Thesis Proposal

Design of Stabilizing Feedback Controls for Strongly Nonlinear Systems

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Nonlinear Systems with Drift

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} f_i(x(t))u_i, \qquad t \in [0, \infty) \qquad \text{(NLS)}$$

$$x(t) \in \mathbb{R}^n,$$
$$u \in \mathbb{R}^m,\ m < n$$
$$f_i, \in \mathbb{R}^n,\ i = 0, 1, \ldots, m$$

Bilinear Systems $\longrightarrow f_i \stackrel{def}{=} A_i x$ with $A_i \in \mathbb{R}^{n \times n}$ (BLS)

Research Objective

Develop systematic and computationally feasible methods for the construction of controls $u_i(x,t) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$, $i = 1, \ldots, m$ *such that system (NLS)/(BLS) is Lyapunov asymptotically stable.*

Motivation

- NLS *with drift and* $m < n$ represent a wide class of systems often encountered in practice.
- The existence of relatively few general methods for the stabilization of NLS.
- NLS often result in uncontrollable linearizations about their equilibria.
- Many NLS fail to satisfy Brockett's conditions for smooth stabilizability, therefore time-varying (continuous or discontinuous) feedback laws are necessary.

Motivation (contd)

- The lack of computationally feasible methods for discontinuous feedback design and the computational complexity of the current control strategies based on Lie algebraic approaches.
- The lack of constructive approaches to stabilization of general (higher order $n > 2$) multi-input bilinear systems for which the matrix $A_0 + \sum_{i=1}^{m} u_i A_i$ is unstable for all choices of constants u_i.

Proposed Approaches (Original Contributions)

M1. Construction of time-varying stabilizing controls for (NLS) based on Coron's approach of critical orbits for driftless systems.

M2. Construction of computationally feasible time-varying discontinuous feedback controls for (NLS) based on Lie algebraic techniques.

M3. Development of stabilizing feedback laws for (BLS) with unstable drift.

$\Longrightarrow$ M1-M3 required the development of a software package for symbolic manipulation of elements in Lie algebraic theory: Lie Tools Package (LTP).

Existing Approaches - NLS

- Methods based on the construction of a suitable control Lyapunov function that can be decreased to zero by an adequate choice of controls in (NLS) or its linearization.
- Lie algebraic methods, mostly for drift-free systems.
- Methods based on particular characteristics of the system (physical or representation related).

Existing Approaches - BLS

- Several approaches exist, but none is general enough.
- Many approaches assume the drift term is stable, i.e. $eig(A_0) \notin \mathbb{C}_+$.
- Stabilizability of homogeneous (BLS) on $\mathbb{R}^2$ has been fully analyzed.
- Sufficient conditions for stabilizability of (BLS) in $\mathbb{R}^n$ by piece-wise constant controls are given by H. Wang, however no constructive approach is provided.

Main Hypotheses

H1. The vector fields $f_0, \ldots, f_m$ are complete and analytic/Lipchitz continuous.

H2. Let $\mathcal{G} \stackrel{def}{=} \{f_0, f_1, \ldots, f_m\}_{LA}$ denote the Lie algebra of vector fields generated by $f_0, f_1, \ldots, f_m$. NLS satisfies the LARC:

$$\dim \mathcal{G}(x) = n \quad \forall x \in \mathbb{R}^n - \{0\}$$

H3. The vector fields $f_0, \ldots, f_r$ form a basis for the algebra $\mathcal{G}$ and the motion in the direction of any Lie bracket $f_i, i = m+1, \ldots, r$ can be realized by piece-wise continuous open loop controls in the original system.

Core Concepts

Lie bracket extension of NLS:

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{r} v_i(t) f_i(x(t)) \qquad \text{(ES)}$$

$f_i, i = m+1, \ldots, r$, are Lie brackets of $f_0,\ f_1, \ldots,\ f_m$ such that H2 and H3 are satisfied $\forall\ x \in B(0; R) - \{0\}$.

Formal equation for the evolution of flows of NLS:

$$\dot{S}(t) = S(t) \left(\sum_{i=0}^{r} f_i\, v_i \right), \quad S(0) = I \qquad \text{(S1)}$$

Wei-Norman representation to the solutions of (S1):

$$S(t) = \prod_{i=0}^{r} e^{\gamma_i(t) f_i} \qquad \text{(S2)}$$

Equations for the Logarithmic Coordinates γ_i:

$$\sum_{k=1}^{r} f_k\, v_k(t) = \sum_{i=1}^{r} \dot{\gamma}_i(t) \prod_{j=1}^{i-1} e^{\gamma_j\, ad_{f_j}} f_i \tag{1}$$

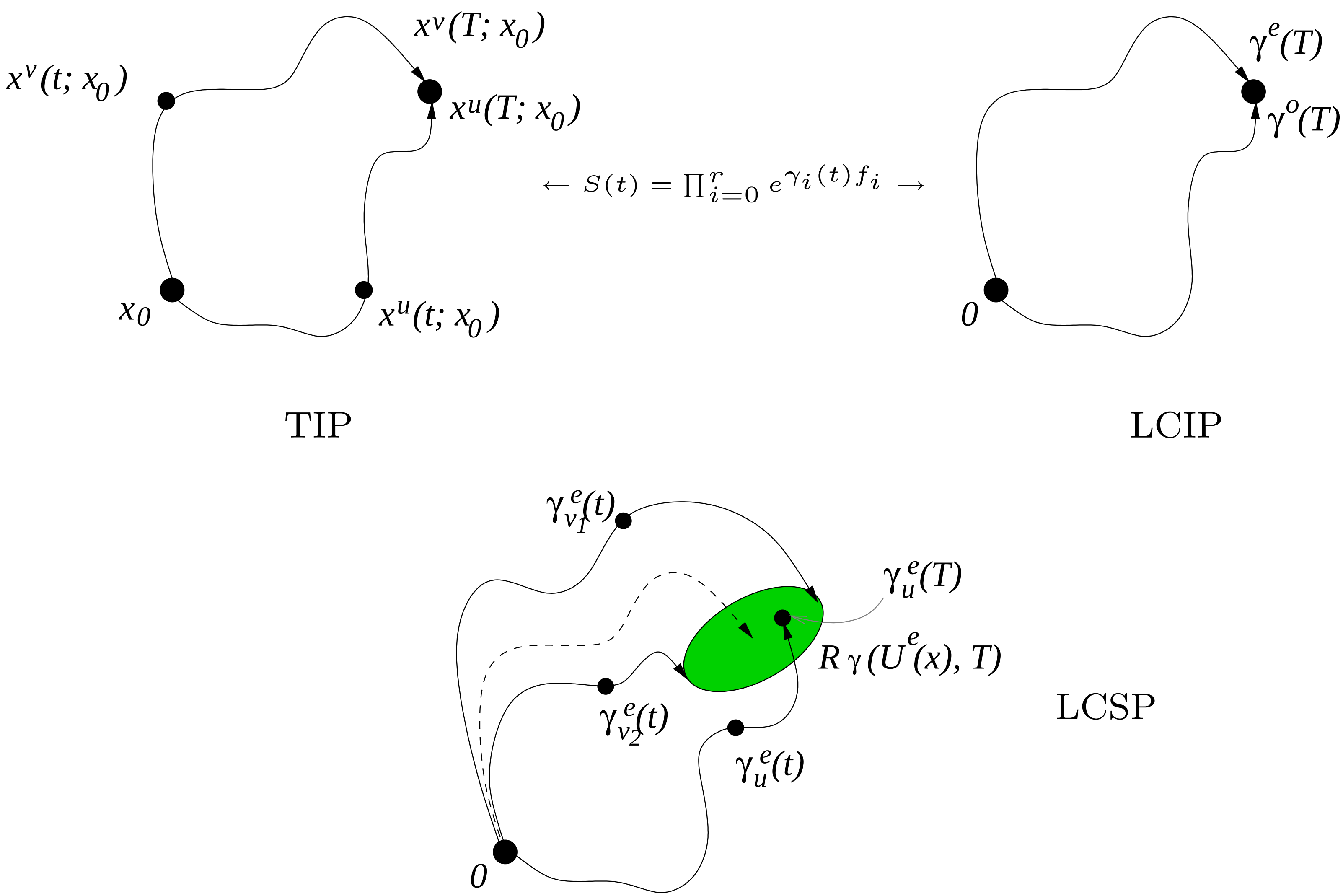
$$= \sum_{i=1}^{r} \sum_{k=1}^{r} \dot{\gamma}_i(t) \xi_{ki} f_k \tag{2}$$

where

$$\begin{aligned} Ad_{e^X} Y &= e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + \dots \\ &= Y + ad_X Y + \frac{1}{2!} ad_X^2 Y + \frac{1}{3!} ad_X^3 Y + \dots \qquad (3) \\ &= e^{ad_X} Y \end{aligned}$$

$$\Longrightarrow \boxed{\dot{\gamma}(t) = \xi^{-1}(\gamma(t))v, \qquad \gamma(0) = 0} \qquad \text{(LE)}$$

TIP/LCIP/LCSP:



Research Progress - M1

Feedback Construction

The asymptotically stabilizing control $u(x,t)$, is computed as the sum

$$u(x,t) \quad = \quad \underbrace{w(x,t)}_{\substack{\text{critically stabilizing} \\ \text{control}}} \quad + \quad \underbrace{\Delta u(x,t)}_{\substack{\text{asymptotically stabilizing} \\ \text{correction}}}$$

A. **Critically Stabilizing Control**

$$v(x) = -\underbrace{[f_1(x) \ \ldots \ f_r(x)]^\dagger}_{G(x)^\dagger} f_0(x) \Rightarrow \dot{x}^e(t) = 0 \rightarrow \boxed{\text{TIP}} \rightarrow w(x,t)$$

Research Progress - M1

B. **Asymptotically Stabilizing Correction**
(draws on idea by Coron and Pomet)

$$V(x,t) = \frac{1}{2}\|\phi_w^{-1}(x,t)\|^2$$

The control $\Delta u(x,t)$ must be such that $\dot{V}(x,t) < 0,\ \forall\, x \in \mathbb{R}^n - \{0\}$.
Choose,

$$\Delta u = -K\left[\nabla_x V(\phi_w(x_0,t),t) \cdot G(x)\right]^T$$

Where, $\phi_w(\phi_w^{-1}(x,t),t) = x$

$$\Longrightarrow \qquad \nabla_x V(x,t) = \underbrace{\left[\phi_w^{-1}(x,t)\right]^T}_{x_0(t)^T} \left[\left.\frac{\partial \phi_w}{\partial x}\right|_{(\phi_w^{-1}(x,t),t)} \right]^{-1}$$

Research Progress - M2

Consider the control Lyapunov function $V(x)$, the proposed control law comprises two modes:

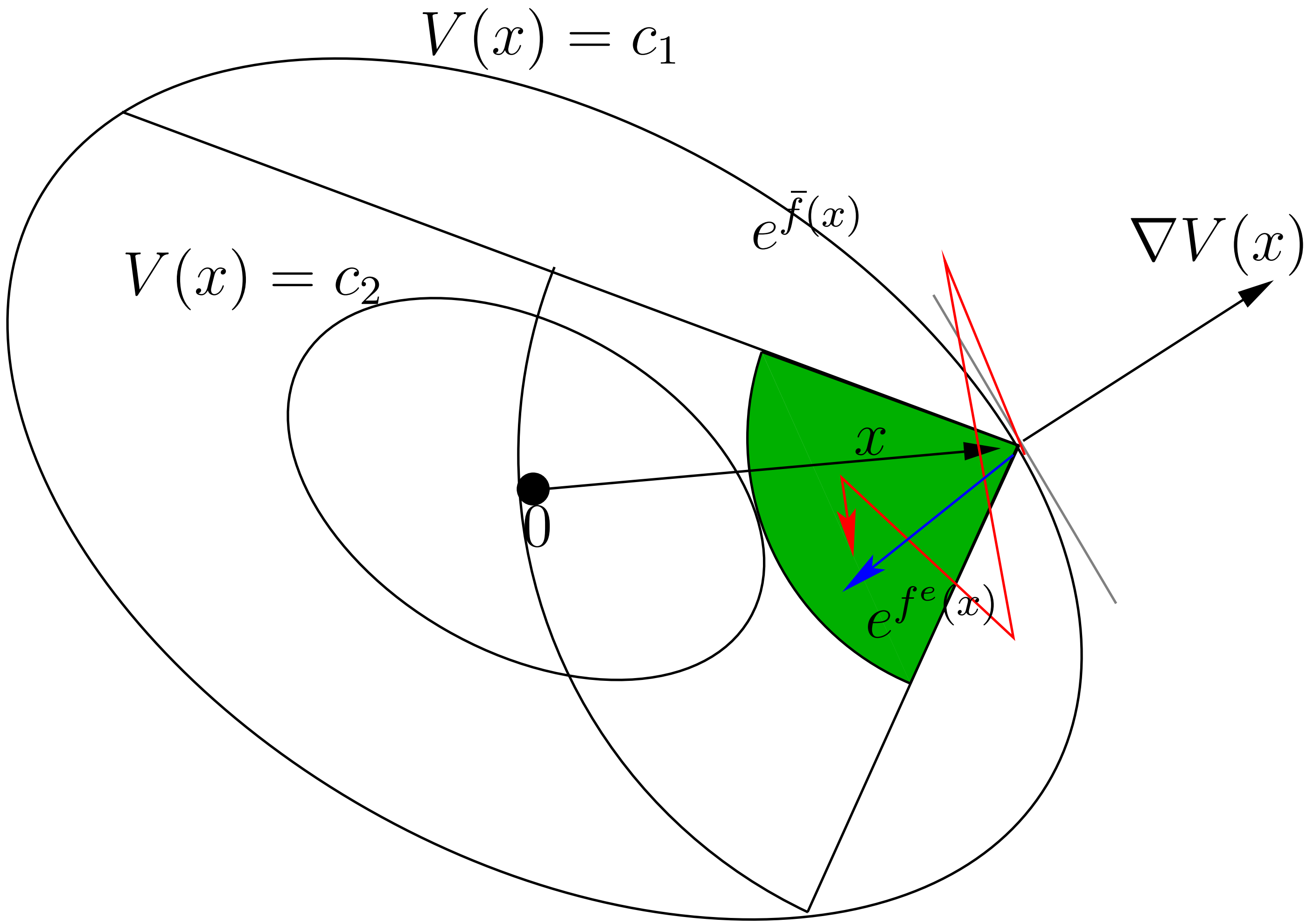
- $u(x)$ that guarantees an instantaneous decrease in $V(x)$, i.e. $\dot{V}(x, u(x)) < 0$:

$$u(x) = \frac{-\nabla V f_0(x) - \eta(x)}{\|\nabla V G(x)\|^2} (\nabla V G(x))^T, \qquad \eta(x) > 0$$

- An open loop piece-wise constant control $\bar{u}(x, t)$ which decreases $V(x)$ in average after a finite period of time T.

The construction of the Lie algebraic control $\bar{u}(x, t)$ is performed by considering a sequence of constant controls whose values are calculated as the solution to a non-linear programming problem (NPP).

Research Progress - M2



Research Progress - M2

NPP:

Idea based on the standard Lyapunov argument: $\dot{V} = \nabla V \bar{f} < 0$, where

$$\bar{f}(x, \bar{u}, \bar{\varepsilon}) = \sum_{i=1}^{r} c_i(\bar{u}, \bar{\varepsilon}) f_i(x)$$

Find a feasible pair $(\bar{u}, \bar{\varepsilon})$ such that:

$$\nabla V \sum_{i=0}^{r} c_i(\bar{u}, \bar{\varepsilon}) f_i(x) \leq -2\eta(x)$$

Research Progress - M2

Approaches to the NPP formulation:

1. Direct application of the CBH for composition of flows.
2. Relaxed control problem in the associated logarithmic coordinates:
   (a) Calculation of the control set $U^e(x) = \{u^e \mid \nabla V f^e(x,u^e) < 0\} = \{u^e \mid \sum_{i=0}^r \nabla V f_i(x) u_i^e < 0\}$
   (b) Calculation of the reachable set of the γ-coordinates $\mathcal{R}_\gamma(U^e(x),T) = \{\gamma(T) \mid \dot{\gamma} = \Gamma(\gamma)u^e,\ \gamma(0)=0,\ u^e \in U^e(x)\}$
   (c) Calculation of the logarithmic coordinates

$$\gamma^o(T,\bar{u},\bar{\varepsilon}) \in \mathcal{R}_\gamma(U^e(x),T).$$

Where γ^o solves, $\quad \dot{\gamma} = \Gamma(\gamma)u^o, \quad \gamma(0)=0, \quad t \in [0,T]$
with $\quad u^o = [1\ \bar{u}_1(x,t)\ \bar{u}_2(x,t)\ \ldots\ \bar{u}_m(x,t)\ 0\ 0\ \ldots\ 0]^T.$

Research Progress - M2

From the logarithmic equation for (ES),

$$\dot{\gamma} = \Gamma(\gamma) u^e, \quad \gamma(0) = 0, \qquad t \in [0, T]$$

we obtain $\gamma = F(u^e, T)$, and hence

$$u^e = F^{-1}(\gamma, T) = F^{-1}(\gamma^o(u^o), T).$$

The NPP may now be formulated with

$$c_i(\bar{u}, \bar{\varepsilon}) = F_i^{-1}(\gamma^o, T)$$

Research Progress - M3

- Two approaches to the stabilization of (BLS).
- Both methods make use of the (ES).
- Approaches:

(a) Based on the trajectory interception approach.

$$v(x) = G(x)^{\dagger}(A_d x - A_0 x) \Rightarrow \dot{x}^e = A_d x \rightarrow \boxed{\text{TIP}} \rightarrow w(x,t)$$

(b) Reaching phase (steering to stable manifold) based on M2 + sliding with (constant) controls.

Research Progress - LTP

- The *Lie Tools Pacakge* (LTP) a software package for symbolic manipulation of elements in Lie algebraic theory.
- Among other functions LTP enables the following automated Lie algebraic manipulations:
  - Construction of Philip Hall bases.
  - Simplification of any Lie bracket expression.
  - Composition of flows via the Campbell-Baker-Hausdorff formula.
  - Set up of the logarithmic-coordinates equation.

Originality of the Proposed Research

- M1:
  - Novel and fairly general.
  - Applicable to a large class of nilpotent systems which do not themselves to successful linearization (Jacobian or by state-feedback).
- M2:
  - First attempt to provide a systematic methodology to discontinuous stabilizing feedback design.
  - More general than M1.
  - Computationally feasible Lie algebraic approach!
- M3:
  - Novel and general.
  - Applicable to systems with unstable drift and systems which are not stabilizable using constant controls.

Originality of the Proposed Research (contd)

- LTP:
  - Provides functionality not encountered in the existing software packages.
  - Essential to the design of stabilizing feedback laws using the proposed methodologies.