Projective Geometry of Image Formation

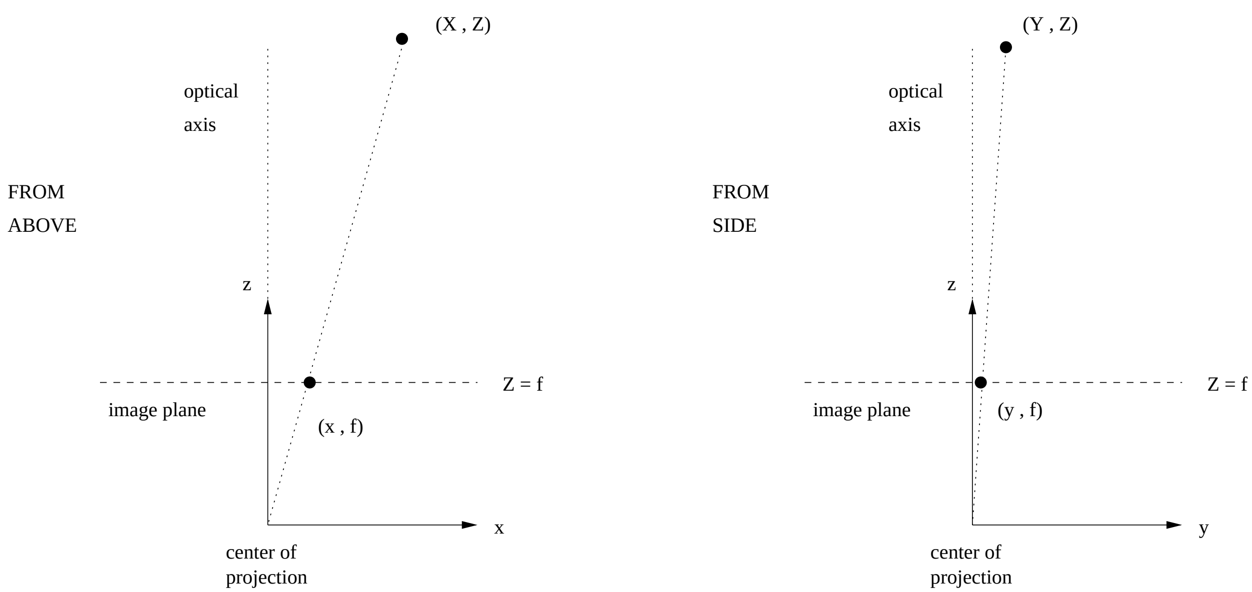
We begin by reviewing some of the basic geometry of image formation. How do points in 3D project to a 2D image? Consider one viewer i.e. a camera. Define the camera coordinate system (X, Y, Z) for the 3D points in the scene as follows. Let the position of the viewer be the origin $(0, 0, 0)$. Let the camera point in direction $(0, 0, 1)$, that is, in the direction of the Z axis. The Z axis is called the *optical axis*.

Next consider a 3D scene point (X_0, Y_0, Z_0) in this camera coordinate system. Consider the line from the origin through this point. We intersect this line with the plane $Z = f$. The plane $Z = f$ is called the *image plane* or *projection plane*. The point of intersection is the *image position*. The origin is the *center of projection*.

Using similar triangles, you can see that the image coordinates of the projected point are

$$(x, y) \;=\; (\frac{X_0}{Z_0}\, f, \frac{Y_0}{Z_0}\, f). \tag{1}$$

Of course, real cameras have the image plane behind the center of projection, so real images are upside down and backwards. We will return to this issue a few lectures from now.



Example: General plane

Suppose the scene consists of a single plane

$$aX + bY + cZ = d. \tag{2}$$

Multiplying both sides of the equation by f/Z, we get

$$\frac{afX}{Z} + \frac{bfY}{Z} + cf = \frac{fd}{Z}$$

and so

$$ax + by + cf = \frac{fd}{Z}$$

which defines a plane in the 3D space defined by coordinates $(x, y, \frac{1}{Z})$.

Notice what happens if we let $Z \rightarrow \infty$, namely we get the line

$$ax + by + cf = 0.$$

This line is sometimes called the *line at infinity.* In more familiar terms, it is called the *horizon.*

Ground plane and the horizon

Let's consider a specific example. Suppose the only visible surface is the ground, which we approximate as a plane. Suppose the camera is above this *ground plane* and is pointing in the Z direction, and so

$$Y = h$$

where $h < 0$.

From Eq. (1), we have

$$y = \frac{hf}{Z} \tag{3}$$

Observe that any fixed value of y defines a horizontal line in the image, and scene points that project to that line have the same depth (independent of x). Similarly, points of a fixed depth $Z = Z_0$ all project to the same y value. In particular, the larger the depth, the nearer the y value is to 0. In the limit as $Z \rightarrow \infty$, we have $y \rightarrow 0$, which define the horizon. Notice that the horizon passes through the center of the image. (This depends on our assumption that the optical axis of the camera is parallel to the plane.)