \text{span}\{f(x) \in \mathbb{R}^n | f \in L(\mathcal{F})\}$ and $\mathcal{F} \stackrel{\text{def}}{=} \{f_0, f_1, \ldots, f_m\}$.

The LARC hence requires the construction of a basis for the Lie algebra of vector fields $L(\mathcal{F})$. To this end the LTP is used as follows. A Hall basis $\{B_1, B_2, \ldots\}$ is first generated for $L(\bar{X}_{m+1})$ and then each Lie product $B_i$, $i = 1, 2, \ldots$ of this basis is mapped into a corresponding Lie product of vector fields in $L(\mathcal{F})$ by using the canonical Lie algebra homomorphism which assigns $f_i$ to $X_i$, $i = 0, \ldots, m$, in any formal Lie product $B_i$, $i = 1, 2, \ldots$.

For systems with drift the LARC only ensures accessibility of the system, i.e. that the reachable set at any $x_0$ has a non-empty interior, see [**6**]. The computation of a basis for $L(\mathcal{F})$ is however still useful since the dimension of the set $\mathcal{G}(x)$ and the highest order of the Lie products appearing in $\mathcal{G}(x)$ reveal the difficulty of controlling (6.14).

Assuming that system (6.14) is completely controllable, a variety of Lie algebraic-based control synthesis methods have been proposed in the literature, see for example [**180, 6**].

Pivotal to controllability considerations, the design, and the derivation of control strategies for system (6.14) is the capability to generate system motions in directions outside the span of the vector fields $f_i$, $i = 0, \ldots, m$. Assuming that piece-wise constant switching controls are employed, such motions can be generated by concatenation of system motions in directions included in the span of $f_i$, $i = 0, \ldots, m$. To this end, the LTP package proves helpful in determining the vector field, which over a given interval of time $T$, yields motions equivalent to the concatenation. More precisely, given $s$ intervals of time $\epsilon_i$, $i = 1, \ldots, s$, the package can determine a vector field $\bar{f}(x, \bar{\epsilon}, T)$, with $\bar{\epsilon} = (\epsilon_1, \ldots, \epsilon_s)$, and $T = \sum_{i=1}^{s} \epsilon_i$, such that for any $x \in \mathbb{R}^n$:

$$xe^{\epsilon_1 z_1} \cdots e^{\epsilon_s z_s} = xe^{T\bar{f}} \tag{6.16}$$

where $z_i = f(x, u_{(i)})$, $i = 1, \ldots, s$, are vector fields corresponding to control actions $u_{(i)}$ employed by system (6.14) over intervals of time $\epsilon_i$, $i = 1, \ldots, s$, respectively. Here, $e^{\epsilon z}$ denotes the flow of the differential equation $\dot{x} = z$ so that $xe^{\epsilon z}$ is the solution of this equation with initial condition $x$, evaluated at time $\epsilon$.

Generally, $\bar{f}$ will be an infinite series in the Lie products involving $f_i$, $i = 0, \ldots, m$, which will only converge for sufficiently small $\epsilon_i$, $i = 1, \ldots, s$. In practice, however, it is often sufficient to evaluate only the first $r$ terms of this series. The LTP package is then employed by repeatedly applying the CBH formula to perform formal calculations associated with the composition of the formal exponential maps so that

$$e^{\epsilon_1 Y_1} \cdots e^{\epsilon_s Y_s} = e^{T\bar{Y}} \tag{6.17}$$

where $Y_i$, $i = 1, \ldots, s$, are members of $L_k(\bar{X}_{m+1})$ and correspond to $f_i$, $i = 0, \ldots, m$, under the Lie algebra homomorphism, $\nu : X_{i+1} \to f_i$, $i = 0, \ldots, m$, between $L(\bar{X}_{m+1})$ and $Ł_k(\mathcal{F})$. Here, $Ł_k(\mathcal{F})$ denotes the nilpotent truncation of $L(\mathcal{F})$ which is obtained by assuming that all brackets of order $k+1$, and higher, are equal to zero. The resulting Lie series $\bar{Y}$ then delivers a $k$-th order approximation $\nu(\bar{Y})$ of $\bar{f}$. If the Lie algebra of vector fields $L(\mathcal{F})$ is already nilpotent, and the vector fields $f_i \in \mathcal{F}$, $i = 1, \ldots, m$, are real, analytic, and complete, then the Lie group $G(\mathcal{F})$ corresponding to $L(\mathcal{F})$, is analytic, nilpotent, and simply connected, thus guaranteeing that (6.16) is valid for arbitrary values of $\epsilon_i$, $i = 1, \ldots, s$, with $\nu(\bar{Y}) = \bar{f}$ holding exactly, see [**158**, p. 95] and [**162**, p. 195].

In trajectory planning problems it is often necessary to calculate the flows of dynamical systems such as (6.14) in terms of specific control functions $u_i$, $i = 0, \ldots, m$. The flow of (6.14) is defined as the map $x \to xe^{tf}$ for $x \in \mathbb{R}^n$, where the time $t$ is a parametrizing variable which corresponds to the total time of motion along the vector field $f$. Maintaining the assumptions about nilpotency of $L(f_1, \ldots, f_m)$, as well as analyticity and completeness of $f_1, \ldots, f_m$, the latter can be achieved by solving the formal equation (6.6) with the assumption that $f_i$ corresponds to $X_{i+1}$, $i = 0, \ldots, m$, through the Lie algebra homomorphism $\nu : X_{i+1} \to f_i$. Once $S(t)$ is calculated in terms of its $\gamma$-coordinates on the Lie group $\hat{G}(\bar{X}_{m+1})$, the value $xe^{tf}$ is calculated as $x\,\tilde{\nu}(S(t))$, where $\tilde{\nu}$ is the Lie group homomorphism $\tilde{\nu} : \hat{G}(\bar{X}_{m+1}) \to G(\mathcal{F})$ induced by the Lie algebra homomorphism $\nu$, see [**44**].

### 6.3.2. Nonlinear Filtering

Lie algebraic methods originally conceived as tools for the analysis of nonlinear systems have also found application in nonlinear filtering problems; the reader is referred to [**165**] for a complete expository review. In the nonlinear filtering problem the objective is to estimate the state of a stochastic process $x(t)$, which cannot be measured directly, but may be inferred from measurements of a related observation process $y(t)$.

Typical filtering problems consider the following signal observation model:

$$\begin{aligned} dx(t) &= f(x(t))dt + g(x(t))dv(t), \qquad x(0) = x_0 \\ dy(t) &= h(x(t))dt + dw(t), \qquad y(0) = 0 \end{aligned} \tag{6.18}$$

where $x, v$ and $y, w$ are $\mathbb{R}^n$ and $\mathbb{R}^m$ valued processes, respectively, and $v$ and $w$ have components which are independent, standard Brownian processes. Furthermore, $f, h$ are smooth and $g$ is an orthogonal matrix.

Essential for the estimation of the state is the conditional probability density of the state, $\rho(t, x)$, given the observation $\{y(s); 0 \leq s \leq t\}$. It is well known, see [**202**], that $\rho(t, x)$ is obtained by normalizing a function $\sigma(t, x)$ which is the solution of the Duncan-Mortensen-Zakai (DMZ) equation:

$$d\sigma(t, x) = L_0\sigma(t, x)dt + \sum_{i=1}^{m} L_i\sigma(t, x)dy_i(t), \qquad \sigma(0) = \sigma_0 \tag{6.19}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{m} h_i^2 \tag{6.20}$$

and where $L_i$ is the operator of multiplication by $h_i$, $i = 1, \ldots, m$, and $\sigma_0$ is the probability density of the initial point $x_0$.

A particularly useful concept associated with the DMZ equation is the *estimation Lie algebra*, as introduced in [**200**], which is defined as the Lie algebra generated by the differential operators $L_0, \ldots, L_m$ with the Lie product defined by $[X, Y]f = X(Yf) - Y(Xf)$, for any smooth function $f$. The structure and dimensionality of the estimation Lie algebra is directly related to the existence of a finite dimensional recursive filter for the computation of $\rho(t, x)$, see [**165**]. It has been shown that if the estimation Lie algebra can be identified with a Weyl algebra of any order, then no non-constant statistics exist for the computation of the conditional density $\rho(t, x)$ with a finite dimensional filter; see references in [**165**]. In this context, the LTP package is helpful in the computation of the generators for the Weyl algebras as it *permits an arbitrary definition of Lie product.*

In the special case when the estimation Lie algebra is finite dimensional and solvable, (i.e. if there exists a chain of ideals $0 \subseteq \mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \cdots \subseteq \mathcal{I}_r = L(H)$ where each $\mathcal{I}_i$ is exactly of dimension $i$; see [**162**] for the definition of solvability), the DMZ equation can be solved via the Wei-Norman formalism. By introducing a transformation of the density $\sigma(t,x)$, [**201**], and constructing a special basis $\{B_1, \ldots, B_r\}$ for the estimation Lie algebra, [**203**], it is possible to bring the DMZ equation into the following "robust" form:

$$\frac{d\xi(t,x)}{dt} = L_0\xi(t,x)dt + \sum_{i=1}^{m}[L_0, L_i]\xi(t,x)dy_i(t) + u(t)\xi(t,x) \tag{6.21}$$

where $u(t) = \frac{1}{2}\sum_{i=1}^{m}[[L_0, L_i], L_i]y_i^2(t)$, and $\xi(t,x)$ is the transformed density.

It can then be shown that equation (6.21) has a solution in the form of a product of exponentials:

$$\xi(t,x) = e^{\gamma_1(t)B_1} \cdots e^{\gamma_r(t)B_r}\sigma_0, \qquad t \geq 0 \tag{6.22}$$

where $\gamma_i$, $i = 1, \ldots, r$, satisfy the Wei-Norman equation (6.13).

The Lie Tools Package is hence also useful for the construction of finite dimensional nonlinear filters.

## 6.4. LTP Functions and Examples

Any Lie product which is written in terms of the algebra generators only, will henceforth be referred to as a *pure Lie product*. By the property of distributivity over scalar multiplication, an *arbitrary* Lie bracket is a product of a symbolic coefficient and a pure Lie bracket.

The main and auxiliary functions provided by LTP are summarized in Table 6.1 and Table 6.2, respectively. Auxiliary functions are invoked by the main functions, but are also made directly available to the user to allow for perusal of intermediate results. Such an organization of the package facilitates the addition of new functions. The reader is referred to the full package documentation and the source code [**174**] for details on the syntax, algorithmic implementation, and other aspects.

Prior to invoking any function in the package, two special variables need to be declared, under arbitrary names, to signify: the number of generators in the Lie algebra $L(\bar{X}_m)$ and its assumed order of nilpotency. The values of these variables are limited only by the available computer memory.

| Function | Purpose |
|---|---|
| cbhexp | Calculates the exponent $Z_3 \in \hat{L}(\bar{X}_m)$ resulting from the composition of exponential mappings in equation (6.2) via the CBH formula (including brackets up to a given order $k$). |
| createLBobjects | Declares the generators $\bar{X}_m$ of the Lie algebra $L_k(\bar{X}_m)$. If needed, it also permits to declare any number of linear combinations of these generators $\sum_{i=1}^{m} a_i X_i$ with symbolic coefficients $a_i$. The LTP assigns a name to each linear combination allowing it to be used by other LTP functions. |
| phb | Declares the generators $\bar{X}_m$ of the free nilpotent Lie algebra $L_k(\bar{X}_m)$ of degree $k$ and constructs a Hall basis for $L_k(\bar{X}_m)$. |
| phbize | Expresses any pure Lie product $X \in L_k(\bar{X}_m)$ in the Hall basis. |
| reduceLB | Reduces a general Lie polynomial $S \in L_k(\bar{X}_m)$ with symbolic coefficients to its simplest form in a given HB. |
| reduceLBT | Given a list of dependencies between the elements of the HB, reduces a general Lie polynomial $S \in L_k(\bar{X}_m)$ with symbolic coefficients to its simplest form. |
| regroupLB | Applies the distributivity properties (over addition and scalar multiplication) of the Lie product to an arbitrary Lie polynomial in $S \in L(\bar{X}_m)$ and collects its terms. |
| simpLB | Applies the distributivity over scalar multiplication property to a given Lie product $X \in L(\bar{X}_m)$ and returns the simplified product $\alpha Y = X$, together with its scalar symbolic component $\alpha$, and the pure Lie bracket $Y \in L(\bar{X}_m)$. |
| wner | Computes the right-hand side of equation (6.10a) and expresses it in the HB, treating $\dot{\gamma}_i$ and $\gamma_i$, $i = 1, \ldots, r$, as symbolic scalars. |
| wnde | Constructs the differential equation for the logarithmic coordinates $\gamma_i$ given by the Wei-Norman equation (6.11). |

TABLE 6.1. Main functions in LTP.

The examples presented in the next two sections consider a set of Lie algebra generators $\bar{X}_3 = (X_1, X_2, X_3)$ and a HB, denoted by $B$, for a nilpotent Lie algebra $L_4(\bar{X}_3)$ with degree of nilpotency $k = 4$. The generators $\bar{X}_3$ and the basis $B$ are easily obtained by executing the package function `phb(3,4)`; the resulting basis $B$ is shown in Section 6.4.2.

### 6.4.1. Simplification of Lie algebraic Expressions

To explain some of the capabilities of the package we consider a few examples.

To simplify the following expression $x := [\alpha X_2, [\alpha X_1, (\alpha + \beta^2) X_0]])$, in which $\alpha$ and $\beta$ are considered to be symbolic scalars, the function `y:=simpLB(x)` is invoked and returns the result: $(\alpha^3 + \alpha^2\beta^2)[X_2, [X_1, X_0]]$, as well as, but separately, the scalar part of it, $(\alpha^3 + \alpha^2\beta^2)$, and the pure Lie product $[X_2, [X_1, X_0]]$. Such an answer form facilitates further calculations; for example when the expression needs to be re-written in terms of elements of the basis $B$. The latter can be accomplished by subsequently invoking the function `phbize(y[3])`, which acts on the third argument of the result.

| Function | Purpose |
|---|---|
| ad | Calculates $(ad_X^n)Y$ for $X, Y \in L(\bar{X}_m)$. |
| bracketlen | Returns the length $l(G)$ of a Lie product $G \in L(\bar{X}_m)$. |
| calcLB | Given the symbolic expressions for two vector fields in the canonical coordinate system, calculates their Lie product. |
| codeCBHcf | Generates code in either Fortran or C for the evaluation of the scalar symbolic coefficients in a given Lie polynomial $S \in L(\bar{X}_m)$. |
| createSubsRel | Creates Maple substitution relations for the the symbolic evaluation of controls $u_i$, $i = 0, \dots, m$, in the dynamic system (6.14). These substitution relations can then be used to permit calculations involving systems with drift and to accommodate for piece-wise constant controls of arbitrary symbolic magnitude, as well as to allow the controls to switch at arbitrary symbolic moments in time, see [**174**] for details. |
| ead | Computes the series expansion of $(e^X)Y(e^{-X}) = (e^{ad_X})Y$. for $X, Y \in L(\bar{X}_m)$ including brackets up to a given order. |
| eadr | Computes the series expansion of $(e^X)Y(e^{-X}) = (e^{ad_X})Y$. for $X, Y \in L_k(\bar{X}_m)$; re-expresses the result in the HB and further simplifies it according to a given list of dependencies involving the elements of the HB. |
| evalLB2expr | Returns a symbolic Maple expression for later evaluation of a Lie product of two vector fields, possibly containing symbolic scalars. |
| pead | Computes the product of exponentials $\prod_{i=1}^n e^{ad_{X_i}} X_{n+1}$ for $X, Y \in L(\bar{X}_m)$ including brackets up to a given order. |
| peadr | Computes the product of exponentials $\prod_{i=1}^n e^{ad_{X_i}} X_{n+1}$ for $X, Y \in L_k(\bar{X}_m)$; re-expresses the result in the HB and further simplifies it according to a given list of dependencies involving the elements of the HB. |
| posxinphb | Returns the position index $i$ of a Lie product $B_i$ in the HB. |
| selectLB | Extract, as a Maple symbolic expression for later use, the part of a given Lie polynomial $S \in L(\bar{X}_m)$ which contains brackets up to, greater than, or equal to a given order. |

TABLE 6.2. Auxiliary functions in LTP.

Another example, where skillful simplification is essential, is provided by the composition of exponential mappings $e^{Z_1}e^{Z_2}$, with $Z_1$ and $Z_2$ declared as two simple Lie polynomials: $Z_1 = a_1X_1 + a_2X_2 + a_3X_3$, $Z_2 = b_1X_1 + b_2X_2 + b_3X_3$, and with $a_i$, $b_i$, $i = 1, 2, 3$, declared as symbolic scalars. To obtain $Z_3$ which satisfies the CBH formula (6.3) the function `cbhexp(`$Z_1$`,`$Z_2$`,n)` is invoked to produce a truncation of $Z_3$ which includes brackets up to order $n \le k$. In terms of Lie products of indeterminates such a formula for $Z_3$, with $n = 4$, would involve 231 terms, but is simplified by executing the function `reduceLB(`$Z_3$`,`$B$`)`. The latter reduces $Z_3$ into its expression in the Hall basis $B$ which, in this particular case, counts only 29 components and are given by:

$$\begin{aligned}
Z_3 \quad := \quad & (a_1 + b_1)X_1 + (a_2 + b_2)X_2 + (a_3 + b_3)X_3 + \frac{1}{2}(a_1b_2 - a_2b_1][X_1, X_2] + \frac{1}{2}(a_1b_3 - a_3b_1)[X_1, X_3] \\
& + \frac{1}{2}(a_2b_3 - a_3b_2)[X_2, X_3] + \frac{1}{12}(a_1^2b_2 - b_1a_1b_2 + b_1^2a_2 - a_1a_2b_1)[X_1, [X_1, X_2]] \\
& + \frac{1}{12}(b_1^2a_3 - a_1a_3b_1 - b_3a_1b_1 + a_1^2b_3)[X_1, [X_1, X_3]] \\
& + \frac{1}{12}(a_1a_2b_2 - b_2^2a_1 + b_1a_2b_2 - a_2^2b_1)[X_2, [X_1, X_2]]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{12}(b_2a_3b_1 - a_3a_2b_1 - b_3a_1b_2 + a_1a_2b_3)[X_2,[X_1,X_3]]\\
&+\frac{1}{12}(b_2^2a_3 - a_3a_2b_2 - b_2a_2b_3 + a_2^2b_3)[X_2,[X_2,X_3]]\\
&+\frac{1}{12}(a_1a_3b_2 - b_3a_1b_2 - a_3a_2b_1 + b_1a_2b_3)[X_3,[X_1,X_2]]\\
&+\frac{1}{12}(b_1a_3b_3 - a_3^2b_1 + a_3a_1b_3 - b_3^2a_1)[X_3,[X_1,X_3]]\\
&+\frac{1}{12}(a_3a_2b_3 - b_3^2a_2 - a_3^2b_2 + b_2a_3b_3)[X_3,[X_2,X_3]]\\
&+\frac{1}{24}(b_1^2a_1a_2 - b_1a_1^2b_2)[X_1,[X_1,[X_1,X_2]]]\\
&+\frac{1}{24}(b_1^2a_2^2 - b_2^2a_1^2)[X_2,[X_1,[X_1,X_2]]]\\
&+\frac{1}{48}(b_1^2a_3a_2 - a_3b_1a_1b_2 - b_2a_1^2b_3 + b_1a_2a_1b_3)[X_3,[X_1,[X_1,X_2]]]\\
&+\frac{1}{24}(a_3b_1^2a_1 - b_1a_1^2b_3)[X_1,[X_1,[X_1,X_3]]]\\
&+\frac{1}{48}(b_1^2a_3a_2 + a_3b_1a_1b_2 - b_1a_2a_1b_3 - b_2a_1^2b_3)[X_2,[X_1,[X_1,X_3]]]\\
&+\frac{1}{24}(b_1^2a_3^2 - b_3^2a_1^2)[X_3,[X_1,[X_1,X_3]]]\\
&+\frac{1}{24}(b_1a_2^2b_2 - b_2^2a_2a_1)[X_2,[X_2,[X_1,X_2]]]\\
&+\frac{1}{48}(a_3b_1a_2b_2 + b_1a_2^2b_3 - b_2^2a_3a_1 - b_2a_2a_1b_3)[X_3,[X_2,[X_1,X_2]]]\\
&+\frac{1}{24}(a_3b_1a_2b_2 - b_2a_2a_1b_3)[X_2,[X_2,[X_1,X_3]]]\\
&+\frac{1}{48}(b_1a_3^2b_2 - a_3b_3a_1b_2 - a_2b_3^2a_1 + a_3b_1a_2b_3)[X_3,[X_2,[X_1,X_3]]]\\
&+\frac{1}{24}(b_2^2a_3a_2 - b_2a_2^2b_3)[X_2,[X_2,[X_2,X_3]]]\\
&+\frac{1}{24}(b_2^2a_3^2 - a_2^2b_3^2)[X_3,[X_2,[X_2,X_3]]]\\
&+\frac{1}{24}(a_3b_1a_2b_3 - a_3b_3a_1b_2)[X_3,[X_3,[X_1,X_2]]]\\
&+\frac{1}{24}(b_1a_3^2b_3 - a_1b_3^2a_3)[X_3,[X_3,[X_1,X_3]]]\\
&+\frac{1}{24}(a_3^2b_2b_3 - b_3^2a_3a_2)[X_3,[X_3,[X_2,X_3]]]
\end{aligned}
$$

Fundamental to the simplification of any Lie polynomial is the ability to express complicated nested Lie products as linear combinations of elements of the HB. To make this procedure computationally efficient the package employs an algorithm similar to that for the construction of the HB itself. The algorithm is summarized below.

Let $l(X)$ denote the length of a Lie product $X$, as explained by Definition 6.1. Define the following operations for any Lie product $X$, such that $l(X) \geq 2$, and any Hall basis $B$:

$$\begin{aligned} lo &: X \to lo(X); && \text{where } lo(X) \text{ is the left-operand of } X. \\ ro &: X \to ro(X); && \text{where } ro(X) \text{ is the right-operand of } X. \\ pos &: (X,B) \to pos(X,B); && \text{where } pos(X,B) \text{ is the position index} \\ & && \text{of a Lie product } X \in B. \end{aligned}$$

For example, let $X := [G,H]$ and $B := \{G, H, [G,H], [G,[G,H]], [H,[G,H]]\}$, then $lo(X) = G$, $ro(X) = H$, and $pos(X,B) = 3$.

Denote by $X$ a pure Lie product to be converted into an expression in terms of the elements in $B$. Notice that if $l(X) \geq 2$ then $X$ can be expressed as $X = [lo(X), ro(X)]$. The steps of the algorithm can now be stated in terms of the following pseudo-code:

**Procedure** $Z := phbize(X,B)$

{

1. If $X \in B$ or $l(X) = 1$ then return $Z := X$.
2. If $lo(X) = ro(X)$ or $X = 0$ then return $Z := 0$.
3. $lx := lo(X)$;

   $rx := ro(X)$;

   If $lx \notin B$ then $lx := phbize(lx, B)$;

   If $rx \notin B$ then $rx := phbize(rx, B)$;

   If $pos(lx,B) > pos(rx,B)$ then

   { commute $lx$ and $rx$:

   $aux := lx$;

   $lx := -rx$;

   $rx := aux$;

   }

   If $pos(lx,B) < pos(lo(rx),B)$ and $l(x) > 2$ then

   return $Z := -[ro(rx), [lx, lo(rx)] + [lo(rx), [lx, ro(rx)]$.

   else

   return $Z := [lx, rx]$.

}

It is worth mentioning that an arbitrary Lie product, which is not necessarily a pure Lie product, can also be expressed in terms of the elements in $B$ by applying the above procedure to its pure Lie bracket component which is extracted using the function `simpLB`. The final result is obtained by multiplying back by the scalar component of the original bracket, also delivered by `simpLB`.

### 6.4.2. Wei-Norman Equations of an Underactuated Rigid Body in Space

The usefulness of LTP for practical applications in control of dynamical systems is illustrated by an example of an underactuated rigid body in space for which, after the application of a suitable feedback transformation, the model equations are (cf. Example 4.5.2, p. 92):

$$\dot{x} = f_0(x) + f_1(x)u_1 + f_2(x)u_2 \tag{6.23}$$

$$\begin{aligned}
\text{where,}\quad f_0(x) \quad &= \quad (\sin(x_3)\,\sec(x_2)\,x_5 + \cos(x_3)\,\sec(x_2)\,x_6)\,\frac{\partial}{\partial x_1} \\
&\quad + (\cos(x_3)\,x_5 - \sin(x_3)\,x_6)\,\frac{\partial}{\partial x_2} \\
&\quad + (x_4 + \sin(x_3)\,\tan(x_2)\,x_5 + \cos(x_3)\,\tan(x_2)\,x_6)\,\frac{\partial}{\partial x_3} + a\,x_4 x_5\,\frac{\partial}{\partial x_6}, \\
f_1(x) \quad &= \quad \frac{\partial}{\partial x_4}, \quad f_2(x) = \frac{\partial}{\partial x_5}, \quad \text{and} \quad \dot{x} = [\dot{x}_1,\ \dot{x}_2,\ \dot{x}_3,\ \dot{x}_4,\ \dot{x}_5,\ \dot{x}_6]^T
\end{aligned}$$

Here $u_1$ and $u_2$ are the actuating controls, $a$ is a scalar constant, $f_0$ is the drift vector field, and $f_1$ and $f_2$ are the input vector fields.

The analysis of the motion of this system and the construction of stabilizing feedback controllers requires the knowledge of the flow of the system. Since the Lie algebra of vector fields $L(f_0, f_1, f_2)$ is not nilpotent, this flow can be computed only approximately by working with some nilpotent approximation of $L(f_0, f_1, f_2)$, see [**69**]. Given such approximation, which is determined by an imposed degree of nilpotency, the LTP package allows to set up the Wei-Norman equations for the computation of the $\gamma$-coordinates of the approximating system's flow for general symbolic inputs $u_1$ and $u_2$. The following steps are involved.

**Step 1: Construction of the Hall basis for the Lie algebra of indeterminates $L_4(X_1, X_2, X_3)$ and the basis for the Lie algebra of vector fields $L_4(f_0, f_1, f_2)$.**

Assuming that the controllability Lie algebra for the system (6.23), $L(f_0, f_1, f_2)$, can be approximated with sufficient accuracy by its nilpotent truncation $\not{L}_4(f_0, f_1, f_2)$, with degree of nilpotency $k = 4$, a Hall basis is first constructed for $L_4(\bar{X}_3)$. This is done by invoking $B$`:=phb(3,4)`, which yields $B$ as a list of 32 elements. The latter are then mapped using the Lie algebra homomorphism into the corresponding Lie products of vector fields. The basis $\tilde{B}$ for the nilpotent Lie algebra $\not{L}_4(f_0, f_1, f_2)$, which approximates $L(f_0, f_1, f_2)$, is obtained by identifying the linear dependencies between all the evaluated products of vector fields, constructing a corresponding list of symbolic dependencies between the elements of $B$, and by assuming that adequately selected brackets are zero.

To evaluate the homomorphic images $g_i \stackrel{def}{=} \nu(B_i)$, $i = 1, \ldots, 32$, of the elements in the Hall basis $B$ the vector fields $f_0, f_1, f_2$ are declared as symbolic expressions in Maple, and the function `calcLB` is invoked, remembering that each $B_i$ corresponds to $f_{i-1}$, $i = 1, 2, 3$.

The 29 brackets computed in this way are:

$$
\begin{aligned}
g_4(x) &= [f_0, f_1] = [0, 0, -1, 0, 0, -a\, x_5]^T \\
g_5(x) &= [f_0, f_2] = [-\sin(x_3)/\cos(x_2), -\cos(x_3), -\sin(x_3)\tan(x_2), 0, 0, -a\, x_4]^T \\
g_6(x) &= [f_1, f_2] = [0, 0, 0, 0, 0, 0]^T
\end{aligned}
$$

$$
\begin{aligned}
g_7(x) &= [f_0, [f_0, f_1]] = \\
&= \begin{bmatrix} (\cos(x_3)\, x_5 - \sin(x_3)\, x_6 + \cos(x_3)\, a\, x_5)/\cos(x_2) \\ -\sin(x_3)\, x_5 - \cos(x_3)\, x_6 - \sin(x_3)\, a\, x_5 \\ \sin(x_2)\,(\cos(x_3)\, x_5 - \sin(x_3)\, x_6 + \cos(x_3)\, a\, x_5)/\cos(x_2) \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
g_8(x) &= [f_0, [f_0, f_2]] =
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{bmatrix} \cos(x_3)\,x_4/\cos(x_2)\,(-1+a) \\ -\sin(x_3)\,x_4\,(-1+a) \\ (\cos(x_2)\,x_6 - \cos(x_3)\,\sin(x_2)\,x_4 + \cos(x_3)\,\sin(x_2)\,a\,x_4)/\cos(x_2) \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
g_9(x) &= [f_1,[f_0,f_1]] = [0,0,0,0,0,0]^T \\
g_{10}(x) &= [f_1,[f_0,f_2]] = [0,0,0,0,0,-a]^T \\
g_{11}(x) &= [f_1,[f_1,f_2]] = [0,0,0,0,0,0]^T \\
g_{12}(x) &= [f_2,[f_0,f_1]] = g_{10}(x) \\
g_{13}(x) &= [f_2,[f_0,f_2]] = [0,0,0,0,0,0]^T \\
g_{14}(x) &= [f_2,[f_1,f_2]] = [0,0,0,0,0,0]^T \\
g_{15}(x) &= [[f_0,f_1],[f_0,f_2]] = [\cos(x_3)/\cos(x_2), -\sin(x_3), \cos(x_3)\,\sin(x_2)/\cos(x_2), 0, 0, 0]^T \\
g_{16}(x) &= [[f_0,f_1],[f_1,f_2]] = [0,0,0,0,0,0]^T \\
g_{17}(x) &= [[f_0,f_2],[f_1,f_2]] = [0,0,0,0,0,0]^T \\
g_{18}(x) &= [f_0,[f_0,[f_0,f_1]]] = \\
&= \begin{bmatrix} -x_4/\cos(x_2)\,(\sin(x_3)\,x_5 + \cos(x_3)\,x_6 + 2\,\sin(x_3)\,a\,x_5) \\ -x_4\,(\cos(x_3)\,x_5 - \sin(x_3)\,x_6 + 2\,\cos(x_3)\,a\,x_5) \\ \left\{ \begin{array}{c} (+\cos(x_2)\,x_5\,x_5 + \cos(x_2)\,a\,x_5\,x_5 - \sin(x_2)\,\sin(x_3)\,x_5\,x_4 \ldots \\ \ldots - \sin(x_2)\,\cos(x_3)\,x_6\,x_4 - 2\,\sin(x_3)\,\sin(x_2)\,a\,x_4\,x_5 \ldots \\ \ldots + \cos(x_2)\,x_6\,x_6)/\cos(x_2) \end{array} \right\} \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
g_{19}(x) &= [f_0,[f_0,[f_0,f_2]]] =
\end{aligned}
$$

$$= \begin{bmatrix} -(-\sin(x_3)\,x_4\,x_4 + \sin(x_3)\,x_4\,x_4\,a + \cos(x_3)\,x_5\,x_6 - \sin(x_3)\,x_6\,x_6)/\cos(x_2) \\ \cos(x_3)\,x_4\,x_4 - \cos(x_3)\,x_4\,x_4\,a + \sin(x_3)\,x_5\,x_6 + \cos(x_3)\,x_6\,x_6 \\ \left\{ \begin{array}{c} (-\sin(x_2)\;\cos(x_3)\,x_5\,x_6 - \sin(x_2)\;\sin(x_3)\,x_4\,x_4\,a \ldots \\ \ldots + \sin(x_2)\;\sin(x_3)\,x_4\,x_4 - x_4\,x_5\;\cos(x_2)\ldots \\ \ldots + \sin(x_2)\;\sin(x_3)\,x_6\,x_6 + 2\;\cos(x_2)\,a\,x_4\,x_5)/\cos(x_2) \end{array} \right\} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned}
g_{20}(x) &= [f_1,[f_0,[f_0,f_1]]] = [0,0,0,0,0,0]^T \\
g_{21}(x) &= [f_1,[f_0,[f_0,f_2]]] = (-1+a)*g_{15}(x) \\
g_{22}(x) &= [f_1,[f_1,[f_0,f_1]]] = [0,0,0,0,0,0]^T \\
g_{23}(x) &= [f_1,[f_1,[f_0,f_2]]] = [0,0,0,0,0,0]^T \\
g_{24}(x) &= [f_1,[f_1,[f_1,f_2]]] = [0,0,0,0,0,0]^T \\
g_{25}(x) &= [f_2,[f_0,[f_0,f_1]]] = (1+a)*g_{15}(x) \\
g_{26}(x) &= [f_2,[f_0,[f_0,f_2]]] = [0,0,0,0,0,0]^T \\
g_{27}(x) &= [f_2,[f_1,[f_0,f_1]]] = [0,0,0,0,0,0]^T \\
g_{28}(x) &= [f_2,[f_1,[f_0,f_2]]] = [0,0,0,0,0,0]^T \\
g_{29}(x) &= [f_2,[f_1,[f_1,f_2]]] = [0,0,0,0,0,0]^T \\
g_{30}(x) &= [f_2,[f_2,[f_0,f_1]]] = [0,0,0,0,0,0]^T \\
g_{31}(x) &= [f_2,[f_2,[f_0,f_2]]] = [0,0,0,0,0,0]^T \\
g_{32}(x) &= [f_2,[f_2,[f_1,f_2]]] = [0,0,0,0,0,0]^T
\end{aligned}$$

The nilpotent approximation to $L(f_0, f_1, f_2)$ (in the neighborhood of the origin) is then derived here by assuming that

$$g_7 = g_8 = g_{18} = g_{19} = [0,0,0,0,0,0]^T, \tag{6.24}$$

as indeed, the values of these brackets evaluated in the neighborhood of the origin are negligibly small. Noting that

$$g_i = [0, 0, 0, 0, 0, 0]^T \tag{6.25}$$

for $i \in \{6, 9, 11, 13, 14, 16, 17, 20, 22, 23, 24, 26, 27, 28, 29, 30, 31, 32\}$, and that the following dependencies among the above Lie products hold:

$$g_{12} = g_{10}, \qquad g_{21} = (-1 + a)g_{15}, \qquad g_{25} = (1 + a)g_{15}, \tag{6.26}$$

the above dependencies translate into the following symbolic dependencies between the elements of $B$:

$$B_i = 0, \quad B_{12} = B_{10}, \quad B_{21} = (-1 + a)B_{15}, \quad B_{25} = (1 + a)B_{15}, \tag{6.27}$$

for $i \in \{6, 7, 8, 9, 11, 13, 14, 16, 17, 18, 19, 20, 22, 23, 24, 26, 27, 28, 29, 30, 31, 32\}$. The relations (6.27) are passed as symbolic arguments to the LTP function and yield a basis for the nilpotent Lie algebra of vector fields $\L_4(f_0, f_1, f_2)$,

$$\{\tilde{B}_0, \tilde{B}_1, \tilde{B}_2, \tilde{B}_3, \tilde{B}_4, \tilde{B}_5, \tilde{B}_6\} = \{g_1, g_2, g_3, g_4, g_5, g_{10}, g_{15}\}$$

It is worth noticing that the ordering of the above basis preserves a condition corresponding to (6.12), which guarantees that the Wei-Norman equation can be given in the explicit form (6.13).

**Step 2: Calculation of the right-hand side of the Wei-Norman equation.**

The derivation of the Wei-Norman equation is carried out in two steps. The product term in the right-hand side of (6.10a) is first computed by invoking the LTP function `wner`, in which the basis elements $B_{i+1}$ need to be replaced by $\tilde{B}_i$, $i = 0, \ldots, 6$. Next, the coefficients corresponding to the basis elements $\tilde{B}_i$ on both sides of equation (6.10a)–(6.10) are equated using the LTP function `wnde`.

More precisely, the LTP function `wner` ought to be invoked with the following parameters: `rhwne:=wner(`$r$`,`$k-1$`,`$B$`,`$\tilde{B}$`,lbdt)`, where $r = 7$ is the dimension of the basis $\tilde{B}$, $k = 4$ is the degree of nilpotency, and `lbdt` is the list of linear dependencies (6.27). The resulting expression is:

$$\begin{aligned} rhwne \quad := \quad & \dot{\gamma}_0 f_0 + \dot{\gamma}_1 f_1 + \dot{\gamma}_2 f_2 + (\dot{\gamma}_1\gamma_0 + \dot{\gamma}_3)[f_0, f_1] + (\dot{\gamma}_2\gamma_0 + \dot{\gamma}_4)[f_0, f_2] \\ & +(\dot{\gamma}_3\gamma_2 + \dot{\gamma}_4\gamma_1 + \dot{\gamma}_5)[f_1, [f_0, f_2]] + (\dot{\gamma}_4\gamma_3 + \dot{\gamma}_5\gamma_0 a + \dot{\gamma}_6)[[f_0, f_1], [f_0, f_2]] \end{aligned}$$

The function `wnde(rhwne,`$r$`,`$B$`,lbdt)` is applied to the above result returning the matrix $\Gamma(\gamma)$ (see equation (6.11)) and the set of equations :

$$\begin{aligned} u_0 &= \dot{\gamma}_0 \\ u_1 &= \dot{\gamma}_1 \\ u_2 &= \dot{\gamma}_2 \\ u_3 &= \dot{\gamma}_1\gamma_0 + \dot{\gamma}_3 \\ u_4 &= \dot{\gamma}_2\gamma_0 + \dot{\gamma}_4 \\ u_5 &= \dot{\gamma}_3\gamma_2 + \dot{\gamma}_4\gamma_1 + \dot{\gamma}_5 \\ u_6 &= \dot{\gamma}_4\gamma_3 + \dot{\gamma}_5\gamma_0 a + \dot{\gamma}_6 \end{aligned}$$

The inversion of $\Gamma(\gamma)$ results in the following Wei-Norman equation:

$$\begin{bmatrix} \dot{\gamma}_0 \\ \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \\ \dot{\gamma}_4 \\ \dot{\gamma}_5 \\ \dot{\gamma}_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\gamma_0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\gamma_0 & 0 & 1 & 0 & 0 \\ 0 & \gamma_0\gamma_2 & \gamma_0\gamma_1 & -\gamma_2 & -\gamma_1 & 1 & 0 \\ 0 & -a\gamma_0^2\gamma_2 & \gamma_0\gamma_3 - a\gamma_0^2\gamma_1 & a\gamma_0\gamma_2 & a\gamma_0\gamma_1 - \gamma_3 & -a\gamma_0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}$$

with $\gamma_i(0) = 0$, $i = 0, 1, \ldots, 6$.

The flows of the original system (6.23) in a neighborhood of the origin $x = 0$ can now be analyzed in an approximate way using the above Wei-Norman equation. Such analysis is helpful for the construction of stabilizing control laws for (6.23), as shown in the previous chapters and in [**76, 77, 78**].

# CHAPTER 7

## Conclusions and Future Research

The problem of stabilizing feedback synthesis for strongly nonlinear systems with drift was investigated in this work, whose main contributions are summarized as follows:

- The development of a continuous time-varying stabilization approach.
- The derivation of two approaches to the design of time-varying discontinuous state feedback stabilizers.
- The implementation of a software package that simplifies symbolic Lie algebraic calculations, many of which are practically impossible to perform by hand.

The contributions of the preceding chapters are briefly reviewed in the following section of this conclusion; some general remarks about each approach are also given. In the last two sections a short qualitative comparison of the feedback strategies and some suggestions for future research are given.

### 7.1. Review of the Results and General Remarks

#### 7.1.1. Chapter 3: Continuous Time-Varying Stabilization Feedback Approach

This chapter presented a feasible approach to the synthesis of time-varying stabilizing feedback controls for nilpotent nonlinear systems with drift. The synthesis method is general and is applicable to a large class of nilpotent systems, however, it is computationally expensive. The latter is not surprising, as stabilization of systems which do not lend themselves to successful linearization (be it through state-feedback transformations, or else simply around some operating points) is well recognized to be difficult.

### 7.1.2. Chapter 4: Discontinuous Time-Varying Stabilization Feedback Approach

The approaches to the design of stabilizing feedback controls presented in Chapter 4 apply to general systems with drift for which controllable linearizations, as well as continuous stabilizing state feedback laws, may not exist.

The advantages and drawbacks of each approach are summarized as follows:

- Compared to other Lie algebraic approaches, the ones proposed here are simple to implement[1]; for example, they do not require the often difficult solution of a trajectory interception problem on the Lie group as in [**75**].
- Both approaches are computationally expensive[2] as they require the solution of a nonlinear programming problem through numerical optimization algorithms, but less so as compared with [**77**].
- As compared with the control procedure of [**68**], both methods provide a more systematic tool for generation of complicated Lie bracket motions of the system which might be necessary in the process of stabilization.
- The advantage of Method 2 (Sec. 4.3.2) over Method 1 (Sec. 4.3.1) is that it is simpler to implement.
- The approaches do not deliver the feedback uniquely as they employ arbitrary solutions to a satisficing nonlinear programming problem. This can be viewed as the strength of the methods, as it leaves the designer much freedom to accommodate for other goals.

[1]The meaning of *simple to implement* is used here in a broad qualitative sense rather than in a quantitative sense. By *simple to implement* it is meant that the mathematical expressions for the control law can be obtained after a small number of steps. It is also meant that the mathematical expressions themselves are simple and involve few terms, and therefore their actual implementation, in the form of a computer routine, is simple. It should be noted, though, that the concept of *simplicity of implementation* is also relative to the existing tools, such as software tools that automatize the mathematical derivations and the programming process.

[2]The terms *computationally expensive* or *computationally complex* are used here in a qualitative sense when referring to a method or procedure that involves a large number of computations or a longer simulation time. Although the concept of *computational complexity* can be given a precise quantitative definition in terms of the number of arithmetic operations, the number of CPU cycles or the amount of computation time, a rigorous quantitative analysis is impractical and is beyond the scope of this research because the feedback laws involve the solution of optimization problems that require an indeterminate (variable) number of iterations and because, for practical purposes, it suffices to notice that longer simulation times already imply an increased computational complexity.

For example, it may be desired to solve SP1 or SP2 while simultaneously minimizing the number of discontinuities in the open-loop control or else to construct a time-varying continuous control.

### 7.1.3. Chapter 5: Stabilization of Bilinear Systems with Unstable Drift

#### 7.1.3.1. TIP Approach

The construction of time-varying feedback stabilizers for homogeneous bilinear systems was investigated in this chapter. The approach relies on the solution of a flow interception problem in terms of a set of parameters which represent the values of the stabilizing controls for the extended system. Essentially, a closed form parametric solution of this problem is required. In some cases, such as the one presented in the example, a solution to the flow interception problem can be obtained analytically. Analytic solutions are usually impossible when the system has a more complicated Lie algebraic structure since then the order and complexity of the equations describing the evolution of the $\gamma$-coordinates of flows increases. This fact motivated the development of the more computationally practical approaches presented in Chapter 4, and the software package for symbolic Lie algebraic calculations described in Chapter 6.

However, an undeniable advantage of this approach is that it applies to homogeneous bilinear systems of a general form, without any specific assumptions concerning the stability of the drift term or the dimension of the system. Considering that no alternative methods of similar generality exist as yet, this study has been worthwhile.

#### 7.1.3.2. Steering to the Stable Manifold Approach

Switching stabilizing control of bilinear systems has previously been proposed only for non-homogeneous bilinear systems [**102**], or for single input systems with a dyadic $A_1$ matrix [**104**].

The contribution of this chapter is the novel approach to the synthesis of stabilizing feedback control for homogeneous bilinear systems with an unstable drift. The method applies to systems in which the drift cannot be stabilized by any constant control.

It is shown that the methods proposed in Chapter 4 are not limited to systems whose controllability Lie algebra is nilpotent and that these methods offer some flexibility in the sense that they can be modified to serve for other purposes such as is steering to a manifold. The examples presented confirm the effectiveness of the approach.

### 7.1.4. Chapter 6: A Software Package for Symbolic Lie Algebraic Computations

The development of LTP was motivated by computationally intense applications in nonlinear control system design and nonlinear filter theory. The present version of the package is thus an effective tool for:

(i) Performing general Lie algebraic calculations on Lie algebras of indeterminates, including construction of bases and simplification of arbitrary Lie algebraic expressions involving symbolic scalars.

(ii) The analysis of the structure of general dynamical systems and the structure of estimation Lie algebras.

(iii) The solution of differential equations evolving on Lie groups.

The importance of the development of computational tools to assist in difficult symbolic and numerical calculations has already been recognized, see [**194, 173**], and will hopefully encourage further advances in mathematics.

## 7.2. Comparison of the Stabilization Strategies

The *beauty* of the proposed approaches is in that they apply to rather general systems. Moreover, systems with the same Lie algebraic structure need the derivation of the feedback law only once. However, it must also be acknowledged that the generality comes at the cost of an elevated computational complexity of the feedback laws as compared to methods which exploit specific model properties.

The continuous time-varying approach is conceptually interesting, however it has a severe practical limitation due to the necessity of finding, *at each integration step*, the starting point $x_0(t)$ corresponding to current state $x(t)$ which is reached by the critical trajectory $\Phi_w(t, x_0(t))$ at time $t$. The latter means that in practice the system equation must be repeatedly integrated to solve a two-end-point constrained problem, for which limited numerical procedures currently exist.

The discontinuous time-varying approaches are simple and, given enough computational power, they perform reasonably well compared to the continuous time-varying approach. Again, a drawback is in the severely oscillatory nature of the trajectory which might be undesirable in a number of applications. This behaviour is in part due to the well recognized difficulty of steering underactuated systems along directions of adequately chosen Lie bracket vector fields which would counteract the system motion along an unstable or non-cyclical drift vector field.

## 7.3. Future Research Topics

### 7.3.1. Continuous Time-Varying Feedback Approach

As pointed out earlier and shown by Example 3.4, certain assumptions made in Chapter 3, such as the following:

- the continuity with respect to time of the solution to the trajectory interception problem,
- the nilpotency of the Lie algebra generated by the vector fields in the system model,
- and the assumption about the origin being an isolated equilibrium of the unforced system

can possibly be relaxed, which is a topic for future research.

### 7.3.2. Discontinuous Time-Varying Feedback Approach

Future research is aimed at:

- Replacing the on-line computation of SP1 or SP2 by an a priori calculation of controls (4.51), for each $C_s \subset B(0, R)$, in the collection of compact non-overlapping subsets covering $B(0, R)$.
- Exploring possible simplifications which might originate from a particular structure of the Wei-Norman equations describing the evolution of the flow of the system on the Lie group.

### 7.3.3. Improvement of the LTP Package

Continuing work aiming at further development of the LTP package is concerned with:

(i) Investigating the possibility of introducing functions that employ the chronological algebra formalism to simplify calculations leading to explicit evaluation and simplification of the generalized Campbell-Baker-Hausdorff-Dynkin formula (the logarithm of the Chen-Fliess series), see [**144, 147**].

(ii) Implementing a procedure for the construction of the Wei-Norman equations for flows on arbitrary matrix Lie groups. The difficulty in obtaining a closed formula for the Wei-Norman equation for finite dimensional, rather than nilpotent Lie algebras, resides in the fact that the exponentials, $e^{ad_{B_i}}$, of the basis elements, $ad_{B_i}$, in the adjoint representation of the Lie algebra, are generally given as infinite series. The new procedure will bypass this difficulty by employing finite expressions for these exponentials in terms of the structure constants of the Lie algebra, as suggested in [**141**].

### 7.3.4. Other Ideas for Further Developments

The following ideas are also worth considering and exploring in future developments.

- A more general extended system $\Sigma^e$ given by:

$$\Sigma^e : \qquad \dot{x} = \sum_{i=0}^{r-1} g_i(x) v_i \stackrel{\text{def}}{=} g^v(x) \tag{7.1}$$

  could possibly be used instead of the extended system of equation (2.1). Notice that the former drift term $g_0$ in (2.1) is now treated as an input vector field and also that it would only be required that dim $L_x(\mathcal{F}) = r = n$, i.e. that the set of vector fields $\{f_0(x), \ldots, f_{r-1}(x)\}$ span $\mathbb{R}^n$ for all $x \in B(0, R)$ (or $x \in \mathbb{R}^n$ for global results). Note, however, that now $v_0$ would not be simply related to the period of integration $T$ as when the extended system is regarded as a system with drift for which $v_0 = 1$.

- Development of an algorithm to solve the TIP off-line using smooth functions, possibly based on the approach proposed in [**62**], which uses controls that are polynomials in time, or the approach in [**51, 52, 47**] employing highly oscillatory sinusoidal controls to generate Lie bracket motions.

- Although the differential geometric approach has been employed extensively in the literature pertinent to the analysis and control of nonlinear systems, its inherent weakness is its non-robustness to modelling errors and the presence of unmodelled dynamics. These errors can be viewed as perturbations to the true vector fields that define the system and may significantly alter the structure of the Lie algebra $L(\mathcal{F})$. Hence, exploring the changes in the characteristics of $L(\mathcal{F})$ and the sensitivity of the stability properties of system $\Sigma$ to modelling errors, as well as to disturbances in the inputs, is a topic worth considering as these results should be useful to the researcher looking to apply the proposed stabilization approaches to engineering applications.

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# REFERENCES


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# APPENDIX A

## Notation and Mathematical Background

The work presented in this dissertation requires familiarity with some basic concepts which, for convenience of the reader, are presented in this appendix. The notation employed in this thesis is described in the first section. All definitions can be found in standard books, see for instance [**152, 155, 157, 160, 162**] for concepts of Lie theory, [**153, 156**] for concepts of group theory and [**183, 185, 186**] for concepts of differential geometry, [**181, 182**] for notions in differential topology or [**187, 189**] for the fundamentals of analysis. Most definitions employed here can also be found in standard nonlinear control systems literature such as [**2, 7, 6**].

### A.1. Notation

#### General

| | |
|---|---|
| $\stackrel{\text{def}}{=}$ | Denotes definition. |
| $\mid$ | Denotes "such that". |
| $^T$ | Superscript $T$ denotes *transposition*. |

#### Sets

| | |
|---|---|
| $\emptyset$ | Empty set. |
| $A^\circ$ | Interior of set $A$. |
| $\bar{A}$ | Closure of $A$. |

| | |
|---|---|
| $\flat(A)$ | Boundary of $A$. |
| $A \setminus B$ | Set $A$ excluding those elements that belong to set $B$. |

**Algebra**

| | |
|---|---|
| $\mathbb{N}$ | Set of strictly positive integers $\{1, 2, 3, \ldots\}$. |
| $\mathbb{Z}$ | Ring of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. |
| $\mathbb{Z}_+$ | Set of non-negative integers, $\{0, 1, 2, \ldots\}$, note that $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$. |
| $\mathbb{R}$ | Field of real numbers. |
| $\mathbb{R}_+$, $(\mathbb{R}_-)$ | Set $[0, \infty)$ of non-negative (non-positive) reals. |
| $\mathbb{R}^n$ | $n$-dimensional Euclidean space. |
| $\mathbb{C}$ | Field of complex numbers. |
| $\mathbb{C}_+$, $(\mathbb{C}_-)$ | Set of complex numbers in the right (left) half plane, including the imaginary axis. |
| $\mathbb{C}_+^\circ$, $(\mathbb{C}_-^\circ)$ | Interior of $\mathbb{C}_+$, i.e. $\mathbb{C}_+^\circ \stackrel{\text{def}}{=} \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$, (respectively, the interior of $\mathbb{C}_-$). |
| $\mathbf{i}$ | The imaginary number $\sqrt{-1}$. |

**Analysis**

| | |
|---|---|
| $\lVert x \rVert$ | Euclidean norm of $x \in \mathbb{R}^n$. |
| $B(x, r)$ | Open ball of radius $r$ centered at $x$: $$B(x, r) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid \lVert x \rVert < r,\ r > 0\}$$ |
| $\mathrm{eig}_i(A)$ | $i$-th eigenvalue of $A$. |
| $\dot{f}$ | Time derivative $\frac{df}{dt}$ of the scalar or vector function $f$. |

| | |
|---|---|
| $\nabla_x$ | Gradient: Differential operator of first order partial derivatives with respect to $x$ defined as $\begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \dots & \frac{\partial}{\partial x_n} \end{bmatrix}$. |
| $H_x$ | Hessian: Differential operator of second order partial derivatives with respect to $x$. |
| $\mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n)$ | The family of all $\mathcal{C}^\infty$ vector fields $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ defined on $\mathbb{R}^n$. |

**Lie Theory**

| | |
|---|---|
| $L_f$ | Lie derivative operator. |
| $[\cdot, \cdot]$ | Lie product (or bracket). |
| $L(G)$ | Lie algebra associated with the Lie group $G$. |
| $G(L)$ | Lie group associated with the Lie algebra $L$. |

**Control and Dynamical Systems**

| | |
|---|---|
| $\Sigma$ | Original system with drift of equation (1.1). |
| $f_i$ | Vector fields on $\mathbb{R}^n$ of $\Sigma$. |
| $u_i$ | $i$-th control input of $\Sigma$. |
| $f^u$ | Right-hand side of $\Sigma$ defined by equation (1.1). |
| $\Sigma^e$ | Lie algebraic extension of $\Sigma$, see eq. (2.1). |
| $g_i$ | Vector fields on $\mathbb{R}^n$ of $\Sigma^e$. |
| $v_i$ | $i$-th control input of $\Sigma^e$. |
| $g^v$ | Right-hand side of $\Sigma^e$ defined by eq. (2.1). |
| $\mathcal{F}$ | Family of vector fields $\{f_0, \dots, f_m\}$ defined on $\mathbb{R}^n$. |
| $\mathcal{P}^m$ | Family of piece-wise constant functions, continuous from the left, and defined on $\mathbb{R}^m$. |

| | |
|---|---|
| $L(\mathcal{F})$ | Lie algebra of vector fields generated by $\mathcal{F}$. |
| $L_x(\mathcal{F})$ | Set $\{f(x) \mid f \in L(\mathcal{F})\} \subset \mathbb{R}^n$, of vectors resulting from the evaluation of $L(\mathcal{F})$ at $x$. |
| $\mathcal{G}$ | Basis $\{g_0, \ldots, g_{r-1}\}$ for $L(\mathcal{F})$. |
| $\bar{\mathcal{G}}(x)$ | Span of $f \in L(\mathcal{F})$ (or *accessibility distribution*) evaluated at $x$, i.e., $\bar{\mathcal{G}}(x) \stackrel{\text{def}}{=} \text{span}\{f(x) \in \mathbb{R}^n \mid f \in L(\mathcal{F})\}$. |
| $exp : L(G) \to G$ | Standard exponential mapping (cf. [**162**]) between elements of the Lie algebra $L(G)$ an the associated Lie group $G$. |
| $\exp(tf)$ | One-parameter group of transformation generated by the vector field $f$, (the flow of the differential equation $\dot{x} = f(x)$). |
| $\mathcal{R}_{\mathcal{F}}(T, x)$ | Reachable set of states of system $\Sigma$ at time $T$ from $x$ (by piece-wise constant controls). |
| $\mathcal{R}_{\mathcal{G}}(T, x, U^e(x))$ | Reachable set of states of the extended system $\Sigma^e$ at time $T$ with controls in the set of admissible controls $U$. |
| $\mathcal{R}_{\gamma}(T, U^e(x))$ | Reachable set of $\gamma$-coordinates attainable at time $T$ with controls in the set of admissible controls $U$. |
| $\mathit{diff}(\mathbb{R}^n)$ | Group (under composition) of diffeomorphisms on $\mathbb{R}^n$. |
| $G$ | Global group of diffeomorphisms under the composition of $exp(t_i f_i)$. |
| $G_T$ | Subgroup of $G$, such that $t_i \geq 0$ and $\sum_{i=1}^{j} t_i = T$. |
| $H$ | Analytic (connected), simply connected, Lie subgroup of $\mathit{diff}(\mathbb{R}^n)$, whose Lie algebra $L(H)$ is isomorphic to $L(\mathcal{F})$. |
| $\phi$ | Analytic mapping $H \times \mathbb{R}^n \ni (h, p) \to h(p) \in \mathbb{R}^n$ which induces an isomorphism between $L(H)$ and $L(\mathcal{F})$. |
| $\phi_L^+$ | Lie algebra isomorphism $\phi_L^+ : L(H) \to L(\mathcal{F})$ (the infinitesimal generator of the mapping $\phi$, see Theorem 2.1). |
| $\phi_G^+$ | Group isomorphism between the groups $H$ and $G$ induced by the isomorphism $\phi_L^+$. |

| | |
|---|---|
| $\Sigma_H$ | Dynamical system $\Sigma$ on the Lie group $H$, see eq. (2.7). |
| $\Sigma_H$ | Dynamical system $\Sigma^e$ on the Lie group $H$, see eq. (2.9). |
| $x(t, x_0, u)$ | Trajectory of a system $\dot{x} = f(x, u)$ on $\mathbb{R}^n$ at time $t$ starting from a state $x_0$ at time $t_0$ and resulting from the application of control $u$. |
| $x(\cdot, x_0, u)$ | Trajectory through $x_0$ and due to control $u$. |
| $\Phi_u(t, x_0)$ | equivalent to $x(t, x_0, u)$. |
| $\Phi(t)$, $\Phi^e(t)$ | Flows of systems $\Sigma$ and $\Sigma^e$, respectively. With the above notation $\Phi(t) = \exp(tf^u)$, $\Phi^e(t) = \exp(tg^v)$. |

## A.2. Groups, Fields, Vector Spaces and Algebras

### A.2.1. Group Related Notions

The notion of a Lie group relies on the definition of a *topological or continuous group.* A topological or continuous group has two different kind of structures on it: a topological structure and an algebraic structure.

Algebraically, a continuous group satisfies the axioms defining a group:

**Definition A.1. - Group ($G$).** *A* group *$G$ is a set with a binary operation $(\cdot)$: $G \times G \to G$, such that $\forall\ a, b, c \in G$, the following properties hold:*

i. *Closure:* $a \in G,\ b \in G \Rightarrow a \cdot b \in G$
ii. *Associativity:* $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
iii. *Identity $e$:* $\exists e : a \cdot e = e \cdot a = a$
iv. *Inverse $a^-1$:* $\exists a^{-1} : a \cdot a^{-1} = a^{-1} \cdot a = e$

*A group $G$ is called Abelian if $a \cdot b = b \cdot a$, $\forall\ a, b \in G$.*

Topologically, a continuous group is a manifold. The algebraic and topological properties are combined by the following two additional axioms for any $a,\ b \in G$:

A1. The mapping $a \times b \to a \cdot b$ is continuous.

A2. The mapping $a \to a^{-1}$ is continuous.

**Definition A.2. - Topological Group or Continuous Group.** *A* topological group *or a* continuous group *consists of:*

(i) *An underlying $n$-dimensional manifold $M$.*

(ii) *A operation $\phi$ mapping each pair of points $(x, y)$ in the manifold into another point $z$ in the manifold.*

(iii) *In terms of coordinate systems around the points $x$, $y$, $z$, one writes*

$$z_k = \phi_k(x_1, \ldots, x_n; y_1, \ldots, y_n) \quad \textit{for} \quad k = 1, \ldots, n$$

**Definition A.3. - Homomorphism.** *A* homomorphism *between groups, $\phi : G \to H$, is a map which preserves the group operation:*

$$\phi(a \cdot b) = \phi(a) \cdot \phi(b)$$

**Definition A.4. - Isomorphism.** *An* isomorphism *is a homomorphism which is bijective.*

### A.2.2. Algebra Related Notions

The definition of a Lie algebra rests upon some other basic notions which are presented first: the definition of a field, a vector space, and a linear algebra.

**Definition A.5. - Field ($K$).** *A* field *$K$ is a set with two binary operations: addition $(+)$ and multiplication $(\cdot)$, such that:*

i. *$K$ is an Abelian group under $(+)$, with identity $0$.*

ii. *$K - \{0\}$ is an (Abelian) group under $(\cdot)$, with identity $1$.*

iii. *$(\cdot)$ distributes over $(+) : a \cdot (b + c) = a \cdot b + a \cdot c$.*

*Some examples of fields are $\mathbb{R}$ and $\mathbb{C}$.*

**Definition A.6. - Vector Space ($V$) over a field $K$.** *Given a field $K$, a* vector space *is a non-empty set of elements, called* vectors*, with rules of addition and scalar multiplication defined on it as:*

i. *Addition: $u, v \in V$, the sum $u + v \in V$.*

ii. *Multiplication (by an element $k$ of a field $K$): $k \in K, v \in V$, the product $kv \in V$.*

*A vector space is a commutative group under the addition operation (+) and the scalar multiplication distributing over the addition, i.e. its elements satisfy the following set of addition and multiplication axioms:*

A1. *Associativity: For any vectors $u, v, w \in V, (u + v) + w = u + (v + w)$.*

A2. *Additive Identity: A vector denoted by 0, called the* zero vector*, for which $v + 0 = v$ for any $v \in V$.*

A3. *Additive Inverse: A vector denoted $-v$ for each $v \in V$ such that $v + (-v) = 0$.*

A4. *Commutativity: For any $u, v \in V, u + v = v + u$.*

M1. *Distributivity over $K$: For any $k \in K$ and any $u, v \in V$, $k(u + v) = ku + kv$.*

M2. *Distributivity over $V$: For any $a, b \in K$ and any $v \in V$, $(a + b)v = av + bv$.*

M3. *Associativity: For any scalars $a, b \in K$ and any $v \in V$, $(ab)v = a(bv)$.*

M4. *Scalar Identity: The element $a \in K$, such that for any $v \in V$, $av = v$. The scalar identity $a$ is denoted by 1.*

**Examples of vector spaces**

$n$-tuple Space $K^n$: The set of $n$-tuples of elements in $K$, explicitly denoted by $(k_1, k_2, \ldots, k_n)$ or $[k_1, k_2, \ldots, k_n]$.

Matrix Space $M^{m \times n}$: The set of all $m \times n$ matrices over an arbitrary field $K$. Note that a linear map $\varphi : K^n \to K^n$, has a matrix representation $M_n(K) \in K^{n \times n}$, and $M_n(K)$ is also a vector space over $K$.

Polynomial Space $P(s)$: The set of polynomials $a_0 + a_1 s + ... + a_k s^k, k = 1, 2, ...$ with coefficients $a_i$ in some field $K$.

Function Space $F(X)$: The set of functions $F(X)$ mapping elements of a non empty set $X$ into $K$.

**Definition A.7. - Normed Vector Space.** *A vector space $V$ over the field of reals is said to be a* normed vector space *if it can be endowed with a norm, denoted by $|\cdot|$, a function from $V \to \mathbb{R}_+$ satisfying:*

i. $|x| \geq 0$, *for all $x \in V$ and $|x| = 0 \Leftrightarrow x = 0$.*

ii. $|\alpha x| = |\alpha||x|$ *for all $\alpha \in \mathbb{R}$.*

iii. $|x + y| \leq |x| + |y|$.

**Definition A.8. - Induced Norm of an Operator.** *Let $X$, $Y$ be normed linear spaces with norms $|\cdot|_X$, $|\cdot|_Y$, respectively. The space $T(X,Y)$ of operators mapping $X \to Y$ has a norm induced by the norms on $X$, $Y$ as follows. Let $P \in T(X,Y)$, then*

$$|P|_i \overset{\text{def}}{=} \sup_{|x|_X \neq 0,\, x \in X} \frac{|Ax|_Y}{|x|_X} \tag{A.1}$$

*If the $P$ is a linear map in $L(X,Y)$, the space of linear maps, then the above norm is equivalent to:*

$$|P|_i \overset{\text{def}}{=} \sup_{|x|_X = 1,\, x \in X} |Ax|_Y \tag{A.2}$$

**Definition A.9. - Linear Algebra ($\mathcal{A}$).** *An algebra $\mathcal{A}$ over a field $K$ is a vector space over $K$ that, in addition to the vector addition $(+)$ and the scalar multiplication operations and the corresponding axioms, it has a vector multiplication $\odot$ such that the following distributivity and associativity laws are satisfied for every $F, G, H \in A$ and every $k \in K$:*

i. *Closure:* $F \odot G \in A$

ii. *Distributivity of $\odot$ over addition in $A$:* $F \odot (G + H) = F \odot G + F \odot H$

iii. *Distributivity of addition in $A$ over $\odot$:* $(F + G) \odot H = F \odot H + G \odot H$

iv. *Distributivity of scalar multiplication:* $k(G \odot F) = (kG) \odot F = G \odot (kF)$

*Properties ii. and iii. are also referred to as* bilinearity*. Different varieties of algebras may be obtained depending on which additional postulates are satisfied. The algebra may be* associative, symmetric (sometimes called commutative), antisymmetric (sometimes called anti-commutative, skew-commutative or skew-symmetric)*, non-commutative (if it is not symmetric), have an identity, and/or satisfy the derivative property (or Jacobi identity). The following postulates describe these properties:*

v. *Associativity of* $\odot$*:* $(F \odot G) \odot H = F \odot (G \odot H)$

vi. *Commutativity:* $F \odot G = G \odot F$

vii. *Anti-commutativity:* $F \odot G = -G \odot F$

viii. *Existence of identity* $1$*:* $F \odot 1 = F$

ix. *Derivative property:* $F \odot (G \odot H) = (F \odot G) \odot H + G \odot (F \odot G)$

*Jacobi Identity:* $F \odot (G \odot H) + G \odot (H \odot F) + H \odot (F \odot G) = 0$

**Definition A.10. - Algebraic Ideal.** *Given an algebra* $(\mathcal{A}, \odot)$*, a subspace* $\mathcal{I} \subset \mathcal{A}$ *is called an* algebraic ideal *if* $x \in \mathcal{I}$*,* $y \in \mathcal{A}$*, implies that* $x \odot y, y \odot x \in \mathcal{I}$*.*

**Definition A.11. - Lie Algebra.** *A linear space* $\mathcal{G}$ *over a field* $K$ *(usually real or complex numbers) with a multiplication* $\mathcal{G} \times \mathcal{G} \to \mathcal{G} : \{X, Y\} \to [X, Y] \in \mathcal{G}$ *is called a Lie algebra if it satisfies the properties:*

i. $[X, \alpha Y] = \alpha[X, Y] = [\alpha X, Y], \ \alpha \in \mathbb{K}$

ii. $[X, Y + Z] = [X, Y] + [X, Z]$

iii. $[X, Y] = -[Y, X]$

iv. $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$ *(Jacobi Identity)*

Property (*i.*) corresponds to distributivity over scalar multiplication. Property (*ii.*) corresponds to distributivity over the algebra generators. Properties (*i.*) and (*ii.*) are expressed by some authors as a single property under the name of bilinearity: $[\alpha X, \beta Y + \gamma Z] = \alpha\beta[X, Y] + \alpha\gamma[Y, Z]$, for $\alpha, \beta, \gamma \in \mathbb{K}$ (i.e. the Lie product is linear in both operands separately, see for instance p. 430 in [**2**], or p. 12 and p. 40 in [**151**]). In the case of algebras of vector fields on $\mathbb{R}^n$, if $\alpha$ and $\beta$ are not in $\mathbb{K}$, but rather are $\mathcal{C}^\infty$ functions such that $\alpha, \beta : \mathbb{R}^n \to \mathbb{R}$, then the distributivity over scalar multiplication is normally expressed as the *chain rule* [**6**]:

$$[\alpha f_0, \beta f_1] = \alpha\beta[f_0, f_1] + \alpha(L_{f_0}\beta) f_1 - \beta(L_{f_1}\alpha) f_0$$

where $f_0, f_1$ are some vector fields on $\mathbb{R}^n$, and $L_{f_0}\beta = \frac{\partial \beta}{\partial x} f_0, L_{f_1}\alpha = \frac{\partial \alpha}{\partial x} f_1$ represent the Lie derivatives of $\beta$ and $\alpha$ along the vector fields $f_0$ and $f_1$, respectively. Property (*iii.*) receives different names: skew-commutativity [**2**], anticommutativity [**151**], skew-symmetry [**180**], or antisymmetry (cf. operator theory texts). The Jacobi identity, Property (*iv.*), plays the same role the associative law plays for associative algebras, for which the multiplication operation $x \cdot (y \cdot z) = (x \cdot y) \cdot z$. In general, any vector space equipped with a bilinear vector multiplication is called a *nonassociative algebra* (cf. p. 12 [**151**]), which is the case for Lie algebras, since $[X, [Y, Z]]$ is not equal to $[[X, Y], Z]$, except in the case $[Y, [X, Z]]$ is zero as can be observed by direct application of the Jacobi identity.

The most familiar Lie algebra is the real 3-dimensional vector space with multiplication operation defined as the vector cross product or outer product.

## A.3. Matrix Groups

Matrix groups, denoted by $M_n(K)$, is a class of groups whose elements are $n \times n$ matrices. Examples of matrix groups are:

- General linear($GL(n, K)$): $GL(n, K) \stackrel{\text{def}}{=} \{M \in M_n(K) \mid \det(M) \neq 0\}$, where 0 is the additive identity of $K$.
- Special linear ($SL(n, K)$): $SL(n, K) \stackrel{\text{def}}{=} \{M \in M_n(K) \mid \det(M) = 1\}$. Note that $SL(n, k) \subset GL(n, K)$, i.e. the special linear group is a subgroup of the general linear group.
- Orthogonal group ($O(n, K)$): $O(n, K) \stackrel{\text{def}}{=} \{M \in M_n(K) \mid \bar{A}^T = A^{-1}\}$, where $\bar{A}^T$ is the complex conjugate of $A$. $O(n, K)$ is also a subgroup of $GL(n, K)$. $O(n) \stackrel{\text{def}}{=} O(n, \mathbb{R})$ is called the *orthogonal group*, and $U(n) = O(n, \mathbb{C})$ is called the *unitary group*.
- Special Orthogonal ($SO(n)$): $SO(n) \stackrel{\text{def}}{=} O(n, \mathbb{R}) \cap SL(n, \mathbb{R})$ is the set of all orthogonal matrices of determinant 1.
- Euclidean group ($E(n)$):

$$E(n) \stackrel{\text{def}}{=} \{M \in \mathbb{R}^{(n+1)\times(n+1)} \mid A = \begin{bmatrix} R & p \\ 0_{1\times p} & 1 \end{bmatrix},\ R \in GL(n),\ p \in \mathbb{R}^n\}$$

- Special Euclidean ($SE(n)$):

$$SE(n) \stackrel{\text{def}}{=} \{M \in \mathbb{R}^{(n+1)\times(n+1)} \mid A = \begin{bmatrix} R & p \\ 0_{1\times p} & 1 \end{bmatrix},\ R \in SO(n),\ p \in \mathbb{R}^n\}$$

## A.4. Elementary Notions of Topology

For further details regarding the definitions in this section the reader is refer to [**183, 182, 186, 181, 185**].

**Definition A.12. - Topological Structure or Topology.** *Let $S$ be a set. A* topological structure, or a topology*, on $S$ is a collection of subsets of $S$, called open sets, satisfying the axioms:*

i. *The union of any number of open sets is open.*
ii. *The intersection of any finite number of open sets is open.*
iii. *The set $S$ and the empty set $\emptyset$ are open.*

**Definition A.13. - Topological Space.** *A set $S$ with a topology is a* topological space.

**Definition A.14. - Basis for a Topology.** *A* basis *for a topology is a collection of open sets, called* basic open sets*, with the following properties:*

i. *$S$ is the union of basic open sets.*
ii. *A nonempty intersection of two basic open sets is an union of basic open sets.*

**Definition A.15. - Neighborhood.** *A* neighborhood *of a point $p$ of a topological space is any open set which contains $p$.*

**Definition A.16. - Closed Subset.** *A subset $U$ of a topological space is said to be closed if its complement $\bar{U}$ in $S$ is open. The intersection of any number of closed sets is closed, the union of any finite number of closed sets is closed, and both $S$ and $\emptyset$ are closed.*

**Definition A.17. - Interior Subset.** *If $S_0$ is a subset of a topological space $S$, there is a unique open set, called the* interior *of $S_0$ and denoted $S_0^\circ$, which is contained in $S_0$ and contains any other open set contained in $S_0$. Actually, $S_0^\circ$ is the union of all open sets contained in $S_0$.*

**Definition A.18. - Closure Subset.** *If $S_0$ is a subset of a topological space $S$, there is a unique closed set, denoted by $cl(S_0)$ and called the* closure *of $S_0$, which contains $S_0$ and is contained in any other closed set which contains $S_0$. The intersection of all closed sets which contain $S_0$ is in fact the set $cl(S_0)$.*

**Definition A.19. - Dense Subset.** *A subset of $S$ is said to be* dense *in $S$ if its closure coincides with $S$.*

### A.4.1. Classes of Mappings of Topological Spaces

Different types of mappings may be defined according to the properties it exhibits. To this end, let $S_1$ and $S_2$ be topological spaces and denote by $F$ a mapping $F : S_1 \to S_2$, the mapping $F$ is said to be a:

**Definition A.20. - Continuous Mapping.** *A mapping $F$ such that the inverse image of every open set of $S_2$ is an open set of $S_1$.*

**Definition A.21. - Open Mapping.** *A mapping $F$ such that the image of an open set of $S_1$ is an open set of $S_2$.*

**Definition A.22. - Homeomorphism.** *A mapping $F : S_1 \to S_2$ that is a bijection and both continuous and open, i.e. for a metric space $X$ and $A, B \subset X$ an homeomorphism of $A$ onto $B$ is a continuous one-to-one mapping of $A$ onto $B$, $F : A \to B$, such that $F^{-1} : B \to A$ is continuous. The topological spaces $S_1$ and $S_2$ are said to be* homeomorphic *or* topologically equivalent *if there is a homeomorphism of $S_1$ onto $S_2$. If $F$ is an homeomorphism, the inverse mapping $F^{-1}$ is also an homeomorphism.*

**Definition A.23. - Smooth Mapping.** *Let $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^l$ be open sets. A mapping $F : U \to V$ is called* smooth *if all the partial derivatives $\partial^n F/\partial x_{i_1}, \ldots, \partial^n F/\partial x_{i_n}$ exist and are continuous. If $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^l$ are arbitrary subsets of Euclidean spaces (not necessarily open), then $F : X \to Y$ is called smooth if there exists an open $U \subset \mathbb{R}^k$ containing $X$ and a smooth map $\mathcal{F} : U \to \mathbb{R}^l$ that coincides with $F$ in $U \cap X$. If $f : X \to Y$ and $g : Y \to Z$ are smooth, the the composition $g \circ f : X \to Z$ is also a smooth mapping.*

**Definition A.24. - Diffeomorphism.** *Let $U \in \mathbb{R}^n$ and $V \in \mathbb{R}^n$ be open sets. The map $F : U \to V$ is a* diffeomorphism *if it is a homeomorphism (i.e. is bijective: one-to-one and onto), and if both $F$ and $F^{-1}$ are smooth. Two sets $X$ and $Y$ are said to be* diffeomorphic *if there exists a diffeomorphism of $X$ onto $Y$.*

A closed interval is diffeomorphic to other smooth closed intervals, similarly a circle is diffeomorphic to any closed curve and a sphere is diffeomorphic to any smooth simply connected surface (i.e. which can be contracted to a point, unlike surfaces with *holes*, such as tori). Polygons and circles are not diffeomorphic, since at the corners of the polygon there is no smooth neighborhood. The same observations can be extended to higher dimensional structures, for example, a cone is not diffeomorphic to a plane, neither polyhedra are diffeomorphic to spheres.

**Definition A.25. - Product Topology $S_1 \times S_2$.** *The Cartesian product $S_1 \times S_2$ of two topological spaces $S_1$ and $S_2$ can be given a topology taking as basis the collection of all small subsets of the form $U_1 \times U_2$, with $U_1$ and $U_2$ basic open sets of $S_1$ and $S_2$, respectively.*

**Definition A.26. - Subset Topology.** *A subset $S_1$ of a topological space $S$ can be given a topology taking as open sets the subsets of the form $S_1 \cap U$ with $U$ any open subset in $S$. This is sometimes called a* subset topology*.*

**Definition A.27. - Induced Topology.** *Let $F : S_1 \to S_2$ be a continuous mapping, and let the $F(S_1)$ denote the image of $S_1$. Then, the mapping $F' : S_1 \to F(S_1)$ with $F'(p) = F(p)$ is also continuous, but might fail to be open. However, the set $F(S_1)$ can be given another topology, called* induced topology*, such that the open sets of $F(S_1)$ are the images of the open sets in $S_1$. This new topology contains the subset topology (i.e. any subset which is open in the subset topology is also open in the induced topology), and the mapping $F'$ is now open. If $F$ is an injection then $S_1$ and $F(S_1)$ provided with the induced topology are homeomorphic.*

**Definition A.28. - Hausdorff Space.** *A topological space $S$ for which the neighborhoods of any two points $p_1, p_2 \in S$ are disjoint is said to satisfy the* Hausdorff separation axion *and is called a* Hausdorff space *for short.*

### A.4.2. Smooth Manifolds

**Definition A.29. - Locally Euclidean Space.** *A* locally Euclidean space *$X$ of dimension $n$ is a topological space such that, for each $p \in X$, there exists a homeomorphism $\phi$ mapping some open neighborhood of $p$ onto an open set in in $\mathbb{R}^n$.*

**Definition A.30. - Manifold ($M$).** *A* manifold $M$ *of dimension* $n$ *is a topological space which is locally Euclidean of dimension* $n$*, is Hausdorff and has a countable basis.*

In this sense, a manifold may be any connected metric space with an open covering $\{U_\alpha\}$, i.e. $M = \bigcup_\alpha U_\alpha$, such that for all $\alpha$, $\{U_\alpha\}$ is homeomorphic to the open unit ball $B(0,1) \in \mathbb{R}^n$, that is for all $\alpha$ there exists a homeomorphism of $U_\alpha$ onto $B(0,1) \in \mathbb{R}^n$, $h_\alpha : U_\alpha \to B(0,1)$.

Brouwer's theorem of invariance of domain establishes that it is not possible that an open subset $U$ of $\mathbb{R}^n$ be homeomorphic to an open subset $V$ of $\mathbb{R}^m$ if $n \neq m$. Thus, the dimension of a locally Euclidean space is well defined with respect to the dimension of the manifold.

**Definition A.31. - (Coordinate) Chart of a Manifold $M$.** *A* coordinate chart *on an* $n$*-dimensional manifold* $M$ *is a pair* $(U, h)$*, where* $U$ *is an open set of* $M$ *and* $h$ *is a homeomorphism of* $U$ *onto an open set of* $\mathbb{R}^n$*.*

If $h$ is a diffeomorphism it is sometimes called *system of coordinates* on the neighborhood $U$ of $M$, and its inverse $h^{-1}$ is called *parametrization*. The coordinate chart may also receive the name of *coordinate map*. The mapping $h$ is sometimes represented as a set $(h_1, h_2, \ldots, h_n)$ and $h_i : U\mathbb{R}$ is called the $i$-th *coordinate function*. A coordinate chart $(U, h)$ is called a *cubic* coordinate chart if $h(U)$ is an open cube about the origin in $\mathbb{R}^n$. If $h(p) = 0$ for $p \in U$, the the coordinate chart is said to be centered at $p$. The set $(h_1(p), h_2(p), \ldots, h_n(p))$ for $p \in U$ is called the set of local coordinated of $p$ in the coordinate chart $(U, h)$.

**Definition A.32. - $C^k$-compatible Charts.** *Two coordinate charts* $(U_\alpha, h_\alpha)$ *and* $(U_\beta, h_\beta)$ *on an* $n$*-dimensional manifold* $M$ *are* $C^k$*-compatible, if whenever* $U_\alpha \bigcap U_\beta \neq \emptyset$*, if the transformation*

$$h = h_\alpha \circ h_\beta^{-1} : h_\beta(U_\alpha \bigcap U_\beta) \to h_\alpha(U_\alpha \bigcap U_\beta)$$

*is a diffeomorphism, i.e. is differentiable (or of class* $C^k$*) and for all* $x \in h_\beta(U_\alpha \bigcap U_\beta)$ *the Jacobian determinant* $\det Dh(x) \neq 0$*.*

Notice that $h_\beta(U_\alpha \bigcap U_\beta)$ and $h_\beta(U_\alpha \bigcap U_\beta)$ are subsets of $\mathbb{R}^n$ since $U_\alpha \bigcap U_\beta \neq \emptyset$. The mapping $h$ takes for every $p \in U_\alpha \bigcap U_\beta$, the set of local coordinates $(h_{\beta_1}(p), \ldots, h_{\beta_n}(p))$ into the set of local coordinates $(h_{\alpha 1}(p), \ldots, h_{\alpha n}(p))$ and is called a *coordinates transformation* on $U_\alpha \bigcap U_\beta$. The inverse mapping $h^{-1} = h_\beta \circ h_\alpha^{-1}$ allows to express the coordinates $(h_{\beta_1}(p), \ldots, h_{\beta_n}(p))$ in terms of $(h_{\alpha 1}(p), \ldots, h_{\alpha n}(p))$. If the coordinates $h_\beta$ and $h_\alpha$ are represented by vectors $x$ and $y$, respectively. Then, the coordinates transformation $h = h_\alpha \circ h_\beta^{-1}$ can be represented in the form $y = [y_1(x) \ \ldots \ y_n(x)]^T = y(x)$ and similarly, the inverse coordinate transformation $h^{-1} = h_\beta \circ h_\alpha^{-1}$ in the form $x = x(y)$. Two $C^k$-compatible charts are sometimes called *smoothly overlapping* charts or are said to have a *smooth overlap*.

**Definition A.33. - Atlas of a Manifold $M$.** *The collection of all $C^k$-compatible charts for a manifold $M$ is called an* atlas *on the manifold $M$. In other words, an atlas is the set $\mathfrak{A} = \{(U_\alpha, h_\alpha), \ \forall \ \alpha\}$ of pairwise $C^k$-compatible coordinate charts that completely cover $M$, i.e. $M = \bigcup_\alpha U_\alpha$.*

An atlas is *complete* if not properly contained in any other atlas. A complete atlas is also called a *maximal atlas.*

**Definition A.34. - Differentiable Manifold, [2].** *A* differentiable manifold *is a manifold with an associated complete atlas, (i.e. if for each $x \in M \subset \mathbb{R}^n$ there is a neighborhood $W \bigcap M$, $W \subset \mathbb{R}^k$, which is diffeomorphic to an open subset $U \subset \mathbb{R}^n$ ).*

Note that this definition essentially means that $M$ a differentiable manifold of dimension $n$ is (i) homeomorphic to $\mathbb{R}^n$, since it is a manifold, and (ii) $M$ is equipped with a differentiable coordinate transformation which is invertible.

**Definition A.35. - Analytic Manifold.** *A differentiable manifold $M$ is said to be* analytic *if the maps $h = h_\alpha \circ h_\beta^{-1}$, associated with the pairwise $C^k$-compatible charts forming its atlas, are analytic.*

**Definition A.36. - Orientable Manifold.** *A differentiable manifold $M$ is said to be* orientable *if there is an atlas with* $\det D(h_\alpha \circ h_\beta^{-1})(x) > 0$, $\forall \ \alpha, \beta$ *and* $x \in h_\beta(U_\alpha \bigcap U_\beta)$.

An equivalent definition of differentiable manifold found in [**146**] is presented next since it encapsulates all the previous concepts required by the above definition of differentiable manifold, including the definitions of analytic and orientable manifolds. Thus the following form of the definition of a differentiable manifold serves the purpose of presenting a summary that shows how all the *bits and pieces* fit together.

**Definition A.37. - Differentiable Manifold,** [**146**]. *An* $n$-dimensional differentiable manifold $M$ *(or a manifold of class* $\mathcal{C}^k$*, is a connected metric space with an open covering* $\{U_\alpha\}$*, i.e.,* $M = \cup_\alpha U_\alpha$*, such that*

i. *for all* $\alpha$*,* $U_\alpha$ *is homeomorphic to the open unit ball in* $\mathbb{R}^n$*,* $B = \{x \in \mathbb{R}^n \mid |x| < 1\}$*, i.e., for all* $\alpha$ *there exists a homeomorphism of* $U_\alpha$ *onto* $B$*,* $h_\alpha : U_\alpha \to B$*, and*

ii. *if* $U_\alpha \cap U_\beta \neq \emptyset$ *and* $h_\alpha : U_\alpha \to B$*,* $h_\beta : U_\beta \to B$ *are homeomorphisms then* $h_\alpha(U_\alpha \cap U_\beta)$ *and* $h_\beta(U_\alpha \cap U_\beta)$ *are subsets of* $\mathbb{R}^n$ *and the map*

$$h = h_\alpha \circ h_\beta^{-1} : h_\beta(U_\alpha \cap U_\beta) \to h_\alpha(U_\alpha \cap U_\beta)$$

*is differentiable (or of class* $\mathcal{C}^k$ *and for all* $x \in h_\beta(U_\alpha \cap U_\beta)$*, the Jacobian determinant* $\det Dh(x) \neq 0$*. The manifold* $M$ *is said to be* analytic *if the maps* $h = h_\alpha \circ h_\beta^{-1}$ *are analytic.*

*The pair* $(U_\alpha, h_\alpha)$ *is called a* chart *for the manifold* $M$ *and the set of all charts is called an* atlas *for* $M$*. The differentiable manifold* $M$ *is called* orientable *if there is an atlas with* $\det Dh_\alpha \circ h_\beta^{-1}(x) > 0$ *for all* $\alpha$*,* $\beta$ *and* $x \in h_\beta(U_\alpha \cap U_\beta)$*.*

Examples of manifolds are:

- The unit sphere $S^2 \subset \mathbb{R}^3$ defined by $\{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ is a smooth manifold of dimension 2, since $S^2$ can be covered by the diffeomorphism $(x_1, x_2) \to (x_1, x_2, \sqrt{1 - x_1^2 - x_2^2})$, which parametrizes the the region $S^2 \bigcap \{x_3 > 0\}$ and the reminder may be covered by interchanging the roles of $x_1, x_2, x_3$ and the sign of the radical.

- The space of orthogonal matrices in $\mathbb{R}^{2\times 2}$ with determinant 1, $SO(2)$, is a manifold of dimension 1, since every $2\times 2$ matrix in $SO(2)$ can be written as:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

## A.5. Derivatives and Differentials

Some basic terminology related to differential operators is introduced in section. For further details on differentiability notions of multivariable mappings see [**187**].

**Definition A.38. - Derivative of a mapping** $f$**.** *Let $f : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^m$, be a mapping between the open sets $U$ and $V$. Then, the linear map $Df : \mathbb{R}^n \to \mathbb{R}^m$ satisfying:*

$$Df_x(h) = \lim_{t\to 0} \frac{f(x+th) - f(x)}{t}$$

*for $x \in U$ and $h \in \mathbb{R}^n$. In fact, $Df_x$ is the $m \times n$ matrix of partial derivatives $\frac{\partial f_i}{\partial x_j}$ evaluated at $x$ and known as Jacobian of $f$ (cf. definition below).*

**Definition A.39. - Gradient Operator.** *The* gradient operator, *$\nabla_x$, is defined as the row vector of partial derivatives $\nabla_x \overset{\text{def}}{=} \begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \dots & \frac{\partial}{\partial x_n} \end{bmatrix}$ with respect to space coordinates $x = [x_1, \dots, x_n]^T$.*

**Definition A.40. - Jacobian.** *The* Jacobian *$J_x$ of a differentiable map $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathcal{V} \subseteq \mathbb{R}^m$ is the matrix:*

$$J_x f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Note that the Jacobian can also be viewed as the matrix of componentwise gradients of $f$, that is:

$$J_x f = \begin{bmatrix} \nabla_x f_1 \\ \vdots \\ \nabla_x f_m \end{bmatrix}$$

## A.6. Tangent Vectors, Tangent Spaces

These definitions are mainly based on those found in [**2**]. Let $M$ be a smooth manifold of dimension $n$ and let $p$ be a point in $M$. The set of all smooth real-valued functions defined on a neighborhood of $p$ is denoted by $\mathcal{C}^\infty(p)$.

**Definition A.41. - Tangent Vector or Derivation.** *A* tangent vector *or* derivation $X_p$ *at* $p$ *is a map* $X_p : \mathcal{C}^\infty(p) \to \mathbb{R}$ *that satisfies the following properties for all* $\alpha, \beta \in \mathbb{R}$ *and* $f, g \in \mathcal{C}^\infty(p)$*:*

i. *Linearity:* $X_p(\alpha f + \beta g) = \alpha\, X_p(f) + \beta\, X_p(g)$.
ii. *Leibniz Rule:* $X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$.

**Definition A.42. - Tangent Space.** *The* tangent space *to* $M$ *at a point* $p$*, denoted* $T_pM$*, is the set of all tangent vectors or derivations* $X_p : \mathcal{C}^\infty \to \mathbb{R}$ *at* $p$*.*

Remark A.1. *The set of all derivations* $X_p : \mathcal{C}^\infty \to \mathbb{R}$*, or tangent vectors* $T_pM$*, forms a vector space over the field* $\mathbb{R}$*, with*

$$(\alpha X_p + \beta Y_p)(f) = \alpha X_p(f) + \beta Y_p(f)$$

*for* $\alpha, \beta \in \mathbb{R}$*,* $X_p : \mathcal{C}^\infty \to \mathbb{R}$*,* $Y_p : \mathcal{C}^\infty \to \mathbb{R}$ *and* $f \in \mathcal{C}^\infty$*.*

Let $(U, \phi)$ be a (fixed) coordinate chart on $M$ around $p$, with local coordinates $(x_1, \ldots, x_n)$. Then, the set of derivations $\left\{\frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n}\right\}$ forms a basis for $T_pM$ and hence we can write

$$X_p = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}$$

The vector $(X_1, \ldots, X_n) \in \mathbb{R}^n$ is a local coordinate representation of $X_p \in T_pM$. The above representation is useful to emphasize that tangent vectors are operators.

**Definition A.43. - Tangent Bundle.** *The* tangent bundle *of* $M$*, denoted* $TM$*, is the collection of all tangent spaces* $T_pM$ *at all points* $p \in M$ *together with the points of tangency; i.e. since* $T_pM$ *is the set of all tangent vectors to* $M$ *at* $p$*, the tangent bundle is the collection of all tangent vectors,* *along with the information of the point to which they are tangent:*

$$TM \stackrel{\text{def}}{=} \{(p, v) \mid p \in M,\ v \in T_pM\}$$

## A.7. Vector Fields and Flows

**Definition A.44. - Vector Field.** *A* vector field *on a manifold $M$ of dimension $n$ is a smooth map, $f : M \to TM$ which assigns to each point $p \in M$* a tangent vector *$f(p) \in T_pM$. Consider the coordinate chart $(U, \phi) = (U, [x_1, \ldots, x_n])$, then a local representation of the vector field, $f(p)$, consistent with the coordinate chart may be given as*

$$\hat{f}(x) = \sum_{i=1}^{n} \hat{f}_i(x) \frac{\partial}{\partial x_i} = \begin{bmatrix} \hat{f}_1(q) \\ \vdots \\ \hat{f}_n(q) \end{bmatrix}$$

*where $\hat{f}_i(x) = (f_i \circ \phi^{-1})(p)$ are the local representation of the unique set of functions $f_i$ given the coordinate homeomorphism is $x = \phi(p)$.*

*For notational convenience, $\hat{f}_i$ will simply be denoted as $f_i$, unless the explicit distinction is necessary.*

*A vector field is smooth if each $f_i(p)$ is smooth.*

**Definition A.45. - Integral Curve.** *A smooth curve, $\sigma : (t_1, t_2) \to M$, on a manifold $M$ is called an* integral curve *of the vector field $f$ if:*

$$\dot{\sigma}(t) = f(\sigma(t))$$

*Thus, vector fields characterize differential equations on manifolds. The curves obviously being solutions to the differential equation on the manifold.*

**Definition A.46. - Flow $\Phi$ of a Vector Field $f$.** *Let $f$ be a smooth vector field on a manifold $M$. Then for each $p \in M$ there exists an interval $I_p = (t_1(p), t_2(p)) \subset \mathbb{R}$, such that $0 \in I_p$ and a smooth mapping*

$$\Phi : W \to M$$

*defined on* $W \subset \mathbb{R} \times M$*, where*

$$W = \{(t, p) \in \mathbb{R} \times M \mid t \in I_p\}$$

*with the following properties:*

i. $\Phi(0, p) = p$

ii. *For each* $p$ *the mapping* $\sigma_p : I_p \to M$ *defined by*

$$\sigma_p(t) = \Phi(t, p)$$

*is an integral curve of* $f$*.*

iii. *If* $\mu : (t_1, t_2) \to M$ *is another integral curve of* $f$ *satisfying* $\mu(0) = p$*, then* $(t_1, t_2) \subset I_p$ *and the restriction of* $\sigma_p$ *to* $(t_1, t_2)$ *coincides with* $\mu$*.*

iv. $\Phi(s, \Phi(t, p)) = \Phi(s + t, p)$ *whenever both sides are defined.*

v. *whenever* $\Phi(t, p)$ *is defined, there exists an open neighborhood* $U$ *of* $p$ *such that the mapping* $\Phi_t : U \to M$ *defined by*

$$\Phi_t(q) = \Phi(t, q)$$

*is a diffeomorphism onto its image, and*

$$\Phi_t^{-1} = \Phi_{-t}$$

*The family of diffeomorphisms* $\Phi_t(p) : M \to M$ *corresponds to a parametrization in* $t$ *of the integral curves* $\sigma$ *generated by* $f$*. The mapping* $\Phi$ *is called the* flow *of* $f$*. The flow* $\Phi_t$ *is sometimes written as* $\Phi_t^f$ *to stress the dependence on* $f$*. The dependency on the initial condition* $p$ *is emphasized with the notation* $\Phi_t^f(p)$*.*

Remark A.2. *Properties [i.] and [ii.] say that* $\sigma_p$ *is an integral curve of* $f$ *passing through* $p$ *at time* $t = 0$*. Thus,*

$$\frac{d\Phi_t^f(p)}{dt} = f(\Phi_t^f(p)), \qquad p \in M$$

*Property [iii.] says that this curve is unique and that the domain $I_p$ on which $\sigma_p$ is defined is maximal. Property [iv.] and [v.] say that the family of mappings $\Phi_t$ is a one-parameter group of local diffeomorphisms, under the operation of composition.*

**Definition A.47. - Complete Vector Field.** *A vector field $f$ is said to be* complete *if, for all $p \in M$, the interval $I_p$ coincides with $\mathbb{R}$, i.e. the domain of definition of the integral curves is $(-\infty, \infty)$. In other words, if the flow $\Phi$ of $f$ is defined on the whole Cartesian product $\mathbb{R} \times M$. Since integral curves of a complete vector field are defined for any initial point $p$, for all $t \in \mathbb{R}$, it is said that the integral curve does* not *have* finite escape times.

## A.8. Lie Theory Notions

**Definition A.48. - Lie Derivative.** *The rate of change with respect to time of a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ along the flow of a vector field $f$ is given by*

$$\dot{V} = \sum_{i=1}^{n} \frac{\partial V}{\partial q_i} \frac{\partial q_i}{\partial t} = \nabla_q V \dot{q} \tag{A.3}$$

$$= \sum_{i=1}^{n} \frac{\partial V}{\partial q_i} f_i = \nabla_q V f \tag{A.4}$$

$$\tag{A.5}$$

*The above time derivative of $V$ along the flow of $f$ is referred to as* directional *or* Lie derivative *of $V$ along $f$ and is denoted $L_f V$, $L_f V \stackrel{\text{def}}{=} \nabla_q V\, f(q)$.*

One may view the *Lie derivative* as the result of an operator $L_f : \mathbb{R} \to \mathbb{R}$ acting on smooth functions $V(q)$.

Two vector fields $f, g \in \mathbb{V}(\mathcal{O})$, where $\mathbb{V}(\mathcal{O})$ is the set of smooth vector fields defined on a open subset $\mathcal{O} \subset \mathbb{R}^n$, are equal if and only if $L_f = L_g$. Because if $L_f V = L_g V$ holds for every $V$, then it holds in particular for each of the $n$ coordinate functions $\phi_i(x) = x_i$, and hence the coordinates of $f$ and $g$ coincide: $f_i(x) = L_f \phi_i = L_g \phi_i = g_i(x)$. While $L_f$ is a first order operator, it can be easily verified that the composition of Lie derivatives, denoted by $L_f L_g$, is a second order operator:

$$L_f L_g V = [(\nabla) L_g V] f \tag{A.6}$$

$$= [(\nabla) \nabla V g] f \tag{A.7}$$

$$= [g^T H_V + \nabla V \nabla g] f \tag{A.8}$$

where $H_V$ is the *Hessian matrix* of $V$ of second order derivatives with entries $\frac{\partial^2 V}{\partial x_i \partial x_j}$. Note that $\nabla g$ is a Jacobian matrix, while $\nabla V$ is a gradient vector.

By symmetry of the Hessian,

$$L_f L_g V - L_g L_f V = L_{\nabla g f - \nabla f g} V$$

**Definition A.49. - Lie Product or Bracket.** *The* Lie product *or* Lie bracket *of $f, g \in \mathbb{V}(\mathcal{O})$ is defined as:*

$$[f, g] \stackrel{\text{def}}{=} \nabla g f - \nabla f g \in \mathbb{V}(\mathcal{O})$$

From the section on Lie derivatives it follows that:

$$L_{[f,g]} = L_f L_g - L_g L_f \qquad \forall\ f, g$$

### A.8.1. The *ad* Operator

It is convenient to write:

$$ad_X Y \stackrel{\text{def}}{=} [X, Y] \tag{A.9}$$

If we regard the the Lie algebra $L(G)$ as a vector space (as is the case for Lie algebras of smooth vector fields, i.e. $X, Y \in \mathbb{V}(\mathcal{O})$). Then $ad_X$ can be viewed as linear operator $ad_X : L(G) \to L(G)$ for each fixed $X \in L(G)$. As a matter of fact, $ad_X$ is the linear space of maps, called *endomorphisms* denoted by End$L(G)$ from $L(G)$ into itself.

In [**162**, §2.2] it is shown that $ad_X$ corresponds to a derivation of the Lie algebra $L$ for all $X \in L$, and that the map $ad : X \to ad_X$ is a representation of $L$, the so-called *adjoint representation of* $L$, written $ad_L = \{ad_X \mid X \in L\}$.

The $ad_X$ operator can be composed, i.e. $(ad_X)(ad_X)Y = (ad_X)[X,Y] = [X,[X,Y]]$. The composition of the $ad_X$ operator $k$ times is simply denoted by

$$(ad_X)^k g = [\underbrace{X,[X,[\ldots,[X}_{k\text{times}},Y] \tag{A.10}$$

This operator is a *differentiation* operator with respect to the Lie bracket [**7**]:

$$ad_X[Y,Z] = [ad_X Y, Z] + [Y, ad_X Z]$$

for all $X, Y, Z$. The above formula is known as the Jacobi identity, especially when written as presented in the properties of a Lie algebra (see Definition A.11 in p. 202).

**Definition A.50. - Lie Algebra $L(\mathcal{F})$ of vector fields.** *A* Lie algebra *(of vector fields on $\mathcal{O} \subseteq \mathbb{R}^n$) is a linear subspace $S \subseteq \mathbb{V}(\mathcal{O})$ that is closed under the Lie bracket operation, that is, $[f,g] \in S$ whenever $f, g \in S$.*

For any subset $\mathcal{F} \subseteq \mathbb{V}(\mathcal{O})$, one defines $L(\mathcal{F})$, the Lie algebra generated by $\mathcal{F}$, as the intersection of all the Lie algebras of vector fields which contain $\mathcal{F}$. This set is nonempty, since it includes $\mathbb{V}(\mathcal{O})$. An intersection of a family of Lie algebras is also a Lie algebra, thus $L(\mathcal{F})$ is the smallest Lie algebra of vector fields which contains $\mathcal{F}$.

**Definition A.51. - Connected Lie Group $G(L)$ associated with $L(\mathcal{A})$.** *The connected Lie group $G(L)$ associated with the Lie algebra $L(\mathcal{A})$ is defined as the set*

$$G(L) \stackrel{\text{def}}{=} \left\{\Phi \in \mathbb{R}^{n\times n} = e^{\Lambda_1} e^{\Lambda_2} \cdots e^{\Lambda_r},\, \Lambda_i \in L,\, i = 1,2,\ldots,m;\, m = 1,2,\ldots\right\}$$

Examples of connected Lie groups are:

$G_n$: The group of nonsingular matrices in $\mathbb{R}^{n\times n}$.

$G_n^+$: The subgroup of nonsingular matrices in $G_n$ with positive determinant.

$G_n(L)$: The connected subgroup of $G_n^+$ associated with $L(\mathcal{A}) = L(\{A_i \in \mathbb{R}^{n\times n}\})$.

### A.8.2. The Exponential Map

The *exponential map* for a general Lie group $G$ is the function:

$$e : X \in T_eG \to e^X \in G$$

mapping elements in of the Lie algebra $L(G)$ into $G$. Where $T_eG$ denotes the tangent space of $G$ about the identity $e$ element associated with $G$. The connection between a Lie algebra and its Lie group established via the exponential map will be a useful tool, for example, by allowing to reduce the study of Lie algebras to their corresponding Lie groups.

By the inverse function theorem, (see Theorem C.2 on p. 246), the exponential mapping is a local diffeomorphism, $e : L(G) \to G$, from a neighborhood of zero in $L(G)$ onto a neighborhood of the identity $e \in G$. Then, denoting by $\{X_1, X_2, \ldots, X_n\}$ a basis for the Lie algebra $L(G)$, the mapping $\sigma : \mathbb{R}^n \to G$ defined by

$$g \stackrel{\text{def}}{=} e^{\sum_{i=1}^n \sigma_i X_i}$$

is a local diffeomorphism between $\sigma \in \mathbb{R}^n$ and $g \in G$, for $g$ in a neighborhood of the identity $e$ of $G$. The $\sigma_i$ are called the *Lie-Cartan coordinates of the first kind* relative to the basis $\{X_1, X_2, \ldots, X_n\}$. Another way of writing the coordinates on a Lie group using the same basis is to define $\theta : \mathbb{R}^n \to G$ by:

$$g \stackrel{\text{def}}{=} \prod_{i=1}^n e^{\theta_i X_i}$$

for $g$ in a neighborhood of $e$. The functions $\theta_i$ are known as *Lie-Cartan coordinates of the second kind* or *logarithmic coordinates.* If the basis $\{X_i, i = 1, \ldots, n\}$ is a Philip Hall basis, then the $\theta_i$ are referred to as *Philip Hall coordinates.*

### A.8.3. The Baker-Campbell-Hausdorff Formula

The notions of conjugation and adjoint maps will be described before introducing the Baker-Campbell-Hausdorff or Campbell-Baker-Hausdorff (CBH) formula.

If $M$ is a differentiable manifold and $G$ is a Lie group, a *left action* of $G$ on $M$ is defined as smooth map $\Phi : G \times M \to M$ such that [**161**], [**160**, p. LG 4.11], [**162**, p. 74] or [**152**, p. 217]:

(i) $\Phi(e, x) = x$ for all $x \in M$.

(ii) For every $g, h \in G$ and $x \in M$, $\Phi(g, \Phi(h, x)) = \Phi(gh, x)$.

where $e$ is the identity element in $G$.

The left action of $G$ on itself defined by $C_g : G \to G$:

$$C_g(h) : ghg^{-1} = R_{g^{-1}} L_g h$$

is called the *conjugation map* associated with $g$.

**Definition A.52. - The** $Ad$ **operator.** *The derivative of the conjugation map at e is called the* adjoint map*, defined as* $Ad_g : L(G) \to L(G)$ *such that, for* $\xi \in L(G)$*,* $g \in G$*,*

$$Ad_g(\xi) = (T_e(C_g))(\xi) = T_e(R_{g^{-1}} L_g)(\xi).$$

*If* $G \subset GL(n, \mathbb{C})$*, then*

$$Ad_g(\xi) = g\xi g^{-1}. \tag{A.11}$$

Once again, viewing Lie algebra $L(G)$ as a vector space, the map $Ad_g$ for $g \in G$ is an element of the Lie group of bijective linear maps from $L(G)$ into itself, called *automorphisms* and denoted by $\mathrm{Aut}L(G)$. The map $Ad : g \to Ad_g$ is called the *adjoint representation of* $G$.

The differential, or the Lie algebra, of the Lie group $\mathrm{Aut}L(G)$, is the space $\mathrm{End}L(G)$, the space of linear maps (endomorphisms), from $L(G)$ into itself.

The facts showing the relation between the $Ad$ operator and the $ad$ operator in (A.9) may be found in [**7**] (lemmas 4.4.2 and 4.4.3). As stated in [**6**], the $ad$ operator may be regarded as the differential of $Ad$ in the following sense. Denote by $\gamma(t) \in G$ a curve in $G$ with $\gamma(0) = I$ and $\left.\frac{d\gamma}{dt}\right|_{t=0} = \xi$, then:

$$\frac{d}{dt} Ad_{\gamma(t)}(\eta) = ad_\xi(\eta)$$

More formally, this basic result concerning the adjoint representation of $G$ is the following.

**Lemma A.1. - Exponential Formula.** *Let $G$ be a Lie group and $L$ its associated Lie algebra. Then the differential of the adjoint representation of $G$ is the adjoint representation of $L$. Moreover, considering $X, Y \in L$, then $Ad_{e^X} Y \in L$ and is given by*

$$\begin{aligned} Ad_{e^X} Y &= e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + \ldots \\ &= Y + ad_X Y + \frac{1}{2!} ad_X^2 Y + \frac{1}{3!} ad_X^3 Y + \ldots \\ &= e^{ad_X} Y \end{aligned} \tag{A.12}$$

Proof. See Magnus [**145**], Varadarajan [**162**, Thm. 2.13.2, p. 104]. A sketch of the proof is found in [**6**, Lemma 8.29, p. 372]. □

The above result may be regarded as a measure of how much $X$ and $Y$ fail to commute over the exponential, since if $[X, Y] = 0$, then $Ad_{e^X} Y = Y$.

**Theorem A.1. - Campbell-Baker-Hausdorff (CBH) Formula for the Composition of Exponential Mappings.** *Let $G$ be a Lie group and $L$ its Lie algebra, then for $X, Y \in L$, and for all sufficiently small $t$:*

$$\begin{aligned} e^{tX} e^{tY} &= e^{F(t,X,Y)} \\ &= e^{tX + tY + \frac{t^2}{2}[X,Y] + \frac{t^3}{12}([[X,Y],Y] - [[X,Y],X]) - \frac{t^4}{48}([Y,[X,[X,Y]]] + [X,[Y,[X,Y]]]) + \ldots} \end{aligned} \tag{A.13}$$

Proof. See Varadarajan [**162**]. □

Remark A.3.

- *The above CBH formula is given for exponential maps acting on the right (i.e. $xe^{tX}e^{tY} = xe^{F(t,X,Y)}$ with the above definition of $F(t, X, Y)$ for some $x \in M$, where $M$ is the manifold on which $L$ is defined).*
- *For Lie algebras of vector fields, the exponential maps correspond to the flows generated by the vector fields.*

The CBH formula provides another measure of the commutativity of $X$ and $Y$ over their exponential. If $X$ and $Y$ commute, then $[X, Y] = 0$ and $e^X e^Y$ is simply $e^{X+Y}$. However, in general $X$ and $Y$ fail to commute and $e^X e^Y$ must be calculated through the CBH formula.

As pointed out by Varadarajan [**162**], the calculation of the terms in the CBH expression of equation (A.13) becomes complicated very rapidly, unfortunately.

**Dynkin form of the CBH.**

Consider a neighborhood $U$ of the identity element $I$ of a Lie group $G$, such that every $g \in G$ can be represented as $\exp(X)$ for some $X \in L(G)$. Then for any two elements $\exp(X)$ and $\exp(Y)$ in $U$ there exists an element $Z \in L(G)$ such that $\exp(Z) = \exp(X)\exp(Y)$. Formally denoting $Z = \log(\exp(X)\exp(Y))$. The element $Z$ can be expressed in a more explicit form of the CBH, known as the Dynkin form [**154, 147, 160**].

$$Z = \sum_{m=1}^{\infty} \sum \frac{(-1)^{m-1} ad_Y^{q_m} ad_X^{p_m} \cdots ad_Y^{q_1} ad_X^{p_1}}{m \left(\sum_{i=1}^{m} (p_i + q_i)\right) \prod_{i=1}^{m} (p_i! q_i!))} \tag{A.14}$$

where the inner sum is over all m-tuples of pairs of nonnegative integers $(p_i, q_i)$ such that $p_i + q_i > 0$, and in order to simplify the notation, the terms $ad_X = X$ and $ad_Y = Y$.

**Power form of the CBH.**

The CBH can also be expressed in terms of powers in $X$ or $Y$ as:

$$\begin{aligned} Z &= Y + \frac{ad_Y}{\exp(ad_Y) - 1} X + \ldots, \\ Z &= X + \frac{ad_X}{\exp(ad_X) - 1} Y + \ldots, \end{aligned}$$

**Structure Constants of a Lie Algebra and Multiplication Table.**

If $B \stackrel{\text{def}}{=} \{X_1, X_2, \ldots, X_n\}$ is a basis for the Lie algebra $L(G)$, the *structure of constants* with respect $B$ are the values $c_{ij}^k \in \mathbb{R}$ defined by:

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k$$

for $i, j = 1, 2, \ldots, n$. The above expression constitutes the $(i, j) - th$ entry of the *multiplication table* for the products of elements of $B$.

**Lemma A.2. - Wei-Norman Lemma.** *Consider $L(G)$ with basis $B \stackrel{\text{def}}{=} \{X_1, X_2, \ldots, X_n\}$ and structure constants $c_{ij}^k$ with respect to $B$. Then*

$$\left(\prod_{j=1}^{r} e^{g_j X_j}\right) X_i \left(\prod_{j=r}^{1} e^{-g_j X_j}\right) = \sum_{k=1}^{n} \xi_{ki} X_k \tag{A.15}$$

$$r = 1, 2, \ldots, n \tag{A.16}$$

*where each $\xi_{ki} = \xi_{ki}(g_1, g_2, \ldots, g_r) \in \mathbb{R}$ is an analytic function of $g_1, g_2, \ldots, g_r \in \mathbb{R}$.*

Proof. Repeated application of lemma A.1 shows that the left-hand side of (A.16) is in $L(G)$, and hence can be written as a linear combination of $X_1$ to $X_n$ as asserted. It remains to show that $\xi_{ki}$ are analytic. It is sufficient to prove the lemma for $r = 1$, since an analytic function of an analytic function is again analytic. From (A.12) we have for $r = 1$ that:

$$\begin{aligned} e^{g_1 X_1} X_i e^{-g_1 X_1} &= e^{g_1 ad_{X_1}} X_i \\ &= X_i + \sum_{k=1}^{\infty} \frac{g_1^k}{k!} (ad_{X_1})^k X_i \end{aligned} \tag{A.17}$$

The terms $(ad_{X_1})^k X_i$ are calculated using the structure constants:

$$\begin{aligned} ad_{X_1} X_i &= \sum_{n_1=1}^{n} c_{1i}^{n_1} X_{n_1} \\ (ad_{X_1})^2 X_i &= \sum_{n_1=1}^{n} \sum_{n_2=1}^{n} c_{1i}^{n_1} c_{1n_1}^{n_2} X_{n_2} \\ &\vdots \end{aligned} \tag{A.18}$$

$$(ad_{X_1})^k X_i = \sum_{n_1=1}^{n} \sum_{n_2=1}^{n} \cdots \sum_{n_k=1}^{n} c_{1i}^{n_1} c_{1n_1}^{n_2} \cdots c_{1n_{k-1}}^{n_k} X_{n_k} \tag{A.19}$$

Substituting $(ad_{X_1})^k X_i$ into (A.17) and noting that the $c_{ij}^k$ are finite and letting $M$ be the maximum of $|c_{ij}^k|$, $i, j, k = 1, 2, \ldots, n$. Estimating each $c_{1n_{s-1}}^{n_s}$ in (A.19) by $M$ we obtain

$$\left| \sum \cdots \sum c_{1i}^{n_1} \cdots c_{1n_{k-1}}^{n_k} \right| \leq (nM)^k$$

Since $(nM)^k/k!$ is the general term of a convergent series the $\xi_{ki}$ are bounded and are functions of $c_{ij}^k$, $k!$ and $g_1^k$, consequently the lemma is proven. □

### A.8.4. Wei-Norman Equation and the Logarithmic-Coordinates

Following [**44, 39**], we introduce some relevant notation, before presenting the main results by Wei and Norman [**149**]. Let $A(X_1,\ldots,X_m)$ denote the algebra of noncommutative polynomials in $(X_1,\ldots,X_m)$, which are purely formal *noncommuting indeterminates*. Defining the Lie product as $[X,Y] = YX - XY$, then $A(X_1,\ldots,X_m)$ is also a Lie algebra. We denote by $L(X_1,\ldots,X_m)$ the Lie subalgebra of $A(X_1,\ldots,X_m)$ generated by $(X_1,\ldots,X_m)$. The elements of $L(X_1,\ldots,X_m)$ are known as *Lie polynomials* in $(X_1,\ldots,X_m)$. We denote by $\hat{A}(X_1,\ldots,X_m)$ and $\hat{L}(X_1,\ldots,X_m)$, respectively, the set of *noncommutative formal power series* and the set of *Lie series* in the $X_i$. The *free nilpotent associative algebra of order* $k$ and the *free nilpotent Lie algebra* denoted by $A_k(X_1,\ldots,X_m)$ and $L_k(X_1,\ldots,X_m)$, respectively, are subsets of the corresponding algebras in which all the monomials in $k+1$ or more indeterminates are zero.

If $Z \in \hat{A}(X_1,\ldots,X_m)$ or $Z \in A_k(X_1,\ldots,X_m)$ then the exponential $e^Z$ is well defined by means of the usual power series. We also define

$$G_k(X_1,\ldots,X_m) \stackrel{\text{def}}{=} \left\{e^Z : Z \in L_k(X_1,\ldots,X_m)\right\} \tag{A.20}$$

Note that $exp(\cdot) = e^{(\cdot)}$ is the exponential mapping taking elements in the Lie algebra $L$ into elements of the Lie group $G$. With this definition, $G_k^m \equiv G_k(X_1,\ldots,X_m)$ is the *analytic simply connected* Lie group with Lie algebra $L_k(X_1,\ldots,X_m)$. The Lie group $G_k^m$ is called the *free nilpotent Lie group of order $k$ with $m$ infinitesimal generators.*

Wei and Norman provided in [**149**] a very useful result concerning the solution of linear differential equations of the form:

$$\begin{aligned} \dot{S}(t) &= \left(\sum_{i=1}^{m} X_i u_i(t)\right) S(t) \\ S(0) &= I \in G_k(X_1,\ldots,X_m) \end{aligned} \tag{A.21}$$

where $m < \infty$ (finite), $X_i$ are indeterminate operators independent of $t$ that generate a finite dimensional Lie algebra $L$ under the commutator product $[X_i, X_j] = X_j X_i - X_i X_j$. The $u_i(t)$ are scalar functions of $t$, $G(X_1, \ldots, X_m)$ is the Lie group associated with $L$, and $S$ is a linear operator in, $\hat{A}(X_1, \ldots, X_m)$, the formal power series of noncommuting indeterminates $X_i$, i.e. $S$ may be expressed in terms of elements in the set of all sums $\sum_I a_I X_I$, where $I$ is the multi-index $I = (i_1, \ldots, i_j)$, with $i_j \in \{0, \ldots, m\}$ for $j = 1, \ldots, k$, and where $a_I$ are real numbers and $X_I = X_{i_1} \cdots X_{i_k}$, see [**39**], p. 164 for further details.

Notice that here $S(t)$ is an *action* on the right. Even if this is not the usual representation employed in the study of control systems, it will be kept for consistency with the majority of the mathematical results found in the literature, and which will be employed next. Some further comments on the considerations that must be made when applying the next results to left-invariant representations (the usual representation for control systems) will be made at the end of this subsection.

It is well known that solutions to (A.21) exist and are unique for all times [**44**], and that the *trajectories* $t \to S(t)$ remain in $G_k(X_1, \ldots, X_m)$ for all times, since $I \in G_k(X_1, \ldots, X_m)$.

The main result states that the solutions to (A.21) may be represented as a product of exponentials in the form:

$$S(t) = e^{\gamma_1(t)X_1} e^{\gamma_2(t)X_2} \cdots e^{\gamma_r(t)X_r} = \prod_{i=1}^{r} e^{\gamma_i(t)X_i} \tag{A.22}$$

where $X_1, X_2, \ldots, X_r$ are the elements forming a basis for the Lie algebra of finite dimension $r$ generated by $X_1, X_2, \ldots, X_m$ of (A.21), and the $\gamma_i(t)$ are scalar functions of time. The $\gamma_i(t)$ are formally Lie-Cartan coordinates of the second kind, and are also called *$\gamma$-coordinates* or *logarithmic coordinates*. Furthermore, it is shown that the $\gamma_i(t)$ satisfy a set of differential equations which only depend on the Lie algebra and the $u_i(t)$.

The result is local, in the sense that the representation (A.22) is valid only for a neighborhood of $t = 0$, unless the Lie algebra is solvable. This representation is thus more advantageous than the one proposed by Magnus [**145**], of the form $S(t) = e^{\sum_{i=1}^{r} \gamma_i(t)X_i}$, which is only local. Note in the latter, the $\gamma_i(t)$ correspond to Lie-Cartan coordinates of the first kind.

The maps $S : \gamma \in \mathbb{R}^r \to G_k^m$ constitute global coordinate charts on $G_k^m$, and establish global diffeomorphisms between $\mathbb{R}^r$ and $G_k^m$.

Defining an *evaluation homomorphism* for the Lie algebra (a Lie algebra homomorphism) as a mapping $\nu : L(X_1, \ldots, X_m) \to L(f_1, \ldots, f_m)$ which assigns to each element of $L(X_1, \ldots, X_m)$ a vector field obtained by substituting the $X_i$ by the corresponding $f_i$, $i = 1, \ldots, m$ in $L(X_1, \ldots, X_m)$, the above equations may be regarded as the equations of a control system with right-invariant vector fields on a (matrix) Lie group $G = \nu(G_k^m)$, with state $\nu(S(t)) \in G$. In fact, the trajectory $t \to x(t)$, defined by $x(t) \stackrel{\text{def}}{=} x(0)\nu(S(t))$ is a unique trajectory of the control system

$$\dot{x} = \sum_{i=1}^{r} f_i(x)u_i(t) \tag{A.23}$$

where $f_i = \nu(X_i)$. Note that for $i = 1, 2, \ldots, m$ the $f_i$ correspond to the generators $(X_1, \ldots, X_m)$ of the Lie algebra $L_k(X_1, \ldots, X_m)$, while for $i = m + 1, \ldots, r$, the $f_i$ correspond to the $\nu$-evaluated Lie brackets of $X_i$'s.

The fact that a set of explicit equations relating the Lie-Cartan coordinates of the second kind, $\gamma_i(t)$, and the inputs $u_i(t)$ can be derived makes of these results highly valuable tool to the synthesis of stabilizers for a wide class of control systems in the form of (A.23).

Using the previous results regarding the exponential map and the CBH, the relations between the $\gamma$-coordinates and the $u_i$ may be obtained as follows.

Let $S(t)$ be of the form (A.22) then,

$$\dot{S}(t) = \sum_{i=1}^{r} \dot{\gamma}_i(t) \prod_{j=1}^{i-1} e^{\gamma_j X_j} X_i \prod_{j=i}^{r} e^{\gamma_j X_j} \tag{A.24}$$

Substitution of (A.24) into (A.21) and post-multiplication by $S^{-1}(t)$, yields:

$$\sum_{k=1}^{r} X_k u_k(t) = \dot{S}(t)S^{-1}(t) \tag{A.25}$$

$$= \sum_{i=1}^{r} \dot{\gamma}_i(t) \prod_{j=1}^{i-1} e^{\gamma_j X_j} X_i \prod_{j=i}^{r} e^{\gamma_j X_j} \prod_{j=r}^{1} e^{-\gamma_j X_j} \tag{A.26}$$

$$= \sum_{i=1}^{r} \dot{\gamma}_i(t) \prod_{j=1}^{i-1} e^{\gamma_j X_j} X_i \prod_{j=i-1}^{1} e^{-\gamma_j X_j} \tag{A.27}$$

$$= \sum_{i=1}^{r} \dot{\gamma}_i(t) \prod_{j=1}^{i-1} e^{\gamma_j \, ad_{X_j}} X_i \tag{A.28}$$

$$= \sum_{i=1}^{r} \sum_{k=1}^{r} \dot{\gamma}_i(t) \xi_{ki} X_k \tag{A.29}$$

where the last two equations are obtained by application of Lemma A.1 and Lemma A.2, respectively. Note that the summation in the first equation has been written from 1 to $r$ instead of 1 to $m$, since, without loss of generality, the $u_k$'s may be assumed to be zero for $k > m$. Finally, since the indeterminates $X_k$ are linearly independent, equating the coefficients of $X_k$ in the first of the above equations to those in the last equation, one obtains the following linear relation between the $u_k(t)$ and the $\dot{\gamma}_i(t)$:

$$\underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} \xi_{11}(\gamma) & \cdots & \xi_{1r}(\gamma) \\ \vdots & \ddots & \vdots \\ \xi_{r1}(\gamma) & \cdots & \xi_{rr}(\gamma) \end{bmatrix}}_{\xi(\gamma)} \underbrace{\begin{bmatrix} \dot{\gamma}_1(t) \\ \dot{\gamma}_2(t) \\ \vdots \\ \dot{\gamma}_r(t) \end{bmatrix}}_{\dot{\gamma}}, \quad \gamma(0) = 0 \tag{A.30}$$

where the $\xi_{ij}(\gamma)$ are analytic functions of the $\gamma_i$'s. Clearly, $\gamma(0) = 0$ since $S(0) = I$. Inverting the matrix of $\xi_{ij}$'s one obtains the *Wei-Norman equation* or *logarithmic-coordinates equation*, $\dot{\gamma} = f(u, \gamma) = \xi^{-1}(\gamma)u$, which may be written as the set of equations:

$$\begin{aligned} \dot{\gamma}_1(t) &= f_1(\gamma(t), u) \\ &\ \vdots \\ \dot{\gamma}_r(t) &= f_r(\gamma(t), u) \end{aligned} \qquad \gamma(0) = 0, \quad t \in N_0 \tag{A.31}$$

where $\gamma(t) \stackrel{\text{def}}{=} (\gamma_1(t), \ldots, \gamma_r(t))$ and $u = (u_1, \ldots, u_r)$. These equations allow to convert the problem of steering from an initial configuration $S(0)$ to a final configuration $S(t)$ into a problem of steering vectors in $\mathbb{R}^r$ from $\gamma(0)$ to $\gamma(t)$.

The existence and uniqueness of a solution to (A.31) is ensured by the fact that $\xi^{-1}$ is analytic in a neighborhood $N_0$ of $t = 0$. The above results hold globally for a particular ordering of the basis if the Lie algebra $L$ is solvable (i.e. if there is a chain of ideals $0 \subset L_r \subset L_{r-1} \subset \ldots \subset L_1 = L$, where each $L_p$ is of dimension $r - p + 1$).

#### A.8.4.1. Comments on the application of the above results to control systems and practical implementation aspects

The results presented so far require little modification in order to be applied to control systems in the standard representation as left-invariant systems on Lie groups, rather than as right-invariant systems. For left-invariant systems, (A.21) has now the form

$$\begin{aligned}
\dot{S}(t) &= S(t)\left(\sum_{i=1}^{m} X_i u_i(t)\right) \\
S(0) &= I \in G_k(X_1, \ldots, X_m)
\end{aligned} \tag{A.32}$$

In this case, equations (A.25)-(A.29) are obtained in a similar way, by substituting (A.24) into (A.32) and pre-multiplication by $S^{-1}(t)$ as:

$$\begin{aligned}
\sum_{k=1}^{r} X_k u_k(t) &= S^{-1}(t)\dot{S}(t) \\
&= \sum_{i=1}^{r} \dot{\gamma}_i(t) \prod_{j=r}^{1} e^{-\gamma_j X_j} \prod_{j=1}^{i-1} e^{\gamma_j X_j} X_i \prod_{j=i}^{r} e^{\gamma_j X_j} \\
&= \sum_{i=1}^{r} \dot{\gamma}_i(t) \prod_{j=r}^{1} e^{-\gamma_j X_j} \prod_{j=1}^{i-1} e^{\gamma_j X_j} X_i \prod_{j=r}^{i-1} e^{-\gamma_j X_j} \prod_{j=1}^{r} e^{\gamma_j X_j} \\
&= \sum_{i=1}^{r} \dot{\gamma}_i(t) \prod_{j=r}^{1} e^{-\gamma_j X_j} \left\{ \prod_{j=1}^{i-1} e^{\gamma_j \, ad_{X_j}} X_i \right\} \prod_{j=1}^{r} e^{\gamma_j X_j} \\
&= \sum_{i=1}^{r} \dot{\gamma}_i(t) \prod_{j=r}^{1} e^{-\gamma_j \, ad_{X_j}} \prod_{j=1}^{i-1} e^{\gamma_j \, ad_{X_j}} X_i
\end{aligned}$$

$$= \sum_{i=1}^{r} \dot{\gamma}_i(t) \prod_{j=r}^{i} e^{-\gamma_j \, ad_{X_j}} X_i$$

$$= \sum_{i=1}^{r} \sum_{k=1}^{r} \dot{\gamma}(t) \xi_{ki} X_k$$

Once again, the last two equations have been obtained by the application of Lemma A.1 and Lemma A.2, respectively. However, the coefficients $\xi_{ki}$ do not correspond to those in (A.29) unless a special selection of the basis and its ordering are assumed.

From a practical consideration, it is enough to obtain the logarithmic equations by following the steps of equations (A.25) through (A.29). Equation (A.28) is in particular relevant because it will allow, together with (A.12), the calculation of the actual coefficients $\xi_{ki}$.

It is worth to mention that when using the above exponential formula and the CBH for the composition of flows, these have been stated for flows acting on the right. Thus, in the practical implementation or simulation of a given system trajectory, say $x_0 e^{a_1 f_1} e^{a_2 f_2}$, this must be interpreted as the trajectory starting from $x_0$ generated by the flow $e^{a_1 f_1} e^{a_2 f_2}$, resulting from the application of a vector field $f_1$ with control magnitude $a_1$, followed by the application of a vector field $f_2$ with control magnitude $a_2$.

## A.9. Basic Notions in Systems and Control

The following standard definitions concerning accessibility, controllability and stabilizability properties of a control system are taken or based on the definitions found in the indicated references.

Some preliminary definitions are presented first, for details see [**7**, pp. 25-28].

**Definition A.53. - Time set $\mathcal{T}$.** *A* time set $\mathcal{T}$ *is a subgroup of* $\mathbb{R}_+$.

**Definition A.54. - System or Machine $\Sigma$.** *A* system *or* machine $\Sigma \stackrel{\text{def}}{=} (\mathcal{T}, \mathcal{X}, \mathcal{U}, \Phi)$ *consists of:*

- *A time set* $\mathcal{T}$.
- *A nonempty set* $\mathcal{X}$ *called the* state space *of* $\Sigma$.
- *A nonempty set* $\mathcal{U}$ *called the* control-value *or* input-value space *of* $\Sigma$.

- *A map* $\Phi : \mathcal{D}_\Phi \to \mathcal{X}$ *the* transition map *of* $\Sigma$*, which is defined on a subset* $\mathcal{D}_\phi$ *of*

$$\left\{(\tau, \sigma, x, \omega) \mid \sigma, \tau \in \mathcal{T}, \sigma \leq \tau, x \in \mathcal{X}, \omega \in \mathcal{U}^{[\sigma,\tau)}\right\},$$

  *such that the properties of nontriviality (i.e. admissibility of* $\omega \in \mathcal{U}^{[\sigma,\tau)}$*), restriction, semigroup and identity hold (see details in* [**7**]*).*

Denote by $d(a, b)$ the distance between two elements of $\mathcal{X}$ and let $d_\infty$ be the uniform distance between two time functions into $\mathcal{X}$; that is if $\gamma_1,\ \gamma_2 : \mathcal{I} \to \mathcal{X}$, for some interval $\mathcal{I} \subseteq \mathcal{T}$, then

$$d_\infty \stackrel{\text{def}}{=} \sup\left\{d(\gamma_1(t), \gamma_2(t)), t \in \mathcal{I}\right\}$$

(essential supremum when dealing with measurable functions and $\mathcal{T} = \mathbb{R}$).

**Definition A.55. - Topological System** $\Sigma$**.** *A* topological system $\Sigma$ *is an object* $(\mathcal{T}, \mathcal{X}, \mathcal{U}, \Phi)$ *such that* $\mathcal{X}$ *is a metric space;* $\Sigma$ *is a system when* $\mathcal{X}$ *is thought of just a set and for each* $\sigma < \tau$ *in* $\mathcal{T}$ *and each* $\omega \in \mathcal{U}^{[\sigma,\tau)}$*,*

$$\Phi(\tau, \sigma, \cdot, \omega)$$

*has an open domain and is continuous there as a map into* $\mathcal{X}^{[\sigma,\tau]}$ *(with metric* $d_\infty$*).*

The property of a state being reachable or attainable by some trajectory of the system starting at an initial state $x$ is now defined as follows; see [**1**].

**Definition A.56. - Reachability or Attainability, [1].** *Consider the system* $\dot{x} = f(x, u)$ *with state space* $M$ *and an admissible control class* $\mathcal{U}$*, a state* $y \in M$ *is* reachable *(or* attainable*) from an initial state* $x \in M$ *at time* $t \geq 0$ *if there is a control* $u(\cdot) \in \mathcal{U}$ *such that* $y = \Phi(t, x, u(\cdot))$*.*

It is now possible to define the set of reachable states at time $t$ from a given point $x$ for each $x \in M$ as the set (see [**1**]):

$$\mathcal{R}(t, x) \stackrel{\text{def}}{=} \{y \mid y \text{ is reachable from } x \text{ in time } t\}$$

and the following reachable sets

$$\begin{aligned}\mathcal{R}([0,t],x) &\stackrel{\text{def}}{=} \bigcup_{s\leq t}\mathcal{R}(s,x)\\ \mathcal{R}(x) &\stackrel{\text{def}}{=} \bigcup_{t\geq 0}\mathcal{R}([0,t],x)\end{aligned}$$

### A.9.1. Accessibility

Another property which is important in the study of controllability is that of *accessibility* - in simple terms this means the ability of the system to reach a full neighbourhood (a set of full dimension) from some point.

**Definition A.57. - Accessibility Property.** *A system $\dot{x} = f(x,u)$ has the* accessibility property *from $x$, if interior$(\mathcal{R}(x)) \neq \emptyset$, and has the* complete accessibility *property if it has the accessibility property from any $x \in M$.*

**Definition A.58. - Strong Accessibility Property.** *The system is said to have the* strong accessibility property (SAP) from $x$ if interior *interior*$(\mathcal{R}(x)) \neq \emptyset$ for some $t > 0$ and the *complete SAP if it has the SAP from any $x \in M$.*

The *accessibility* property is sometimes referred to as *weak controllability*, [**7**].

### A.9.2. Controllability

**Definition A.59. - Controllability, [1].**

*A system $\dot{x} = f(x,u)$ is* controllable *from $x \in M$ if $\mathcal{R}(x) = M$ and is* completely controllable *if it is controllable from any $x \in M$. The system is* locally controllable *at $x \in M$ if there exists an $\epsilon > 0$ such that $\mathcal{R}(x)$ contains the $\epsilon$-neighborhood $\mathcal{N}_\epsilon(x)$.*

Unlike the accessibility property which can be fully established by the LARC (see Theorem B.1 on p. 239), the controllability property for nonlinear continuous-time systems is not easy to characterize completely. However, for some classes of systems, results on complete controllability are available, see Appendix B, p. 234.

### A.9.3. Asymptotic Controllability

This section presents a definition of the notion of *asymptotic controllability*, see [**7**, p. 211].

**Definition A.60. - Asymptotic Controllability to a State.** *Let $y, z \in \mathcal{X}$, and assume $\mathcal{V}$ is a subset of $\mathcal{X}$ containing both $y$ and $z$. Then, $z$* can be asymptotically controlled to $y$ without leaving $\mathcal{V}$ *if there exists some control $v \in \mathcal{U}^{[0,\infty)}$ admissible for $z$ so that:*

- *For the path $\zeta(t) \stackrel{\text{def}}{=} \Phi(z,v)$, $\lim_{t\to\infty} \zeta(t) = y$*
- *$\zeta(t) \in \mathcal{V}$ for all $t \in \mathcal{T}_+$.*

*When $\mathcal{V} = \mathcal{X}$, one just says that $z$* can be asymptotically controlled to $y$.

**Definition A.61. - Asymptotic Controllability to an Equilibrium State $x_0$.** *Let $\Sigma$ be a topological system and $x_0$ an equilibrium state. Then $\Sigma$ is:*

- Locally asymptotically controllable to $x_0$ *if for each neighborhood $\mathcal{V}$ of $x_0$ there is some neighborhood $\mathcal{W}$ of $x_0$ such that each $x \in \mathcal{W}$ can be asymptotically controllable to $x_0$ without leaving $\mathcal{V}$.*
- Globally asymptotically controllable to $x_0$ *if it locally asymptotically controllable and also every $x \in \mathcal{X}$ can be asymptotically controlled to $x_0$.*

*For systems with no control, the standard terminology is to say that the system $\Sigma$ is (locally or globally)* asymptotically stable *with respect to $x_0$, or that $x_0$ is asymptotically stable for that system.*

### A.9.4. Stability

**Definition A.62. - Ultimate Boundedness, [24].** *An autonomous system $\dot{x} = f(x), x \in \mathbb{R}^n$ is said to be* ultimately bounded *provided there is a positive constant $b$ such that for any initial state $x(0)$ there is a $T > 0$ such that*

$$\|x(t)\| \leq b \quad \text{for all } t \in [T, \infty)$$

**Definition A.63. - Practical Stabilization, [24].** *A control system $\dot{x} = f(x,u), x \in \mathbb{R}^n$ is said to be* practically stabilizable *if there exists a control input $u$ such that the system has the ultimate boundedness property.*

### A.9.5. Remark on Stabilizability and Controllability

Stabilizability does not require controllability, an example of this fact is provided in [**11**], where the scalar system:

$$\dot{x} = ax + bxu, \quad a \geq 0$$

is shown to be stabilizable for $b \neq 0$. However no points are attainable from the origin regardless the value of $b$.

### A.9.6. Remark on the Stabilizability of Affine Systems Whose Drift Term Does Not Vanish at the Equilibrium Point

Consider the affine system

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x)u_i(x) \tag{A.33}$$

and assume that the origin is an equilibrium point with a nominal $\bar{u} = [\bar{u}_1, \ldots, \bar{u}_m] \in \mathbb{R}^m$, so that the equality

$$f_0(0) + \sum_{i=1}^{m} f_i(0)\bar{u}_i = 0 \tag{A.34}$$

holds. Them, for each stabilizing $u(x)$, the control value $\bar{u} = u(0)$, must satisfy(A.34). Note that if (A.33) has a stabilizing control law $u(x)$, such that $u(0) = \bar{u}$, then the system

$$\dot{x} = \left(f_0(x) + \sum_{i=1}^{m} f_i(x)\bar{u}_i\right) + \sum_{i=1}^{m} f_i(x)v_i(x) \tag{A.35}$$

can be stabilized setting $v(x) = u(x) - \bar{u}$. Certainly, $v(0) = 0$ and thus we can limit our attention to affine systems whose drift term vanishes at the origin as well as any admissible feedback law. An objection to this reasoning, at least from a practical point of view, is that (A.33) may have many solutions $\bar{u}$ and the existence of a stabilizing feedback for (A.35) may depend on the right choice of $\bar{u}$.

The previous arguments can be extended to the stabilization of the system to points different from the origin. In particular, for linear or bilinear systems, the drift term does not in general vanish at points other than the origin. However, the system may still be stabilized around a point $x^* \neq 0$, if there exists a $u^*$ such that equation (A.33) evaluated at $x^*$, instead of the origin, is satisfied. Then a simple coordinate translation $z = x - x^*$ and the arguments above would allow to handle this case as one of stabilization around the origin (now of the coordinate system $z$) with non vanishing drift at $z = 0$.

## A.10. Some Notions from the Theory of Linear Systems

**Definition A.64. - The stable, unstable and center subspaces, $E^s, E^u$ and $E^c$ respectively, of a linear system, [146].** *Let $w_i = u_i + \mathbf{i}v_i$ be a generalized eigenvector corresponding to the eigenvalue $\lambda_i = a_i + \mathbf{i}b_i$ of a real matrix $A \in \mathbb{R}^{n\times n}$ of a linear system*

$$\dot{x} = Ax \tag{A.36}$$

*And let*

$$B = \{u_1, \ldots, u_k, u_{k+1}, v_{k+1}, \ldots, u_m, v_m\} \tag{A.37}$$

*be a basis of $\mathbb{R}^n$ (with $n = 2m - k$), where $k$ eigenvectors correspond to purely real eigenvalues and $2(m - k)$ correspond to $m - k$ pairs of complex conjugate eigenvalues.*

*Then*

$$\begin{aligned} E^s &= \operatorname{span}\{u_i, v_i \mid a_i < 0\} \\ E^c &= \operatorname{span}\{u_i, v_i \mid a_i = 0\} \\ E^u &= \operatorname{span}\{u_i, v_i \mid a_i > 0\} \end{aligned}$$

*i.e., $E^s, E^u$ and $E^c$ are the subspaces of $\mathbb{R}^n$ spanned by the real and imaginary parts of the generalized eigenvectors $w_i$ corresponding to the eigenvalues $\lambda_i$ with negative, zero and positive real parts, respectively.*

## A.11. Some Notions from Lyapunov Theory

A function $V : x \in \mathbb{R}^n \to V(x) \in \mathbb{R}$ is said to be a *Lyapunov function* for a given system of ordinary differential equations, $\dot{x} = f(x)$, $f(0) = 0$, (where the vector field $f$ is at least of class $\mathcal{C}^1$), if there exists a neighborhood of the origin such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ for } x \in \mathbb{R}^n, x \neq 0 \tag{A.38}$$

$$\dot{V}(x) = \nabla_x V f(x) < 0 \text{ for } x \in \mathbb{R}^n, x \neq 0 \tag{A.39}$$

Furthermore if it satisfies

$$\inf_u [\nabla_x V f(x, u)] < 0 \text{ for each } x \in \mathbb{R}^n,\ x \neq 0 \tag{A.40}$$

then it is called a *control Lyapunov function*, see [**11**].

## A.12. Some Definitions from Dynamics of Mechanical Systems

For further details see [**177, 178, 179**].

**Definition A.65. - Holonomic System, [179].** *A system whose particles are constrained to move on some smooth surface of the form*

$$h(x) = 0 \tag{A.41}$$

*or*

$$h(x, t) = 0 \tag{A.42}$$

*where* $x = [p_{1x}, p_{1y}, p_{1z}, \ldots, p_{nx}, p_{ny}, p_{nz}]$ *is the configuration vector of* $n$ *particles* $\in \mathbb{R}^3$*, is called* holonomic *system (from the Greek, meaning "altogether lawful"). If the system can not be written in either of the previous forms, then it is called* nonholonomic*. Among holonomic constraints one can distinguish between those that do not depend explicitly on time and those that do; these holonomic constraints are called* scleronomic *(meaning "rigid") and* rheonomic *(meaning "flowing"), respectively. This definition also applies to rigid-body systems, but instead of referring to the particles, it refers to a set of coordinates* $q$ *in the configuration space* $\mathcal{Q}$ *of the system.*

# APPENDIX B

## Controllability of Systems with Drift

This section explains some of the existing results on the controllability of nonlinear systems and presents a brief overview of the relevant literature, thus complementing the mathematical background presented in Chapter 1 and Chapter 2. This section is intended to serve as an introductory exposition to some basic concepts, facts and tools from the differential geometric approach for the analysis of nonlinear control systems.

It is convenient to first introduce some basic results and notions, most of which concern drift-free systems, that will be required later in the discussion of results regarding systems with drift and the pertinent literature.

### B.1. Basic Controllability Results

There are many types of controllability notions, see for example [**19, 39**] and the clear exposition in [**5**]. The most natural form of controllability which is probably also the most relevant from a practical perspective is the one that characterizes the system with drift $\Sigma$ as being controllable if for any choice of $x_0$ and $x_f$ in $\mathbb{R}^n$, there exists a finite time $T$ and an input $u : [0, T] \to \mathcal{P}^m$ such that $x(T, x_0, u) = x_f$, i.e. such that there exists a trajectory of $\Sigma$ that joins the starting point $x_0$ to $x_f$ in finite time.

All controllability notions[1] answer some form of the following general question: where can the system be made to move by modifying its inputs. For example, one might wish to know:

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[1]Some formal definitions of the standard types of controllability found in the literature are included for the reader's convenience in sections A.9.2, p. 229 and A.9.3, p. 230 of Appendix A; see definitions A.59, A.60 and A.61.

- To which points can one drive the system from some starting point, i.e. which points can be *reached* or *accessed* from an initial point $x_0$ by steering the trajectories of $\Sigma$.
- From which states can one steer the system to a final given state.

These questions are certainly of geometric nature and related to a perhaps even more basic question which is: if the solutions to $\Sigma$ exist, where do they exist? The latter question was answered by the Frobenius theorem [**2, 14, 19, 6**], which can be regarded as a generalization of the Cauchy-Lipschitz existence theorem for unforced ordinary differential equations $\dot{x} = f(x)$, $x_0 = x(0) \in \mathbb{R}^n$. Although Frobenius' results only indicate the existence of some manifold containing the solutions to (1.1) without giving a more explicit characterization of the sets on which the solutions evolve, it involves realizing that the solutions of $\Sigma$ are not only defined by the span of vector fields $f_i \in \mathcal{F}$, but also by Lie products $[f_i, f_j] \stackrel{\text{def}}{=} \frac{\partial f_j}{\partial x} f_i - \frac{\partial f_i}{\partial x} f_j$ of the vector fields $f_i, f_j \in \mathcal{F}$. To see this, consider a control system of the form

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2, \qquad x(0) = x_0 \tag{B.1}$$

and apply a control $u = [u_1, u_2]$ defined as

$$u = \begin{cases} [\ \ 1,\ \ 0], & 0 \le t < \epsilon \\ [\ \ 0,\ \ 1], & \epsilon \le t < 2\epsilon \\ [-1,\ \ 0], & 2\epsilon \le t < 3\epsilon \\ [\ \ 0, -1], & 3\epsilon \le t < 4\epsilon \end{cases}, \qquad \epsilon > 0 \tag{B.2}$$

The above control generates a system trajectory corresponding to the integration of (B.1) for $\epsilon$ units of time, first along $f_1$, then $f_2$, followed by $-f_1$ and finally $-f_2$. For small $\epsilon$, the approximated trajectory can be calculated by evaluating the Taylor series expansion in $\epsilon$ of $x(t) = \int_0^{4\epsilon} \dot{x} dt$ as follows.

For $t \in [0, \epsilon]$ and denoting by $o(\epsilon^k)$ the terms of order $\epsilon^k$,

$$\begin{aligned} x(t) &= x_0 + \int_0^t f_1(x(s))ds \\ &= x(0) + \dot{x}(0)(t - t_0) + \frac{1}{2}\ddot{x}(0)(t - t_0)^2 + o((t - t_0)^3) \end{aligned}$$

which evaluated at $t = \epsilon$ with $t_0 = 0$ yields,

$$x(\epsilon) = x_0 + \epsilon f_1(x_0) + \frac{\epsilon^2}{2}\frac{\partial f_1}{\partial x}(x_0) f_1(x_0) + o(\epsilon^3)$$

Similarly, for $t \in [\epsilon, 2\epsilon]$,

$$\begin{aligned} x(t) &= x(\epsilon) + \int_{\epsilon}^{t+\epsilon} f_2(x(s))ds \\ &= x(\epsilon) + f_2(x(\epsilon))(t-\epsilon) + \frac{1}{2}\frac{\partial f_2}{\partial x}(x(\epsilon)) f_2(x(\epsilon))(t-\epsilon)^2 + o(\epsilon^3) \end{aligned}$$

yields at time $t = 2\epsilon$,

$$\begin{aligned} x(2\epsilon) &= x_0 + \epsilon f_1(x_0) + \frac{1}{2}\epsilon^2 \frac{\partial f_1}{\partial x}(x_0) f_1(x_0) + \epsilon f_2(x(\epsilon)) + \frac{\epsilon^2}{2}\frac{\partial f_2}{\partial x}(x_0) f_2(x_0) + o(\epsilon^3) \\ &= x_0 + \epsilon\left[f_1(x_0) + f_2(x_0)\right] + \frac{\epsilon^2}{2}\left[\frac{\partial f_1}{\partial x}(x_0) f_1(x_0) + 2\frac{\partial f_2}{\partial x}(x_0) f_1(x_0) + \frac{\partial f_2}{\partial x}(x_0) f_2(x_0)\right] + o(\epsilon^3) \end{aligned}$$

where $f_2(x(\epsilon))$ was approximated by the Taylor series expansion[2]:

$$f_2(x(\epsilon)) = f_2(x_0 + \epsilon f_1(x_0)) = f_2(x_0) + \epsilon \frac{\partial f_2}{\partial x}(x_0) f_1(x_0) + o(\epsilon^2)$$

Repeated use of the second-order Taylor series expansion yields respectively for $t \in [2\epsilon, 3\epsilon]$ and $t \in [3\epsilon, 4\epsilon]$:

$$\begin{aligned} x(t) &= x(2\epsilon) + \int_{2\epsilon}^{t+2\epsilon} -f_1(x(s))ds \\ &= x(2\epsilon) - f_1(x(2\epsilon))(t-2\epsilon) - \frac{1}{2}\frac{\partial f_1}{\partial x}(x(2\epsilon)) f_1(x(2\epsilon))(t-2\epsilon)^2 + o(\epsilon^3) \\ x(3\epsilon) &= x_0 + \epsilon f_2(x_0) + \frac{\epsilon^2}{2}\left[2\frac{\partial f_2}{\partial x}(x_0) f_1(x_0) + \frac{\partial f_2}{\partial x}(x_0) f_2(x_0) - 2\frac{\partial f_1}{\partial x}(x_0) f_2(x_0)\right] + o(\epsilon^3) \end{aligned}$$

and

$$\begin{aligned} x(t) &= x(3\epsilon) + \int_{3\epsilon}^{t+3\epsilon} -f_2(x(s))ds \\ &= x(3\epsilon) - f_2(x(3\epsilon))(t-3\epsilon) - \frac{1}{2}\frac{\partial f_2}{\partial x}(x(3\epsilon)) f_2(x(3\epsilon))(t-3\epsilon)^2 + o(\epsilon^3) \\ x(4\epsilon) &= x_0 + \frac{\epsilon^2}{2}\left[2\frac{\partial f_2}{\partial x}(x_0) f_1(x_0) - 2\frac{\partial f_1}{\partial x}(x_0) f_2(x_0)\right] + o(\epsilon^3) \end{aligned}$$

[2] One might regard the approximation of $f_2(x(\epsilon))$ either as: (a) the Taylor series expansion of $f_2(x)$ about $x = x_0$ and evaluated at $x = x(\epsilon) = x_0 + \epsilon f_1(x_0)$, or (b) the Taylor series expansion of $f_2(x(\epsilon))$ about $\epsilon = 0$.

Thus showing that the resulting system motion

$$\begin{aligned} x(4\epsilon) - x(0) &= \exp(-\epsilon f_2) \circ \exp(-\epsilon f_1) \circ \exp(\epsilon f_2) \circ \exp(\epsilon f_1) x_0 - x_0 \\ &\approx \epsilon^2 [f_1, f_2] \end{aligned}$$

is in the direction of $[f_1, f_2]$ which in not in the linear span of $f_i \in \mathcal{F}$. This type of motion along a Lie product is known as a *Lie bracket motion*. As similar result can be derived for systems with drift, however, involving lengthier and more tedious calculations.

The geometric interpretation of the Lie product as the trajectory of the system $\dot{x} = [f, g](x)$ and the non-commutativity of vector fields $f_1$, $f_2$, i.e. $[f_1, f_2] \neq 0$ is illustrated in Fig. B.1.

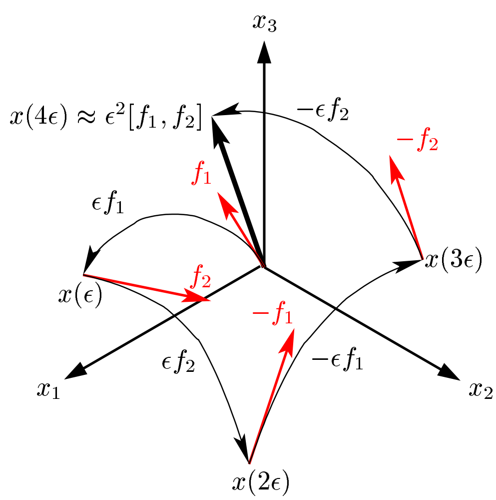


FIGURE B.1. Geometric interpretation of the Lie product as a *Lie bracket motion*.

The relevance of the Lie bracket and the Lie algebra generated by forming repeated Lie products of the vector fields in $\mathcal{F}$, (see definition in Appendix A, p. 202), to the study of controllability should be clear from the above calculation which strongly suggests that not only $\exp(\epsilon f_1)x_0$ and $\exp(\epsilon f_2)x_0$ are points in the reachable set from $x_0$ but also $\exp(\epsilon[f_1, f_2])x_0$. Iterating on this idea, but with a more complicated input sequence than (B.2), it is possible to obtain motion in the direction of higher order brackets, such as $[f_1, [f_1, f_2]]$, see [**184**]. This suggests that the set of maps $\{\exp(tf)\}$ acting on $x_0$ defines the reachable set if $f$ can be expressed as a *bracketed combination* of the $f_i \in \mathcal{F}$. In fact, the Lie algebra $L(\mathcal{F})$ plays an important role in the characterization of the reachable set of states from a given $x_0$ in a fundamental result known as Chow's Theorem (see Theorem B.1, p. 239).

Denote by $f_{\mathcal{F}}$ and $f_{L(\mathcal{F})}$ any vector field $f \in \mathcal{F}$ and $f \in L(\mathcal{F})$, respectively. Let diff$(M)$ denote set of all $\mathcal{C}^\infty$ one-to-one and onto mappings of a $\mathcal{C}^\infty$-manifold $M$ onto itself. The set of such mappings is closed under inversion and composition, and therefore they form a group called the *group of diffeomorphisms* of $M$. Since the system $\Sigma$ is defined on $\mathbb{R}^n$, for simplicity of exposition, it will be assumed from now on that $M = \mathbb{R}^n$.

Under the assumption that the vector fields in $L(\mathcal{F})$ are complete, there exits a group of mappings of $\mathbb{R}^n$ into itself, denoted by $G(\mathcal{F}) \subset \text{diff}(\mathbb{R}^n)$, which is closely related to $L(\mathcal{F})$ and which is obtained by "exponentiating" all the vector fields $f_{\mathcal{F}}$. The exponentiation operation relating elements of the Lie algebra $L(\mathcal{F})$ to those of the group $G(\mathcal{F})$ is defined by the operator $\exp : L(\mathcal{F}) \to G(\mathcal{F})$, which is called *exponential map*. The map exp is an analytic diffeomorphism of $L(\mathcal{F})$ onto $G(\mathcal{F})$ if $G(\mathcal{F})$ is simply connected, see [**162**, Thm. 3.6.2, p. 196]. A rigorous definition of the exponential map is found in [**162**, p. 84], but for the sake of clarity, this concept is informally explained as follows, see for instance [**14**].

Given $f$, for each $t$, $\exp(tf)$ defines a map of $\mathbb{R}^n$ into itself, which is just the mapping produced by the flow on $\mathbb{R}^n$ defined by the differential equation $\dot{x} = f(x)$. In other words, $\exp(tf)$ is the mapping parametrizing the solutions of $\dot{x} = f(x)$. This mapping is called the the flow[3] generated by $f$.

Denoting by $\{\exp(f_{\mathcal{F}})\}_G$ the smallest subgroup of diff$(\mathbb{R}^n)$ which contains $\exp(tf)$ for all $f \in \mathcal{F}$, it is clear that for a drift-free version of system $\Sigma$, any point $x \in \mathbb{R}^n$ of the form $x = \{\exp(f_{\mathcal{F}})\}_G x_0$ can be reached from $x_0$ along solution curves of system because $x$ can be expressed as:

$$x = \prod_{i=1}^{m} \exp(t_i f_{I(i)}) x_0$$

for some indices $I(i) \in \{1, 2, \ldots, m\}$ by setting all but one of the inputs to zero while setting the input associated with the index $I(i)$ to one.

For drift-free systems, Chow's theorem establishes that $\{\exp(f_{\mathcal{F}})\}_G$ and $\{\exp(f_{L(\mathcal{F})})\}_G$ are in fact equal under rather weak assumptions. As there are many versions of the existence theorem for

[3]Recall in linear systems theory the flow is defined by the state transition matrix which parametrizes the solutions of the uncontrolled system $\dot{x} = Ax$ and is calculated as the matrix exponential $e^{At}$, where $A$ is an $n \times n$ matrix. In differential geometry, $\exp(tf)$ is used instead of the standard notation $\Phi^f(t)$ from the theory of differential equations to denote the flow of the differential equation defined by $f$.

ordinary differential equations, there are several versions of Chow's theorem. Here, three versions which are common in the control systems literature are given below.

**Theorem B.1. - Chow's Theorem [14, 6].**

*Let $\mathcal{F}$ be a collection of vector fields such that $L(\mathcal{F})$ is:*

*Version 1. analytic on an analytic manifold $M$. Then given any point $x_0 \in M$, there exists a maximal submanifold $N \subset M$ containing $x_0$ such that $\{\exp(f_{\mathcal{F}})\}_G x_0 = \{\exp(f_{L(\mathcal{F})})\}_G x_0 = N$.*

*Version 2. $\mathcal{C}^\infty$ on a $\mathcal{C}^\infty$ manifold $M$ with $\dim span\, L(\mathcal{F})$ constant on $M$. Then given any point $x_0 \in M$, there exists a maximal submanifold $N \subset M$ containing $x_0$ such that $\{\exp(f_{\mathcal{F}})\}_G x_0 = \{\exp(f_{L(\mathcal{F})})\}_G x_0 = N$.*

*Version 3. is analytic and the vector fields in $\mathcal{F}$ are complete. Then the drift-free system $\Sigma$ (i.e. system (1.1) with $f_0(x) = 0$) is controllable on an open neighborhood of the origin, $N \subset M = \mathbb{R}^n$, if the* Lie algebra rank condition*, (LARC):*

$$span\, L_x(\mathcal{F}) = \mathbb{R}^n, \quad \textit{for all } x \in N \tag{B.3}$$

*is satisfied.*

Chow's theorem gives a conclusive answer about the controllability of driftless systems, but it does not for systems with drift. The main limitation of Chow's theorem is that it does not distinguish between positive and negative time, since the submanifold whose existence is guaranteed by Theorem B.1 may include points which can only be reached by passing backwards along the vector field $f(x)$, i.e. by going forward along $-f(x)$ (see discussion in Section 1.4, p. 10, concerning the difficulties in steering $\Sigma$). This means that while the reachable set $\mathcal{R}_{\mathcal{F}}(x_0) \stackrel{\text{def}}{=} \{\exp(f_{\mathcal{F}})\}_G x_0$ will be always contained in the manifold $N = \{\exp(f_{L(\mathcal{F})})\}_G x_0$ defined by Chow's theorem, it will be a proper subset of this manifold, i.e. $\mathcal{R}_{\mathcal{F}}(x_0) \subset N$, see [**36, 19**].

In the light of this discussion, it is worth pointing out that the strong controllability hypothesis assumed here, imposes that the reachable set $\mathcal{R}_{\mathcal{F}}(T, x) = G_T\, x = \{\exp(T f_{\mathcal{F}})\}_G x$ is equal to $G\, x \stackrel{\text{def}}{=} \{\exp(f_{L(\mathcal{F})})\}_G x$, which in fact demands that backward trajectories in $G\, x$ are realizable by combinations of forward trajectories in $G_T\, x$, see [**20, 22**].

A situation in which the drift term causes no difficulty is when it defines critically stable trajectories for the unforced system $\Sigma$. This result is stated in the following theorem, see [**14**]:

**Theorem B.2. - Reachable Set of a System with Critically Stable Drift.**
*Suppose the vector fields in $\mathcal{F}$ satisfy the conditions of Chow's theorem and suppose that for each initial condition $x_0$ the solution of $\dot{x} = f_0(x)$ is periodic with smallest period $T(x_0)$. Then the reachable set from $x_0$ is $\{\exp(f_{L(\mathcal{F})})\}_G x_0$.*

PROOF. The sketch of the proof is found [**14**]. It based on the fact that if one would require to pass backwards along the drift vector filed $f_0(x)$, one simply needs to set $u_i = 0$, $i = 1, \ldots, m$, and let the free periodic motion bring $x_0$ nearly back to $x_0$ along the integral curve of $\dot{x} = f_0(x)$. If the least period of the periodic motion through $x_0$ is $T$, then by following $\dot{x} = f(x)$ for $T - \epsilon$ units of time, one achieves the same effect as following $\dot{x} = -f(x)$ for $\epsilon$ units of time. Thus, given enough time, any point which is reachable by the driftless system $\dot{x} = \sum_{i=0}^{m} f_i(x)u_i$ can also be reached by the system with drift $\Sigma$ of equation (1.1). □

Due to the lack of general criteria to establish the natural form of controllability for systems with drift, stated at the beginning of this section, all the existing structural characterizations of $\Sigma$ have as yet only been expressed in terms of simple accessibility notions and only go as far as providing sufficient conditions for the verification of the small-time local controllability (STLC) property.

The properties of local accessibility and small-time locally controllability are stated in terms of the following reachable set of $\Sigma$ defined by

$$\mathcal{R}^N_{\mathcal{F}}(x_0) \stackrel{\text{def}}{=} \bigcup_{t \leq T} \mathcal{R}^N_{\mathcal{F}}(t, x_0)$$

where $N$ is a neighbourhood of $x_0$ and

$$\mathcal{R}^N_{\mathcal{F}}(t, x_0) \stackrel{\text{def}}{=} \{y \in \mathbb{R}^n : x(t, x_0, u) = y \text{ and } x(s, x_0, u) \in N, \text{for } s \in [0, t], x_0 \in \mathbb{R}^n, u : [0, t] \to \mathcal{P}^m\}$$

i.e. $\mathcal{R}^N_{\mathcal{F}}(t, x_0)$ is the set of states *reachable* at time $t$ from $x_0$ by trajectories which do not leave the neighbourhood $N$.

The control system $\Sigma$ is thus said to be:

(i) *locally accessible from* $x_0$ if for all neighbourhoods $N$ of $x_0$ and all $T$, the reachable set $\mathcal{R}_{\mathcal{F}}^{N}(x_0)$ contains a non-empty open set.

(ii) *small-time locally controllable from* $x_0$ if for all neighbourhoods $N$ of $x_0$ and all $T$, the reachable set $\mathcal{R}_{\mathcal{F}}^{N}(x_0)$ contains a non-empty neighbourhood of $x_0$.

The Lie algebra rank controllability condition (B.3) of Chow's theorem provides a computable criteria to establish the accessibility property (i) for system $\Sigma$. For analytic vector fields the LARC is in fact a necessary and sufficient condition for accessibility (controllability in the driftless case).

As far as small-time local controllability is concerned, only a sufficient verifiable condition exists, see [**39**, Thm. 7.3]. This condition is given by the following theorem:

**Theorem B.3** (Sussmann, [**39**], p. 185)**.** *Consider $L(\mathcal{F})$, the smallest subset formed by all the Lie brackets of $f_i \in \mathcal{F}$ that contains $\mathcal{F}$, and for any Lie bracket $b \in L(\mathcal{F})$, let $\delta_i(b)$, $i = 0, 1, \ldots, m$, be the number of times the generator $f_i$ occurs in $b$. Then, under the assumptions that:*

- *the vector fields $\mathcal{F}$ satisfy the LARC at $x$, namely* $\dim \, span \, L_x(\mathcal{F}) = n$*, and*
- *whenever $b \in L(\mathcal{F})$ is a bracket for which $\delta_0(b)$ is odd and $\delta_1(b), \ldots, \delta_m(b)$ are even, $b$ can be written as a linear combination of brackets in $L(\mathcal{F})$ of lower degree.*

*the system $\Sigma$ is STLC from $x$.*

## B.2. Overview of the Literature on the Controllability of Systems with Drift

There have been many attempts to improve the meager state of knowledge regarding the reachable set of points for systems with drift and their controllability properties. Although the first significant results pertaining the controllability of nonlinear systems reported in [**18, 27**] consider only driftless systems, these contributions have played a fundamental role in later studies of controllability of systems with drift. The studies in [**18, 27**] make systematic use Lie theory and differential geometry. The work in [**36**] applies to systems with drift unlike [**18, 27**], which essentially deal only with symmetric systems, i.e. systems with the property that $f(x, -u) = -f(x, u)$. Symmetric systems form a very specific class and do not offer major control difficulties since reversing the sign of the control allows to generate backward motions with respect to those generated using the opposite control.

The result by Palais, Theorem 2.1 presented earlier in this chapter (p. 28), is fundamental since it allows to reformulate system $\Sigma$ (under the condition that $L(\mathcal{F})$ is finite dimensional), as a system on a Lie group $G$ for which the vector fields corresponding to constant controls are right-invariant. This is because Palais' theorem allows to give the group $G = \{\exp(f_{\mathcal{F}})\}_G$ the structure of a Lie group with Lie algebra isomorphic to $L(\mathcal{F})$, see [**20**]. The analysis of control systems as systems on Lie groups was brought into attention most notably in [**12, 13, 21**], where issues concerning controllability, observability and realization theory were studied. Another important reference is [**20**], where an explicit characterization of the reachable set $\mathcal{R}_{\mathcal{F}}(T,x) = G_T\, x = \{\exp(T f_{\mathcal{F}})\}_G\, x$ is derived. The result in [**20**, Thm. 3.3] states that the reachable set can be decomposed as the group action defined by the drift vector field and the set of group actions defined by the remaining Lie brackets in $L(\mathcal{F})$, as $\mathcal{R}_{\mathcal{F}}(T,x) = \{\exp(T f_{\mathcal{F}})\}_G\, x = \{\exp(T f_{L^*})\}_G \exp(T f_0)\, x$, where $L^*$ denotes the Lie algebra $L^*(\{f_i, [f_0, f_i], [f_0, [f_0, f_i]], \ldots;\ i = 1, 2, \ldots, m\})$.

These results are of significant importance to the development of the feedback laws proposed in [**72, 45, 60, 57**], which pose the original control problem on the manifold as a problem of steering the corresponding system on the Lie group. In a less direct, but also successful way, these ideas are applied in [**44, 75**] to the development of more general approaches for time-varying feedback stabilization.

Results on (weak) local controllability for general nonlinear systems with drift were obtained in [**19**]. These results basically state that the well known Lie algebra rank condition, which ensures the controllability of driftless systems (Chow's Theorem), only guarantees weak local controllability in a neighborhood of the equilibrium point of $\Sigma$. In [**22**], the controllability of scalar input systems is guaranteed if the LARC in (B.3) holds for every state $x$ in the manifold of solutions $M$, and if the set of points reachable in positive times from every $x$ in a dense subset of $M$ is equal to $M$. In [**8**] criteria for local and global controllability of systems with polynomial drift term are also formulated in terms of properties of their Lie algebra.

More recently, sufficient conditions for small-time local controllability (STLC) of (1.1) were given in [**39**]. Theorem B.3, presented in the previous section, is the main result in [**39**] and provides the most computable criteria for STLC, but it is in general hard to check for non-nilpotent systems, unless the original system is nilpotenized or one assumes some nilpotent approximation that preserves the controllability properties of the original system. The results in [**39**] can also be used to establish the local controllability of systems which have a drift term $f_0$ that vanishes in a connected set $E$, i.e. the uncontrolled system has multiple equilibrium points in a connected set $E$, if the system is controllable to $E$.

# APPENDIX C

## Useful Theorems and Other Results

Some useful theorems and techniques referred to, but not original to this dissertation, are gathered in this appendix for the reader's convenience. The proofs of the Inverse Function Theorem and the Implicit Function Theorem can be found in any textbook of advanced calculus or mathematical analysis, see for example [**189**].

### C.1. Gronwall-Bellman Lemma

The present version of the Gronwall-Bellman Lemma is based on the version found in [**4**].

**Lemma C.1. - Gronwall-Bellman Lemma.** *Let $x : [a, b] \to \mathbb{R}$ be continuous and $y : [a, b] \to \mathbb{R}$ be continuous and nonnegative. Then if a continuous function $z : [a, b] \to \mathbb{R}$ satisfies*

$$z(t) \leq x(t) + \int_a^t y(s)z(s)ds$$

*for $a \leq t \leq b$, then on the same interval*

$$z(t) \leq x(t) + \int_a^t x(s)y(s)e^{\int_s^t y(\tau)d\tau}ds$$

*In particular, if $x(t)$ is nondecreasing, then*

$$z(t) \leq x(t)\, e^{\int_a^t y(\tau)d\tau}$$

*or simply*

$$z(t) \leq \alpha\, e^{\int_a^t y(\tau)d\tau}$$

*for a constant $x(t) \equiv \alpha$. If, in addition, $y(t) \equiv \beta > 0$ is a constant, then*

$$z(t) \leq \alpha\, e^{\beta(t-a)}$$

Proof. Let

$$v(t) = \int_a^t y(s)z(s)ds$$

and

$$w(t) = x(t) + v(t) - z(t) \geq 0$$

Then, $v$ is differentiable and

$$\dot{v}(t) = y(t)z(t) = y(t)\left(x(t) + v(t) - w(t)\right)$$

This is a scalar linear state equation with the state transition function

$$\phi(t,s) = e^{\int_s^t y(\tau)d\tau}$$

Since $v(a) = 0$, we have

$$v(t) = \int_a^t \phi(t,s)\left(x(s)y(s) - w(s)y(s)\right)ds$$

The term $\int_a^t \phi(t,s)w(s)y(s)ds$ is nonnegative, and therefore,

$$v(t) \leq \int_a^t x(s)y(s)e^{\int_s^t y(\tau)d\tau}ds$$

Since $z(t) \leq x(t) + v(t)$, this completes the proof for the general case.

When $x(t)$ is nondecreasing, $x(s) \leq x(t)$ for any $s \leq t$, thus

$$v(t) \leq \int_a^t x(s)y(s)e^{\int_s^t y(\tau)d\tau}ds \leq x(t)\int_a^t y(s)e^{\int_s^t y(\tau)d\tau}ds$$

and applying the Leibniz's rule for the differentiation of an integral we have,

$$\begin{aligned}
\int_a^t y(s)e^{\int_s^t y(\tau)d\tau}ds &= -\int_a^t \frac{d}{ds}\left\{e^{\int_s^t y(\tau)d\tau}\right\}ds \\
&= -\left\{e^{\int_s^t y(\tau)d\tau}\right\}\Bigg|_{s=a}^{s=t} \\
&= -1 + e^{\int_a^t y(\tau)d\tau}
\end{aligned}$$

which proves the lemma when $x(t)$ is nondecreasing or simply a constant. The proof when also $y(t)$ is constant follows by integration. $\square$

## C.2. Inverse Function Theorem

**Theorem C.1. - Inverse Function Theorem.** *Let $A$ be an open subset of $\mathbb{R}^n$ and $f : A \to \mathbb{R}^n$ a continuous $\mathcal{C}^1$ mapping. If $\frac{\partial f}{\partial x}(\bar{x})$ is nonsingular for some $\bar{x} \in A$, then there exists an open neighbourhood $U \subset A$ of $\bar{x}$ such that $V = \{y \in \mathbb{R}^n \mid y = f(x),\, x \in U\}$ is open in $\mathbb{R}^n$ and the restriction of $f$ to $U$ is a diffeomorphism onto $V$.*

Remark C.1. *The map $f$ does not need to be a global diffeomorphism on noncompact sets $U$ even if $\frac{\partial f}{\partial x}$ is nonsingular at any $x$. For example the map $f : (x_1, x_2) \to (e^{x_1}\cos(x_2), e^{x_1}\sin(x_2))$ is nonsingular at every point. However, such $f$ is not a global diffeomorphism, since it is periodic in $x_2$. The globality of the diffeomorphism is ensured by a theorem due to Palais, (see proof in* [**190**]*), if $f$ is a proper map; that is, the inverse image of compact sets is compact.*

## C.3. Implicit Function Theorem

**Theorem C.2. - Implicit Function Theorem.** *Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be open subsets. Let $f : A \times B \to \mathbb{R}^n$ be a $\mathcal{C}^1$ mapping, i.e. $f$ is continuously differentiable at each point $(x, y) \in S \subseteq A \times B$. Let $(\bar{x}, \bar{y})$ be a point in $S$ for which $f(\bar{x}, \bar{y}) = 0$, and for which the Jacobian matrix*

$$\frac{\partial f}{\partial y}(\bar{x}, \bar{y})$$

*is nonsingular. Then, there exist open neighbourhoods of $A_0 \subset A$ of $\bar{x}$ and $B_0 \subset B$ of $\bar{y}$, and a unique $\mathcal{C}^1$ mapping $g : A_0 \to B_0$ such that for each $x \in A_0$ the equation $f(x, y) = 0$ has a unique solution $y \in B_0$ satisfying $y = g(x)$ such that*

$$f(x, g(x)) = 0$$

*for all $x \in A_0$.*

## C.4. The Stable Manifold Theorem

See [**146**] for a proof of the following theorem which is helpful in proving the stabilizing properties of the feedback law proposed in Section 5.5.

**Theorem C.3. -The Stable Manifold Theorem.** *Let $E$ be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in \mathcal{C}^1(E)$, and let $\phi_t$ be the flow of the nonlinear system*

$$\dot{x} = f(x) \tag{C.1}$$

*Suppose that $f(0) = 0$ and that its Jacobian $Df(0) = 0$ has $k$ eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Then there exists a $k$-dimensional differentiable manifold $S$ tangent to the stable subspace $E^s$ of the linear system*

$$\dot{x} = Ax$$

*with $A = Df(0)$ such that for all $t \geq 0$, $\phi_t(S) \subset S$ and for all $x_0 \in S$,*

$$\lim_{t\to\infty} \phi_t(x_0) = 0$$

*and there exists an $n-k$ dimensional differentiable manifold $U$ tangent to the unstable subspace $E^u$ of (C.3) at 0 such that for all $t \leq 0$, $\phi_t(U) \subset U$ and $x_0 \in U$,*

$$\lim_{t\to-\infty} \phi_t(x_0) = 0$$

*$S$ and $U$ are referred to as the* local *stable and unstable manifolds of (C.1) at the origin.*

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