TIME VARYING STABILISING FEEDBACK DESIGN FOR BILINEAR SYSTEMS *

HANNAH MICHALSKA and MIGUEL A. TORRES-TORRITI

Department of Electrical Engineering, McGill University
3480 University Street, Montréal, H3A 2A7 Canada
e-mail: michalsk@cim.mcgill.ca

Abstract

A method is presented for construction of time varying stabilising feedback control for homogeneous bilinear systems. The method is universal in the sense that it is independent of the vector fields determining the motion of the system, and does not require a Lyapunov function. The proposed feedback law is a composition of a stabilising feedback control for a Lie bracket extension of the original system and a periodic continuation of a specific solution to an open loop control problem on the associated Lie group. The latter is posed as a trajectory interception problem in the logarithmic coordinates of flows.

Keywords: stabilising nonlinear feedback control, bilinear systems.

1. Introduction

The problem of feedback stabilisation of bilinear control systems whose equation of motion is given by

$$\dot{x}(t) = A_0x(t) + \sum_{i=1}^{m} u_i(t)A_ix(t) \qquad (1.1)$$

is considered. Here, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $A_i \in \mathbb{R}^{n\times n}, i = 0, 1, ..., m$. Systems of this type are of practical interest as they can be considered to be approximations to other nonlinear systems (such as those obtained by linearising nonlinear systems at a common fixed point with respect to the state x only).

The stabilisation problem for the above class of systems has found much interest in the literature, see [1] for a survey. Several methods for stabilising (1.1) start by finding a suitable Lyapunov function for the free system $\dot{x} = A_0x$ (assuming that A_0 is stable), [2], [3]. The proposed feedback is either linear or quadratic or piece-wise constant, and usually results in slower than exponential asymptotic stability.

Stabilisation of homogeneous bilinear systems in the plane has been fully analysed. Bacciotti and Boieri, [4], using constant, linear, and quadratic feedbacks, and Chabour et al., [5], using feedbacks differentiable except at zero, have given complete classifications of the possibilities for stabilisability of $\dot{x} = A_0x + uA_1x$ on $\mathbb{R}^2 - \{0\}$. The methods of analysis in these papers again involve Lyapunov functions, center manifolds, and properties of plane curves.

For higher dimensional systems, however, relatively few methods for feedback stabilisation are available. Although in [6], Wang gives a sufficient condition for stabilisability of systems in $\mathbb{R}^n$ by piece-wise constant controls, no general procedure for their construction is provided. There seems to be a total lack of constructive approaches to stabilisation of higher order systems for which the matrix $A_0 + \sum_{i=1}^{m} u_iA_i$ is unstable for all choices of constants u_i.

With this motivation we follow the idea, already mentioned in [1], of stabilising (1.1) by employing time-periodic feedback which brings into play the Lie brackets of the system matrices $A_0, A_1, ..., A_m$.

Our method is based on considering of what is known as the Lie bracket extension of the original system (1.1). Under reasonable assumptions, a stabilising feedback control is easy to construct for the extended system. The stabilising time-invariant feedback control for the extended system is then combined with a periodic continuation of a specific solution of an open loop, finite horizon control problem. This open loop control problem is posed in terms of the logarithmic coordinates for flows, [7], and its purpose is to generate open loop controls such that the trajectories of the controlled extended system and the open loop system intersect after a finite time T, independent of their common initial condition. It is hence, a finite horizon interception problem for the flows in the logarithmic coordinates. While the time-invariant feedback for the extended system dictates the speed of convergence of the system trajectory to the desired terminal point,

*Research supported by NSERC of Canada under grant OGPO-138352.

the open loop solution serves the *averaging* purpose in that it ensures that the "average motion" of the original system is that of the controlled extended system.

The construction proposed here demonstrates that synthesis of time varying feedback stabilisers for bilinear systems is possible and can be viewed as a procedure of combining static feedback laws for a Lie bracket extension of the system with a solution of an open loop control problem on the associated Lie group.

2. Problem definition and basic assumptions

The objective is to construct controls $u_i(x,t) : I\!R^n \times I\!R^+ \to I\!R$, $i = 1,...,m$ such that system (1.1) is Lyapunov asymptotically stable. For our construction to be valid, we need to impose the following hypotheses:

H1. System (1.1) is globally asymptotically controllable to zero using piece-wise continuous controls, i.e. for each initial condition $x_0 \in I\!R^n$ there exist piece-wise continuous controls $u_i : [0,\infty) \mapsto I\!R, i = 1,...,m$ such that the corresponding state trajectory converges to the origin.

H2. Let $\mathcal{G} \stackrel{def}{=} \{A_0, A_1, ..., A_m\}_{LA}$ denote the Lie algebra of n-dimensional square and real matrices generated by $A_0, A_1, ..., A_m$. System (1.1) satisfies the Lie algebra rank condition for accessibility, namely that

$$\dim\{A_0, ..., A_m\}_{LA}(x) = n \quad \forall x \in I\!R^n - \{0\}$$

where n is the dimension of the state and

$$\{A_0, ..., A_m\}_{LA}(x) \stackrel{def}{=} \text{span}\{Mx | M \in \mathcal{G}\}$$

H3. The Lie algebra $\{A_0, A_1, ..., A_m\}_{LA}$ has finite dimension k.

3. The feedback law

3.1 Stabilisation of the Lie bracket extension of the original system

We first consider the so called Lie bracket extension of the original system (1.1):

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{r} v_i(t) A_i x(t) \qquad (3.2)$$

where the additional matrices $A_i, i = m+1, ..., r$, are the Lie brackets of the original matrices $A_0, A_1, ..., A_m$ such that $\dim \text{span}\{A_1 x, ..., A_r x\} = n$ for all $x \in B(0;R) - \{0\}$, where $B(0;R)$ denotes a sufficiently large neighbourhood of the origin with $R >> 0$.

The stabilising feedback for the extended system is next defined by

$$v(x) \stackrel{def}{=} [v_1(x), ..., v_r(x)]^T = G(x)^\dagger (A_d x - A_0 x)$$
$$\text{where } G(x) \stackrel{def}{=} [A_1 x, ..., A_r x] \qquad (3.3)$$

and $G(x)^\dagger$ denotes the Moore-Penrose pseudo-inverse of the state dependent matrix G, and A_d is some stable "reference system" matrix. Since $G(x)G(x)^T$ is invertible for all $x \in I\!R^n - \{0\}$ (as it is in fact a Grammian matrix for a set of linearly independent vectors - the rows of $G(x)$), then the pseudo-inverse is calculated as $G(x)^\dagger = G(x)^T [G(x)G(x)^T]^{-1}$ and is a right inverse of $G(x)$ in $B(0;R) - \{0\}$, so that $G(x)G(x)^\dagger = I$ for all $x \in B(0;R) - \{0\}$. It follows that the trajectories of the extended system satisfy

$$\dot{x}(t) = A_0 x(t) + G(x(t)) v(x(t)) = A_d x(t) \qquad (3.4)$$

so that the extended system is stable, as desired.

3.2 The discretised extended control

Although the feedback given in (3.3) is exponentially stabilising for the extended system, it is not stabilising for the original system unless the action of the r feedback controls in (3.3) is somehow "translated" into a corresponding action of the $m < r$ controls of (1.1). To facilitate such a construction, we first consider substituting the controlled extended system by a system with "discretised" feedback control in which the feedback controls $v_i(x)$ are "updated" only at discrete moments of time $nT, n = 0, 1, 2, ...$:

$$\dot{x} = A_0 x(t) + \sum_{i=1}^{r} v_i^n(x) A_i x(t) \qquad (3.5)$$

in which the functions $v_i^n, i = 1, ..., r$ are obtained from $v_i, i = 1, ..., r$ by the formula

$$v_i^n(x(t)) \stackrel{def}{=} v_i(x(nT)) \qquad t \in [nT, (n+1)T),$$
$$n = 0, 1, 2, ... \quad i = 1, ..., r \qquad (3.6)$$

Hence, v_i^n is obtained from v_i by the "sample and hold" operation. It should be pointed out that such a discretisation is only needed if the extended controls are not constant, and is introduced in order to ensure that the feedback controlled extended system (3.5) has the same Lie algebraic structure as the original system (1.1) within each time interval $[nT, (n+1)T)$.

Intuitively, it is clear that sufficiently fine discretisation of the extended controls (reflected by a sufficiently small T) will preserve their stabilising properties; the latter is confirmed by the following result, see [8].

Proposition 1 *Suppose that hypotheses H1-H3 are valid, so that the controlled extended system given by*

(3.4) is globally exponentially stable. Under these conditions, for any region $B(0;R)$ there exists a constant $T>0$ such that the discretised controlled extended system (3.5) is exponentially stable with region of attraction $B(0;R)$.

3.3 An open loop control problem on a Lie group

The task of the open loop control problem discussed below is to generate open loop controls $u_i, i=1,...,m$ for the original system such that its trajectories and the trajectories of the discretised extended system (3.5) intersect periodically with the given frequency of discretisation $1/T$.

To this end, and for simplicity of notation, we define

$$a_i \stackrel{def}{=} v_i^n(x(t)), \quad i=1,...,r \tag{3.7}$$

$$\text{and} \quad a \stackrel{def}{=} [a_1,...,a_r]^T \tag{3.8}$$

which are constant over each interval $[nT,(n+1)T)$.

By virtue of hypothesis H3, without the loss of generality we also assume that

H4. The matrices $A_0,...,A_r$ form a basis for the algebra $\{A_0,...,A_m\}_{LA}$

In order to achieve a type of "point-wise equivalence" of the trajectories of the extended and the original system we pose the following open loop "trajectory interception problem" (TIP):

TIP: Find control functions $w_i(a,t)$, $i=1,...,m$, in the class of functions which are continuous in a, and piece-wise continuous and locally bounded in t , such that for *any initial condition* $x(0)=x\neq 0$ and any constant "control vector" a the trajectory $x^a(t;x,0)$ of the extended system:

$$\dot{x}(t)=A_0x(t)+\sum_{i=1}^{r}a_iA_ix(t), \quad x(0)=x \tag{3.9}$$

intersects the trajectory $x^w(t;x,0)$ of the original system with controls w_i:

$$\dot{x}(t)=A_0x(t)+\sum_{i=1}^{m}w_i(a,t)A_ix(t), \quad x(0)=x \tag{3.10}$$

precisely at time T, so that $x^a(T;x,0) = x^w(T;x,0)$.

Employing the powerful formalism of [9] is now essential as it enables to find a solution of TIP while abstracting from the actual form of the matrices $A_0,...,A_m$, any particular values of the initial condition x, as well as the extended system controls a_i.

To this end, the open loop control problem TIP is restated as a (FIP) - interception problem for the flows (transition matrices) of the original and extended equations. Writing the solution of (3.10) in the form $x^w(t;x,0)=\Phi(t,0)x$ we clearly have

$$\dot{\Phi}(t,0)=[A_0+\sum_{i=1}^{m}w_i(a,t)A_i]\Phi(t,0)$$
$$\text{with} \quad \Phi(0,0)=I$$

A similar equation obviously also holds for the flow of system (3.9), so that (FIP) can be stated as follows:

FIP: Consider the two formal initial value problems:

$$S1: \quad \begin{cases} \dot{S}^e(t)=[A_0+\sum_{i=1}^{r}a_iA_i]S^e(t) \\ S^e(0)=I \end{cases} \tag{3.11}$$

$$S2: \quad \begin{cases} \dot{S}^o(t)=[A_0+\sum_{i=1}^{m}w_i(a,t)A_i]S^o(t) \\ S^o(0)=I \end{cases} \tag{3.12}$$

Find control functions $w_i(a,t)$, $i=1,...,m$, in the class of functions which are continuous in a, and piece-wise continuous and locally bounded in t , such that the above flows (of the extended and original systems, respectively) intersect at time T, i.e. $S^e(T)=S^o(T)$, regardless of the values of the extended controls $a_i, i=1,...,r$.

3.4 Solution of the flow interception problem

It is now the result of [9] which implies that the solution to both initial value problems $S1$ and $S2$ has the same general form:

$$S^e(t)=\Pi_{i=1}^{r}\exp(g_i^e(t)A_i) \tag{3.13}$$

$$S^o(t)=\Pi_{i=1}^{r}\exp(g_i^o(t)A_i) \tag{3.14}$$

where the matrix exponentials are defined in the usual way:

$$\exp(g(t)A) \stackrel{def}{=} I+g(t)A+\frac{g(t)^2}{2!}A^2+... \tag{3.15}$$

and where $A_0,...,A_r$ is a basis for the algebra $\{A_0,...,A_m\}_{LA}$. The functions g_i^e and g_i^o, $i=1,...,r$ will be dependent on the control values $a_i, i=1,...,r$ and $w_i, i=1,...,m$, respectively. The representations (3.13)-(3.14) are generally only local (valid for sufficiently small times t). The latter can be shown to be global if the algebra has special properties (is solvable) and, or else in the case of real 2×2 systems.

A procedure for the construction of the functions g_i^e (or

g_i^o), which are known under the name of "logarithmic coordinates" of the corresponding flow, is also provided in [9]. It relies on substituting of the expression (3.13) into the equation (3.11) which it satisfies, and employing the well known Campbell-Baker-Hausdorff formula for rearranging the equation in such a way as to be able to equate the coefficients which correspond to the same basis elements $A_0, ..., A_r$ on its both sides. The result is a set of ordinary differential equations for the $g^e \stackrel{def}{=} [g_1^e, ..., g_r^e]^T$ of the form:

$$M\left(g^e(t)\right)\dot{g}^e(t) = a \tag{3.16}$$

where M is a real, analytic matrix function of the $g_i^e, i = 1, ..., r$. Generally, M is invertible only in the neighbourhood of zero. If, however, M^{-1} exists for all values of g^e then the representation (3.13) is global and the functions g^e are calculated by explicitly solving

$$\dot{g}^e(t) = M\left(g^e(t)\right)^{-1} a \quad \text{with } g^e(0) = 0 \tag{3.17}$$

The zero initial condition in (3.17) clearly follows from the well known property of transition matrices by which $S^e(0) = I$.

A similar solution procedure can, of course, be applied to the flow equation for the original system (3.14), which yields the final statement of the (TIP) , now with respect to the logarithmic coordinates of the corresponding flows:

LC: Consider the two formal "control systems":

$$CS1: \quad \dot{g}^e(t) = M\left(g^e(t)\right)^{-1} a, \quad g^e(0) = 0 \tag{3.18}$$

$$CS2: \quad \dot{g}^o(t) = M\left(g^o(t)\right)^{-1} w(a,t), \quad g^o(0) = 0 \tag{3.19}$$

where $w(a,t) \stackrel{def}{=} [w_1(a,t), ..., w_m(a,t), 0, ..., 0]^T$. Find control functions $w_i(a,t)$, $i = 1, ..., m$, in the class of functions which are continuous in a, and piece-wise continuous and locally bounded in t , such that the above logarithmic coordinates of flows (of the extended and original systems, respectively) intersect at time T, i.e. $g^e(T) = g^o(T)$, regardless of the values of the extended controls $a_i, i = 1, ..., r$.

Both (FIP) and (LC) are clearly independent of the initial condition $x(0) = x$ but the control functions $w(a,t)$ must be found in terms of the parameter a - the value of the dicretised extended controls.

The existence of solutions to (LC) is not obvious. Solutions to (LC) will however always exist if motion in the direction of any Lie bracket $A_i, i = m+1, ..., r$ can be realised by switching controls in the original system. The solution to (LC) is not unique.

Using Proposition 1, it is now possible to show the following stabilisation result for the closed loop system (1.1), [8].

Theorem 1 *Let a solution to (LC) exist and let $\hat{w}_i(a,t), i = 1, ..., m$ denote its periodic continuation with respect to the time variable. Under the hypotheses H1-H4, and with $v_i^n(x), i = 1, ..., r$ defined in (3.7), for any region $B(0;R)$, there exists a constant $T > 0$ such that the time varying feedback controls*

$$u_i(x,t) = \hat{w}_i(v_1^n(x), ..., v_r^n(x), t), \quad i = 1, ..., m \tag{3.20}$$

are asymptotically stabilising for system (1.1), with region of attraction $B(0;R)$.

The proof is technical and is omitted here for the sake of brevity but is based on the fact that the controls $v_1(0), ..., v_r(x)$ stabilise the the extended system exponentially and hence such a system exhibits some stability robustness margin. What is implied is that any controls $v_1(x) + \delta_1(x), ..., v_r(x) + \delta_r(x)$ with $|\delta_i(x)| \le \epsilon||x||, i = 1, ..., r$, for ϵ sufficiently small also stabilise the extended system. Now, by virtue of the previous discussion, the result of Proposition 1, and the fact that $\hat{w}_i(a,t), i = 1, ..., r$ solve the (TIP), the trajectory of the original system intersects the trajectory of the stabilised extended system with period T, and the distance between the two trajectory decays to zero as $||x|| \to 0$. In a sense then, the controls $\hat{w}_i(v(x),t)$ can be considered to be close to the $v_i(x)$ and stabilisation of the original system follows by virtue of robustness.

The above result can be further generalised to hold for systems which fail to satisfy hypothesis H3, i.e. for systems for which the Lie algebra $\{A_0, ..., A_m\}_{LA}$ is infinite dimensional, but which allow for sufficiently close finite dimensional approximations. The calculation of the parametrised periodic controls $\hat{w}(a,t)$ could then be carried out with respect to the approximate system with a finite dimensional algebra. The resulting control would prove effective when applied to the original system if a sufficiently large stability margin would be provided for in the initial design. Rigorous error estimates are very difficult to derive.

4. Example

For simplicity, the system to be stabilised is defined on the plane:

$$\dot{x}(t) = A_0 x(t) + u(t) A_1 x(t) \tag{4.21}$$

$$x(t) \stackrel{def}{=} [x_1(t), x_2(t)]^T \tag{4.22}$$

with the matrices A_0, A_1 given by:

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & 4 \end{bmatrix}; \quad A_1 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \tag{4.23}$$

It is worth noticing that the above system has the property that there exists no constant control u for which the system matrix A_0+uA_1 becomes stable (in terms of u, the eigenvalues of this matrix are $\lambda_{1/2} = 2\pm\sqrt{3-u}$) so stabilisation of (4.21) is non-trivial.

The Lie algebra $\{A_0, A_1\}_{LA}$ is indeed finite dimensional (actually of dimension equal to four) as shown by the following Lie bracket multiplication table in which the product of any two matrices A and B is calculated as their Lie bracket $[A, B] \stackrel{def}{=} BA - AB$.

	A_0	A_1	A_2	A_3
A_0	0	A_2	A_3	$12A_3$
A_1		0	$-2A_1$	$-2A_2$
A_2			0	$24A_1 - 2A_3$
A_3				0

where the following shorthand notation was used:

$$A_2 \stackrel{def}{=} [A_0, A_1] \quad (4.24)$$

$$A_3 \stackrel{def}{=} [A_0, [A_0, A_1]] \quad (4.25)$$

To facilitate further the derivation of the equations for the logarithmic coordinates of flows we will truncate the series expansions of the exponentials at Lie brackets of order one whenever employing the Campbell-Baker-Hausdorff formula:

$$\exp(\mathcal{A})\mathcal{B}\exp(-\mathcal{A}) = \mathcal{B} + [\mathcal{A}, \mathcal{B}] + \frac{1}{2!}[\mathcal{A}, [\mathcal{A}, \mathcal{B}]] + \frac{1}{3!}[\mathcal{A}, [\mathcal{A}, [\mathcal{A}, \mathcal{B}]]] + ...$$

$$\sum_{k=0}^{\infty} \frac{(\mathrm{Ad}\mathcal{A})^k}{k!}\mathcal{B} \stackrel{def}{=} \exp(\mathrm{Ad}\mathcal{A})\mathcal{B} \quad (4.26)$$

for any $\mathcal{A} = g(t)A$ and $\mathcal{B} = h(t)B$ where, by the bilinear nature of the Lie product, $[g(t)A, h(t)B] = g(t)h(t)[A, B]$, and where the operation Ad can be defined recursively as follows

$$(\mathrm{Ad}\mathcal{A})^k\mathcal{B} = (\mathrm{Ad}\mathcal{A})^{k-1}\mathrm{Ad}(\mathcal{A})\mathcal{B} \quad (4.27)$$

$$(\mathrm{Ad}\mathcal{A})\mathcal{B} \stackrel{def}{=} [\mathcal{A}, \mathcal{B}] \quad (4.28)$$

The latter amounts to stating that all the higher order Lie brackets of A_0 and A_1 can be assumed to be equal to zero. Considering this simplifying assumption, the extended system for (4.21) involves only the first order Lie bracket:

$$\dot{x}(t) = A_0x(t)+v_1(t)A_1x(t)+v_2(t)[A_0, A_1]x(t) \quad (4.29)$$

This simplification is possible due to the fact that

$$\mathrm{span}\ \{A_1x, [A_0, A_1]x\} = I\!R^2 \quad (4.30)$$

for all $x \in \mathcal{S} \stackrel{def}{=} \{x \in I\!R^2 | x_1 \neq 0\}$. Thus the matrix $G(x) \stackrel{def}{=} [A_1x, [A_0, A_1]x]$ is invertible on $\mathcal{S}$. The singularity of $G(x)$ on the complement of $\mathcal{S}$, $\mathcal{S}^C$, does not incur problems as, in this case, the un-forced system escapes $\mathcal{S}^C$.

The extended controls can thus be evaluated as

$$v(x) = G(x)^{-1}[A_dx - A_0x] \quad (4.31)$$

$$v(x) \stackrel{def}{=} [v_1(x), v_2(x)] \quad (4.32)$$

with a suitable choice for the reference system $\dot{x} = A_dx$.

The (FIP) now requires us to find a control $w(a_1, a_2, t)$ such that the flows $S^e(t)$ and $S^o(t)$, respectively satisfying:

$$\dot{S}^e(t) = (A_0 + a_1A_1 + a_2A_2)S^e(t), \quad S^e(0) = I \quad (4.33)$$

$$\dot{S}^o(t) = (A_0 + w(a_1, a_2, t)A_1)S^o(t), \quad S^o(0) = I \quad (4.34)$$

intersect at T.

It remains to derive the equations describing the evolution of the corresponding logarithmic coordinates and to solve the associated (LC). Assuming that the solution to (4.33) is of the form $S^e(t) = \exp(g_0(t)A_0)\exp(g_1(t)A_1)\exp(g_2(t)A_2)$, we calculate its time derivative as:

$$\dot{S}^e = [\dot{g}_0A_0 + \dot{g}_1\exp(g_0\mathrm{Ad}A_0)A_1 + \dot{g}_2\exp(g_0\mathrm{Ad}A_0)\exp(g_1\mathrm{Ad}A_1)A_2]S^e \quad (4.35)$$

Using the Campbell-Baker-Hausdorff formula (4.26) (with higher order brackets taken to be zero) yields:

$$\exp(g_0\mathrm{Ad}A_0)A_1 = A_1 + g_0A_2 \quad (4.36)$$

$$\exp(g_0\mathrm{Ad}A_0)\exp(g_1\mathrm{Ad}A_1)A_2 = A_2 \quad (4.37)$$

Substituting (4.35)-(4.37) into (4.33) and equating coefficients of $A_0, ..., A_2$ gives the control system CS1 of (LC):

$$CS1: \begin{cases} \dot{g}_0^e &= 1 \\ \dot{g}_1^e &= a_1 \\ \dot{g}_2^e &= -g_0^ea_1 + a_2 \end{cases} \quad (4.38)$$

Similarly the control system CS2 is

$$CS2: \begin{cases} \dot{g}_0^o &= 1 \\ \dot{g}_1^o &= w \\ \dot{g}_2^o &= -g_0^ow \end{cases} \quad (4.39)$$

It can be verified that one possible solution of the (LC) is

$$w(a_1, a_2, t) = a_1 + \frac{2\pi a_2}{T}\sin(\frac{2\pi}{T}t) \quad (4.40)$$

defined for $t \in [0, T]$. In terms of the continuous extended feedback controls the final stabilising control law is thus

$$\hat{w}(v_1(x), v_2(x), t) = v_1(x) + \frac{2\pi v_2(x)}{T}\sin(\frac{2\pi}{T}t) \quad (4.41)$$

which is now defined for $t \in [0, \infty)$, due to the periodic continuation of the sine.

One set of simulation results is presented and corresponds to a reference system in which $A_d = -\alpha I$, with gain $\alpha = 8$. The sampling period used was $T = 0.01sec$. Figures 1 and 2 show the extended system trajectory, and the corresponding extended controls, respectively. Figures 3 and 4 show the original system trajectory (in the phase plane) and the respective stabilising control $\hat{w}(v_1(x), v_2(x), t)$ in which the extended controls $v_i(x), i = 1, 2$ have been updated every $T/10sec$. Finally, Figure 5 displays the controlled system state variables vs. time.

5. Conclusions

In this paper we have investigated the possibility of constructing time-varying feedback stabilisers for homogeneous bilinear systems. The approach relies on the solution of a flow interception problem in terms of a set of parameters which represent the values of the stabilising controls for the extended system. Essentially, a closed form parametric solution of this problem is required. In some cases, such as the one presented in the example, a solution to the flow interception problem can be obtained analytically. Analytic solutions are usually impossible when the system has more complicated Lie algebraic structure since then the order and complexity of the evolution of the logarithmic coordinates of flows increases. No computationally practical approaches to this problem have been developed yet.

However, an undeniable advantage of this approach is that it applies to homogeneous bilinear systems of a general form, without any specific assumptions concerning the stability of the drift term nor the dimension of the system. Regarding that no alternative methods of similar generality exist as yet, our study seems worthwhile.

6. References

[1] D. L. Elliot, Bilinear Systems in the *Encyclopedia of Electrical Engineering*, J. Webster Ed., John Wiley & Sons Inc, 1999.

[2] M. Slemrod, Stabilisation of bilinear control systems with applications to nonconservative problems in elasticity. *SIAM J. Control and Optimisation*,Vol. 16, No. 1, 1978, pp. 131-141.

[3] G. Lu, Global asymptotic stabilisation of MIMO bilinear systems with undamped natural response. *Proceedings of the 37th IEEE Conference on Decision and Control*, Tampa, USA, 1998, pp. 1989-1994.

[4] A. Bacciotti and P. Boieri, A characterisation of single input planar bilinear systems which admit a smooth stabiliser. *Systems and control letters*, Vol. 16, 1991, pp. 139-144.

[5] R. Chabour, G. Sallet and J. C. Vivalda, Stabilisation of nonlinear systems: a bilinear approach. *Mathematics of Control, Signals and Systems*, Vol. 6, 1993, pp. 224-246.

[6] H. Wang, Feedback stabilisation of bilinear control systems. *SIAM J. Control and Optimisation*,Vol. 36, No. 5, 1998, pp. 1669-1684.

[7] G. Lafferriere and H. J. Sussmann, A differential geometric approach to motion planning, in *Nonholonomic Motion Planning*, Z. Li, and J. F. Canny Eds., Kluwer Academic Publishers, 1993, pp. 235-270.

[8] H. Michalska, Synthesis of time varying stabilising feedback for drift free systems, *Internal Research Report*, Dept. of Electrical Engineering, McGill University, Montreal, 1996.

[9] J. Wei and E. Norman, On global representations of the solutions of linear differential equations as products of exponentials. *Proceedings of the American Mathematical Society*, 1964, pp. 327-334.

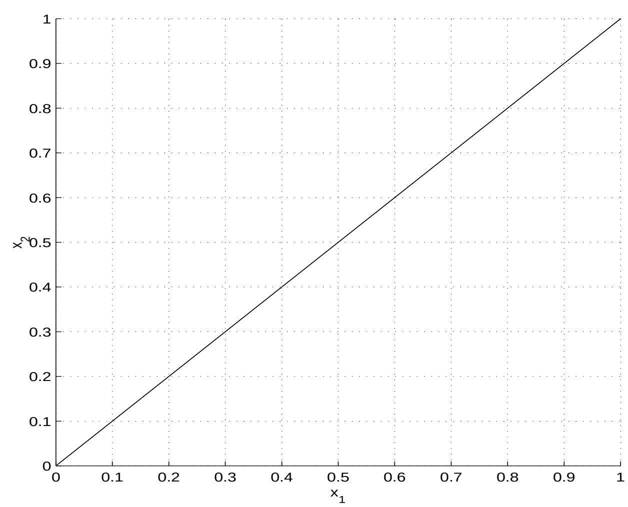

Figure 1: Plot of the extended system trajectory in the phase plane.

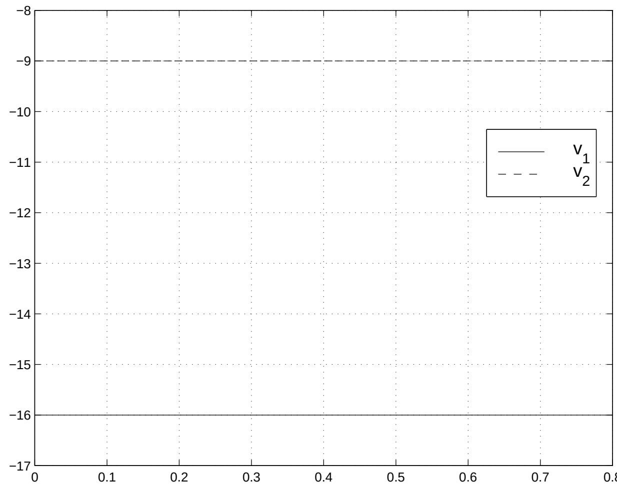

Figure 2: Plot of the extended system controls $v_1(x), v_2(x)$.

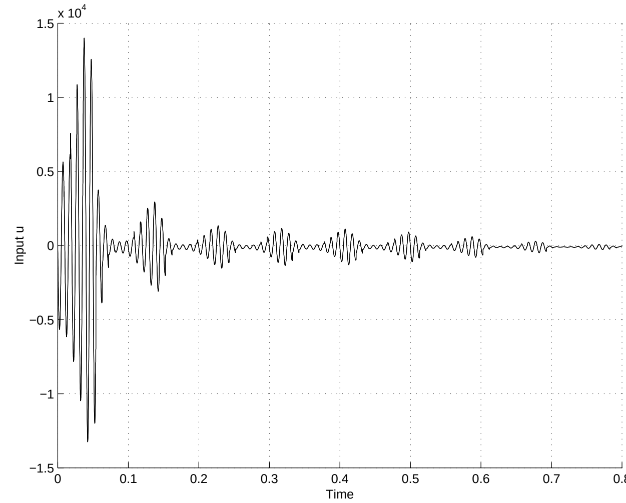

Figure 4: Plot of the stabilising control $u(x,t) = \hat{w}(v_1(x), v_2(x), t)$.

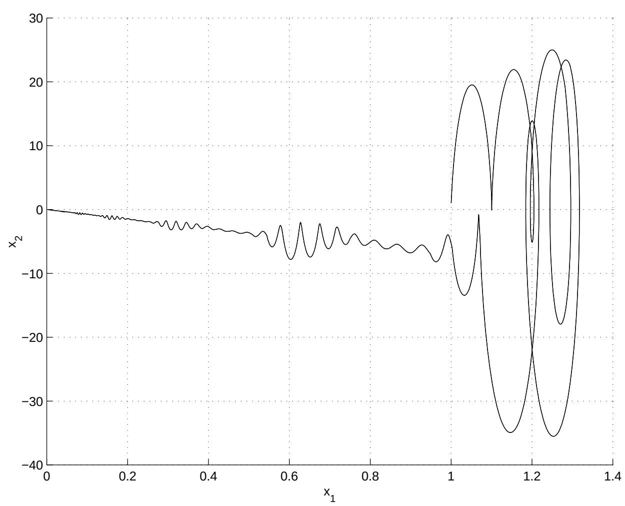

Figure 3: Plot of the original controlled system trajectory in the phase plane.

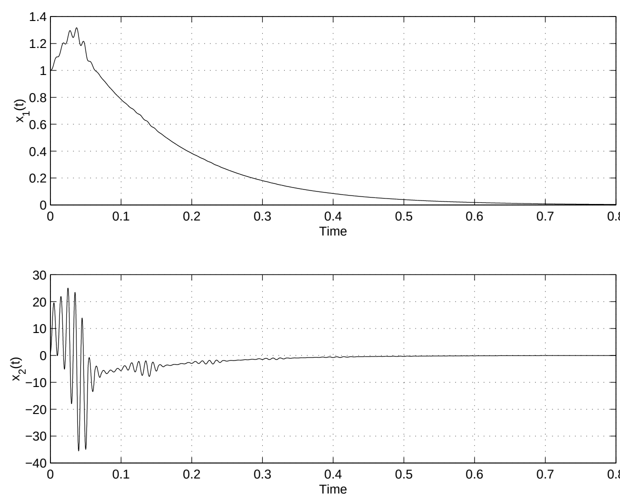

Figure 5: Plot of the original controlled system state variables x_1, x_2 versus time.