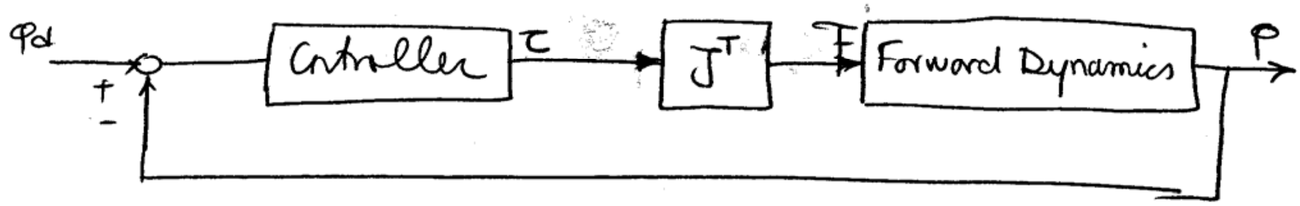
5.4.2 Forward Dynamics

Forward Dynamics is required if the manipulator dynamics needs to be simulated on computers. in Forward Dynamics τ is given and the moving platform position/orientation is calculated.



if the experimental setup for the manipulator exists, the control algorithm can be implemented on the System, without need to calculate the Forward Kinematics, usually only inverse Kinematics is used for Control algorithms as explained in the next chapter.

In general formulating F. Dynamics is very complicated for the parallel manipulators, and No closed-form solution can be found in general. for 6DOF SGP, though, general solution for forward Kinematics is given in [Khalil 04], but because of the details of manipulation it is recommended to be read for interested Students. This paper is available at Course Webpage.

5.5.1 Numerical Method for integration

As explained before using the derivation of Dynamic Eq. as given By Nguyan & Pooran, or as in Lebret et. al. the Dynamic Eq is formulated by given Mass matrix, and the Gravity and Coriolis & Centrifugal Torque vectors as

$$\mathcal{T} = D\ddot{\varphi} + H(\varphi, \dot{\varphi}) + G(\varphi)$$

OR

$$\mathcal{T} = D\ddot{\varphi} + C(\varphi, \dot{\varphi})\dot{\varphi} + G(\varphi)$$

To integrate this equation many numerical method can be used. Runge-Kutta integration method is the most popular one, in which if the Dynamic Eq. is written in form of a State-space representation

$$\dot{x} = f(x, u)$$

with a Set of initial Conditions $X(0) = x_0$ can be integrated using "ode23" or "ode45" of Matlab, the syntax of using this function is:

$$[T, X] = \text{ode45}(\text{odefun}, \text{tspan}, X0)$$

in which the above function $f(x,u)$ is given in odefun & the time of integration is given in tspan $= [t_0, t_f]$ (or tspan $= [t_0, t_1, t_2, \dots t_f]$) hence to use the robot function for integration we need only to generate the function in state-space format as:

$$\ddot{\varphi} = D^{-1}\{F - H(\varphi, \dot{\varphi}) - G(\varphi)\}_{n\times 1}$$

use first derivatives too:

$$\dot{\varphi} = \omega$$
$$\dot{\omega} = \ddot{\varphi} = D^{-1}\{F - H(\varphi, \dot{\varphi}) - G(\varphi)\}$$

$$\rightarrow \quad x = \begin{bmatrix} \varphi \\ \omega \end{bmatrix}_{2n\times 1}, \quad f(x,u) = \begin{Bmatrix} \omega \\ D^{-1}\{F - H(\varphi, \dot{\varphi}) - G(\varphi)\} \end{Bmatrix}_{2n\times 1} \qquad (I)$$

you need to give the time of integration $tspan = [t_0, t_f]$

and the initial values $x_0 = \begin{bmatrix} \varphi_0 \\ \dot{\varphi}_0 \end{bmatrix} = \begin{bmatrix} \varphi_0 \\ \omega_0 \end{bmatrix}_{2n\times 1}$

To avoid using inverse of Mass matrix in the function definition (I)

most of ode solver can accept explicit mass matrix in the definition

of functions as $\quad M(t,x)\,\dot{x} = f(t,x)$

the Mass matrix can be given in a seperate function called Massfun

and set for the Solver using "odeset" function

```
options = odeset('Mass', @Massfun)
[t, x] = ode45(@odefun, tspan, x0, options)
```

Similar Solver exist for Simulink, which is left for student to

explore

Implicit function integrations

As seen in previous sections, usually for parallel manipulators it is hard to obtain M, H and G terms explicitly and the Dynamic Eq of the parallel robot is derived in an implicit format:

$$\mathcal{F} = f(x, \dot{x}, \ddot{x})$$

fortunately ∃ a special function in Matlab to Solve implicit Dy Equations and for cases in which M, H & G terms are not explicitly determined 'ode15i' can be used as a mean for integration,

The Syntax is as follows:

$$[T, X] = \text{ode15i}[(@\text{odefun}, \text{tspan}, x_0, x_{p0})$$

in which the function should contain only first derivatives, but can be implicit: $\quad f(t, x, \dot{x}) = 0$

we use the same algorithm as in other solvers to reduce the 2^{nd} order system into a set of first order functi

$$\begin{aligned} \dot{x} &= v \rightarrow \\ \dot{v} &= \ddot{x} \Rightarrow \end{aligned} \; F = \begin{cases} \dot{x} - v \\ f(x, v, \dot{v}) = 0 \end{cases}$$

your state would be $y = \begin{bmatrix} \underset{\sim}{x} \\ \underset{\sim}{v} \end{bmatrix}$ & your fuction $\Rightarrow F = \begin{bmatrix} \dot{\underset{\sim}{y}}_1 - \underset{\sim}{y}_2 \\ f(y_1, y_2, \dot{y}_2) = 0 \end{bmatrix}$

The ode function is formed as the residue of $F(x) - 0 = res$

hence for the parallel manipulator, we have:

$$f(x, \mathring{x}, \mathring{\mathring{x}}) - \mathcal{F} = 0$$

using $y = \begin{bmatrix} \underset{\sim}{x} \\ \underset{\sim}{v} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{x} \\ \underset{\sim}{\mathring{x}} \end{bmatrix}$ we have

$$res = \begin{cases} \mathring{y}_1 - y_2 \\ f(y_1, y_2, \mathring{y}_2) - \mathcal{F} \end{cases}$$

But since the function is implicit you need to provide x_0, x_{p_0} which corresponds to y_0 & $\mathring{y}_0$ at initial states, usually $y_0 = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$ are known but $y_{p_0} = \begin{bmatrix} v_0 \\ \mathring{v}_0 \end{bmatrix}$ at least for $\mathring{v}_0$ are not known and might be not zero, to have consistent initial conditions the f(Cond) [sic]

differential Equation Consistent Initial Condition: decic

is used with the syntax of:

$$[y_0, y_{p_0}] = \text{decic}[@\text{odefun}, t_0, y_0, [1\ 0\ 1 \ldots], y_{p_0}, [0\ 1 \ldots]]$$

in which y_0 & y_{p_0} is your initial guess, and you use [1]s to keep these values corresponding to y_0 & y_{p_0} & [0] if you allow to change them, hence [ ] allows all the elements to be changed and

[1 1 1 ... 1] does not allow any element to be changed

6.0 Review of Lyapunov Analysis:

As seen in previous chapter the Equations of Motion of a general parallel manipulator can be written as a state-space Nonlinear Eq.

$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}, u)$$

in which $\underset{\sim}{x}$ is the state vector & u is the control input, the objective of this chapter is to design the control input, u, as a function of x, such that the closed loop system becomes or retain its stability, and be able to track a prescribed trajectory $x_d(t)$ with minimum error. hence if

$$u = g(x) \quad \text{is designed}$$

then $$\dot{x} = f(x, g(x)) = f(x)$$

the formulation $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x})$ is a general Nonlinear dynamic equation for a closed loop system or an unforced System $^{u=0}$

to define Stability, first consider the dynamic property of system:

6.0.1. Equilibrium point. x^*

a state x^*, or we call a point x^* is an equilibrium point for ... $\dot{x} = f(x)$ if as $t \to \infty$ the trajectories $x(t) \to x^*$. The equilibrium point can be derived from

$$\dot{x}^* = f(x^*) = 0$$

Since at steady-state no more variation in $x(t)$ is seen

The stability of the Nonlinear system is defined usually locally in the vicinity of each equilibrium point, note that since the above equation can have multiple Solution, and hence we may observe multiple behavior in a nonlinear System.

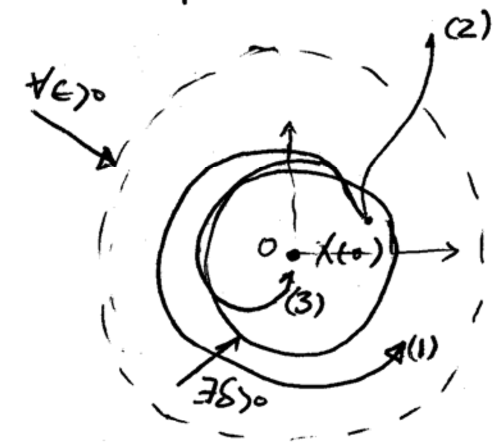
Let us rigourously define different kind of stability for a nonlinear system formulated as $\dot{x} = f(x)$ (1)

Def: The equilibrium point $\underset{\sim}{x} = 0$ of (1) is

- stable if for each $\epsilon > 0$, $\exists\, \delta(\epsilon) > 0 \ni$
  $\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon \quad \forall t \geq 0$
- unstable, if not stable
- asymptotically stable, if it is stable & δ can be chosen $\ni$
  $\|x(0)\| < \delta \Rightarrow \lim x(t) = 0$

Note1: the assumption that x^* is at origin $x^* = 0$ provides no limitation to the problem, since if $x^* \neq 0$ by use of a change of variable $y = x - x^*$, y^* will become at equilibrium @ origin.

Note2: Geometrical Representation



Note 3 : The most useful stability closer to the definition of stability for Linear System is asymptotic stability, and we are seeking asy. stab. for Robotic applications.

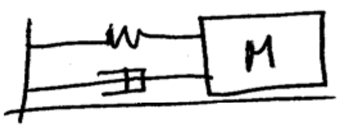
Note 4 :

Lyapunov Direct Method

The philosophy of Lyapunov's direct method for stability analysis of a nonlinear System is the mathematical extension of a fund. physical observation:

If the total energy of a system is continuously dissipating then the System must eventually settle down to an eq. point

Start with an example: M-Spring-damper with Nonlinear stiffness $F_S = K_0 x + K_1 x^3$ and " damping $F_D = b\dot{x}|\dot{x}|$



$$m\ddot{x} + b\dot{x}|\dot{x}| + k_0 x + k_1 x^3 = 0$$

Suppose we have large distances from origin $(x = \dot{x} = 0)$, will the resulting Motion becomes Stable?!!

It is very hard to use the definition of the stability, Examine the Energy (total) of system

Total Energy $V(x) = K_E + P_E = \frac{1}{2} m\dot{x}^2 + \int_0^x F_S \, dx$

$$V(x) = \frac{1}{2} m\dot{x}^2 + \int_0^x (k_0 x + k_1 x^3) dx$$

$$V(x) = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} K_0 x^2 + \frac{1}{4} k_1 x^4$$

Compare the definition of stability corresponding to total Energy

- zero energy corresponds to the equilibrium point ($x=\dot{x}=0$).
- asymptotic stability implies the convergence of total energy to zero.
- instability is related to the growth of energy.

Hence the stability is related to the variation of the total energy

$$\dot{V}(x) = m\dot{x}\ddot{x} + (k_0 x + k_1 x^3)\dot{x}$$

→ here the Equation of motion of system comes into picture.

$$\dot{V}(x) = \dot{x}(m\ddot{x} + k_0 x + k_1 x^3)$$
$$= \dot{x}(-b\dot{x}|\dot{x}|) = -b|\dot{x}|^3 < 0$$

The energy of the system is continuously dissipating until $\dot{x}=0$.

To generalize this approach, look that the energy function has the following properties:

1) $V(x)$ is strictly positive $V(x)>0$ except at origin $V(x)\big|_{\substack{x=0\\ \dot{x}=0}}=0$

2) $\dot{V}(x)$ is monotonically decreasing.

Def: A scalar continuous function $V(x)$ is locally positive definite if $V(0)=0$ and in a Ball B_{R_0} $V(x)>0$ if $x \neq 0$.

Note: if the above holds for the whole space $\rightarrow$ globally positive definite.

Related concepts:

$V(x)$ is negative definite: if $-V(x)$ is pos. def.

$V(x)$ is positive semi definite if $V(0)=0$ and $V(x) \geq 0$ for $\underset{\sim}{x} \neq \underset{\sim}{0}$

Theorem: (Lyapunov Local Stability)

If in a ball B_{R_0}, $\exists$ a function $V(x)$ which is pos. def, and has continuous partial derivatives, and if its time derivative $\left(\dot{V}(x) = \frac{\partial V}{\partial x} \cdot \dot{x}\right)$ along any state trajectory is

- negative semi definite then eq. point $\underset{\sim}{0}$ is stable.
- negative definite then eq. point 0 is asymptotically stable.

Theorem: (Lyapunov Global Stability)

If $\forall x \in \mathbb{R}^n$, $\exists$ a function $V(x)$ with continuous first order derivatives, $\ni$

- $V(x)$ is pos. def
- $\dot{V}(x)$ is neg. def
- $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$ (Radial unboundedness)

then the eq. point $\underset{\sim}{0}$ is "Globally Asymptotically Stable"

Example: Consider a class of nonlinear system, (1^{st} order)

$$\dot{x} + f(x) = 0 \quad \text{where} \quad x f(x) > 0 \text{ for } x \neq 0$$

Lyapunov Candidate
$$\begin{cases} V = x^2 > 0 \\ \dot{V} = 2x\dot{x} = -2x f(x) < 0 \\ V \to \infty \text{ as } \|x\| \to \infty \end{cases}$$

$\Rightarrow$ the System is Globally asymptotically stable at $x=0$.

this example includes systems as $\dot{x} = \sin^2 x - x$
or $\dot{x} = -x^3$

Note1: All Theorems are sufficiency Theorem meaning that

if $\exists\ V(x) > 0 \Rightarrow \dot{V}(x) < 0 \Rightarrow$ eq. point is asy. stable

but if for $V(x) > 0 \to \dot{V}(x)$ is indefinite $\Rightarrow$ No conclusion

Look for another $V(x)$

Note 2: How to find a Lyapunov function:

Start with $V(x)$ as the total energy of the System

$$V(x) = K_E + P_E$$

and check wheather $\dot{V}(x)$ is dissipative (passivity)

As we explained before we are looking forward to asy. Stability what if $\dot{V}(x) \leq 0$?

Theorem (Lasalle's Theorem)

Consider the system $\dot{x} = f(x)$ with f continuous, and let $V(x)$ be a scalar function with continuous partial derivatives, Assume that in a certain neighborhood Ω of the origin

- $V(x)$ is locally positive defenite
- $\dot{V}(x)$ is negative semi-defenite
- Let R defined by $\dot{V}(x) = 0$ the set of points where $\dot{V}(x)$ becomes zero. Consider R contains no trajectories of the system other than the trivial one $x \equiv 0$.

Then the eq. point $\underline{0}$ is locally asymptotically stable.

Example: pendulum with viscous damping

$$\ddot{\theta} + \dot{\theta} + \sin\theta = 0$$

equilibrium point $x^* = \begin{bmatrix} \theta^* = 0 \\ \dot{\theta}^* = 0 \end{bmatrix}$ = origin $\quad x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$

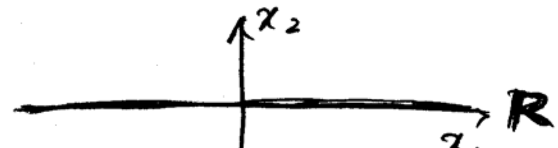
Lyapunov Candidate: $V(x) = (1 - \cos\theta) + \dfrac{\dot{\theta}^2}{2}$ (How? Total energy)

$$\dot{V}(x) = \dot{\theta}\sin\theta + \dot{\theta}\ddot{\theta} = \dot{\theta}(\ddot{\theta} + \sin\theta) = \dot{\theta}(-\dot{\theta}) = -\dot{\theta}^2$$

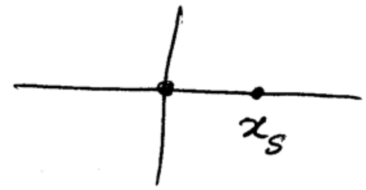
$$\dot{V}(x) = -\dot{\theta}^2 \leq 0$$ neg. semi defenite, why?

because $\dot{V}(x) = 0$ at $x = \begin{bmatrix} \text{any } \theta \\ 0 \end{bmatrix}$ and not only at $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R = \{ x \in \mathbb{R}^2 \mid x_1 = \text{anything},\ x_2 = 0 \}$



Can any trajectory be in R?



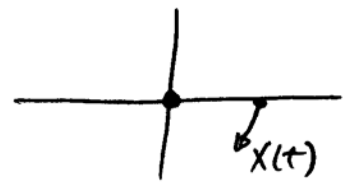
Consider a point on R

$\underset{\sim}{x}_s = \begin{bmatrix} x_s \\ 0 \end{bmatrix} \in R$ @ x_s we have $\begin{cases} \theta_s \neq 0 \\ \dot{\theta}_s = 0 \end{cases}$

$\Rightarrow \ddot{\theta}_s + \cancel{\dot{\theta}_s}^{\,0} + \sin\theta_s = 0 \Rightarrow \ddot{\theta}_s = -\sin\theta_s \neq 0$

hence at an incremental time Δt $\dot{\theta} = -\sin\theta_s \cdot \Delta t < 0$

the trajectory have the shape of



hence No trajectory other than the equilibrium point is included in R

$\rightarrow$ Lasalle's Theorem conditions holds

$\Rightarrow x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an asymptotically stable point.