Time-Varying Stabilizing Feedback Control for Nonlinear Systems with Drift [1]

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Abstract

A novel method is presented for the construction of time varying stabilizing feedback control for nonlinear systems with drift. The proposed feedback law is a composition of a periodic time-varying control, and an asymptotically stabilizing feedback "correction" term. The periodic control is obtained through a solution of an open loop control problem on the associated Lie group which is posed as a trajectory interception problem in the Philip Hall coordinates of flows.

Keywords: Stabilization, time-varying feedback control, systems with drift.

1 Problem Definition and Basic Assumptions

We consider the problem of feedback stabilization of control systems with drift whose equation of motion is given by

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)) \tag{1.1}$$

Here, $x(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}$, and $f_i, i = 0, 1, ..., m$ are smooth vector fields on $\mathbb{R}^n$. The objective is to construct time varying feedback controls $u_i(x,t) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$, $i = 1, ..., m$ such that system (1.1) is Lyapunov asymptotically stable. For our construction to be valid, we need to impose the following hypotheses:

H1. System (1.1) is globally asymptotically controllable to zero using piece-wise continuous controls, i.e. for each initial condition $x_0 \in \mathbb{R}^n$ there exist piece-wise continuous controls $u_i : [0, \infty) \mapsto \mathbb{R}, i = 1, \ldots, m$ such that the corresponding state trajectory converges to the origin.

H2. The vector fields $f_0, ..., f_m$ are complete, analytic and the origin is an isolated equilibrium state of the unforced system $\dot{x} = f_0(x)$.

H3. Let $\mathcal{G} \overset{def}{=} \{f_0, f_1, ..., f_m\}_{LA}$ denote the Lie algebra of vector fields generated by $f_0, f_1, ..., f_m$. $\mathcal{G}$ is assumed nilpotent and system (1.1) satisfies the LARC:

$$\dim\{f_0, ..., f_m\}_{LA}(x) = n \quad \forall x \in \mathbb{R}^n - \{0\} \tag{1.2}$$

$$\{f_0, ..., f_m\}_{LA}(x) \overset{def}{=} \mathrm{span}\{f(x) | f \in \mathcal{G}\} \tag{1.3}$$

H4. The vector fields $f_0, ..., f_r$ form a basis for the algebra $\mathcal{G}$ and the motion in the direction of any Lie bracket $f_i, i = m+1, ..., r$ can be realized by piecewise continuous open loop controls in the original system.

2 The Synthesis and Properties of the Feedback Control

The asymptotically stabilizing control $u(x,t)$, will be computed as the sum

$$u(x,t) = w(x,t) + \Delta u(x,t) \tag{2.4}$$

where $w \in \mathbb{R}^m$ makes (1.1) periodic, and $\Delta u \in \mathbb{R}^m$ is an additional corrective term which provides for asymptotic stabilization.

2.1 Construction of Critically Stabilizing Periodic Control

Consider the *Lie bracket extension* of (1.1):

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{r} v_i(t) f_i(x(t)) \tag{2.5}$$

defined on $x \in B(0; R) - \{0\}$, where $B(0; R)$ denotes a neighbourhood of the origin in which zero is the only equilibrium state for $\dot{x} = f_0(x)$.

A "critically" stabilizing feedback for the extended system is next defined by

$$v(x) \overset{def}{=} [v_1(x), ..., v_r(x)]^T = -G(x)^{\dagger} f_0(x)$$
$$\text{where } G(x) \overset{def}{=} [f_1(x), ..., f_r(x)] \tag{2.6}$$

with $G(x)^{\dagger}$ denoting the Moore-Penrose pseudo-inverse of G. It follows that the trajectories of (2.5) satisfy $\dot{x}(t) = 0$, which is a stable, trivially periodic system. Moreover, for any initial condition $x_0 = x(t_0)$, the control law (2.6) is constant, i.e. $v(x(t)) = v(x_0) \forall\ t \geq t_0$. The periodic control $w(x,t)$ is finally obtained from $v(x_0)$ by solving an open loop control trajectory interception problem (TIP) on an associated Lie group formulated in terms of the logarithmic coordinates of flows for the original and the extended systems, as described in [1].

2.2 Asymptotically Stabilizing Correction to the Periodic Control

Drawing on the idea proposed by Coron and Pomet in [2] for systems without drift, the control correction term

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Δu is found by requiring that the following Lyapunov function:

$$V(x,t) = \frac{1}{2}\|\phi_w^{-1}(x,t)\|^2 \tag{2.7}$$

decreases along the trajectories of the closed loop system using the combined control (2.4). Here, $\phi_w(x_0,t) = x^w(t;x_0,0)$ with $x^w(t;x_0,0)$ being the solution of the TIP in logarithmic coordinates and $\phi_w^{-1}(x,t) = x_0$ is the inverse function of ϕ_w which *retrieves* the starting point x_0 for the trajectory $x^w(t;x_0,0)$, given the current state x at time t. Since $\dot{V}(x,t) = \nabla_x V(\phi_w(x_0,t),t)\cdot G(x)\Delta u$, then

$$\Delta u \quad = \quad -K\left[\nabla_x V(\phi_w(x_0,t),t)\cdot G(x)\right]^T \tag{2.8}$$

yields

$$\dot{V}(x,t) = -K\|\left[\nabla_x V(\phi_w(x_0,t),t)\cdot G(x)\right]^T\|^2 \leq 0 \tag{2.9}$$

which makes Δu a good candidate for the asymptotically stabilizing correcting control. Clearly,

$$\nabla_x V(x,t) = \underbrace{\left[\phi_w^{-1}(x,t)\right]^T}_{x_0(t)^T}\left[\left.\frac{\partial\phi_w}{\partial x}\right|_{(\phi_w^{-1}(x,t),t)}\right]^{-1} \tag{2.10}$$

which makes the calculation of Δu computationally feasible.

2.3 The combined time-varying feedback control

Under our assumptions the following stabilization result is valid.

Theorem 2.1 *[3] Suppose that hypothesis H1-H4 are valid and that the TIP problem can be solved yielding $w(x,t)$. Further suppose that for any $x_0 \in B(0;R)$, the linearization of system (1.1), along the closed loop system trajectory corresponding to the periodic control $w(v(x_0),t)$, is a controllable time-varying system. If the gradient $\nabla_x V$ can be calculated at any point $(\phi_w^{-1}(x,t),t)$, then the control $u(x,t)$ given by (2.4), is a locally asymptotically stabilizing feedback law for system (1.1).*

The proof of can be found in [3], and uses a similar argument to that of Coron and Pomet [2], shown to be valid for systems with no drift.

3 Example

Consider the following single-input dynamical system Σ_o, on $\mathbb{R}^3$:

$$\Sigma_o : \qquad \dot{x} = f_0(x) + f_1(x)u_1$$

where,

$$f_0(x) = \begin{bmatrix} -x_2 + x_3^2 \\ -x_3 \\ 0 \end{bmatrix} \quad f_1(x) = \begin{bmatrix} 0 \\ 2x_3 \\ 1 \end{bmatrix}$$

System Σ_o is nilpotent, r=3, with $f_2 \stackrel{\text{def}}{=} [f_0, f_1]$ and $f_3 \stackrel{\text{def}}{=} [f_0,[f_0,f_1]]$. Following Lafferriere's approach, [4], a particular piece-wise constant solution to the TIP on the interval $[0,T]$, formulated in [1], is found to be:

$$w(v(x_0),t) = \begin{cases} \frac{9x_3(0)}{2T} + \frac{27}{T^2}\left(x_2(0) - x_3(0)^2\right) & t \in I_1 \\ -\frac{54}{T^2}\left(x_2(0) - x_3(0)^2\right) & t \in I_2 \\ -\frac{9x_3(0)}{2T} + \frac{27}{T^2}\left(x_2(0) - x_3(0)^2\right) & t \in I_3 \end{cases}$$

with the time intervals $I_k \stackrel{\text{def}}{=} \{t : nT + (k-1)\epsilon \leq t < nT + k\epsilon\}$ for $\epsilon = T/3,\ k = 1,2,3$ and $\forall\ n = 0,1,2,\ldots$.

The simulation results obtained by applying the proposed stabilizing feedback control to our example system are shown in Figure 1.

The numerical evaluation of the gradient involves retrieving the starting point x_0 at time t of the corresponding orbit generated with the control w which passes through the current state $x(t)$. The problem of numerically finding x_0 is solved here by seeking:

$$x_0 = \arg\min_{x_0\in\mathbb{R}^n}\|x(t) - \phi_w(x_0,0)\|^2 \tag{3.11}$$

The Levenberg-Marquardt modification of the Gauss-Newton method is employed as minimization procedure. The gradient $\nabla_x V$ is calculated here by using finite difference approximations to the partial derivatives needed.

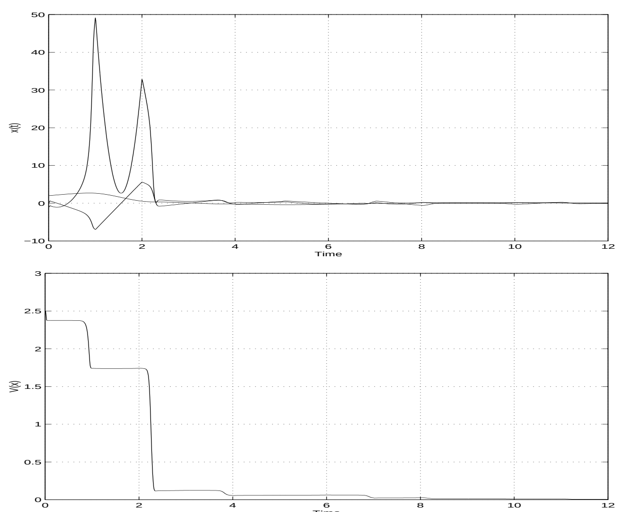

Figure 1: Closed loop trajectories $x(t)$ of the stabilized system and the Lyapunov function $V(t)$.

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