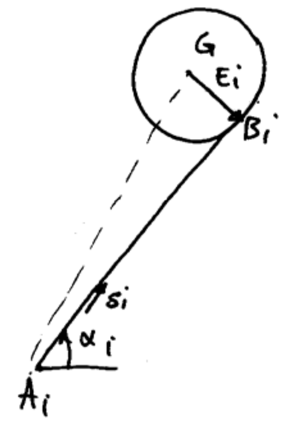
Project Part II Solutions

Dynamic Analysis Based on Newton-Euler Formulation

1) Acceleration analysis

Start with the same loopclosure as before



$$\overrightarrow{A_iG} + \overrightarrow{GB_i} = \overrightarrow{A_iB_i} \qquad (1)$$

differentiate: Note that $\dot{s}_i = \dot{\alpha}_i (k \times s_i)$, $\dot{E}_i = \dot{\varphi}(k \times E_i)$

$$\underset{\sim}{x}_G + E_i = L_i \hat{s}_i$$

$$v_G + \dot{\varphi}(k \times E_i) = \dot{L}_i s_i + L_i \dot{\alpha}_i (k \times s_i) \qquad (2)$$

Dot multiply with $\hat{s}_i$ to cancell $\dot{\alpha}_i$

$$v_G \cdot \hat{s}_i + \dot{\varphi} k (E_i \times s_i) = \dot{L}_i \longrightarrow \boxed{\dot{L}_i = \underbrace{\begin{bmatrix} s_{ix} & s_{iy} & E_{ix} s_{iy} - s_{ix} E_{iy} \end{bmatrix}}_{J_x} \begin{bmatrix} v_{Gx} \\ v_{Gy} \\ \dot{\varphi} \end{bmatrix} \; (\mathrm{I})}$$

Cross multiply to $\hat{s}_i$ to find $\dot{\alpha}_i$

$$s_i \times v_G + \dot{\varphi} \overbrace{s_i \times (k \times E_i)}^{k(E_i \cdot s_i)} = L_i \dot{\alpha}_i \overbrace{s_i \times (k \times s_i)}^{k}$$

$$\downarrow$$

$$k(v_{Gy} s_{ix} - v_{Gx} s_{iy}) + k(\dot{\varphi}(E_i \cdot s_i)) = k(L_i \dot{\alpha}_i) \rightarrow \boxed{\dot{\alpha}_i = \overbrace{\frac{1}{L_i} \begin{bmatrix} -s_{iy} & s_{ix} & E_i \cdot s_i \end{bmatrix}}^{J_\alpha} \begin{bmatrix} v_{Gx} \\ v_{Gy} \\ \dot{\varphi} \end{bmatrix} \; (\mathrm{II})}$$

differentiate (2) and note again that $\dot{s}_i = \dot{\alpha}_i (k \times s_i)$; $\dot{E}_i = \dot{\varphi}(k \times E_i)$

$$a_G + \ddot{\varphi}(k \times E_i) + \dot{\varphi}^2 \overbrace{k \times (k \times E_i)}^{-E_i} = \ddot{L}_i \hat{s}_i + 2\dot{L}_i \dot{\alpha}_i (k \times \hat{s}_i) + L_i \ddot{\alpha}_i (k \times \hat{s}_i) + L_i \dot{\alpha}_i^2 \overbrace{k(k \times \hat{s}_i)}^{-s_i}$$

$$a_G + \ddot{\varphi}(k \times E_i) - \dot{\varphi}^2 E_i = (\ddot{L}_i - L_i \dot{\alpha}_i^2)\hat{s}_i + (2\dot{L}_i \dot{\alpha}_i + L_i \ddot{\alpha}_i)(k \times \hat{s}_i) \qquad (3)$$

Dot multiply (3) with s_i to cancell $\ddot{\alpha}_i$

$$a_G \cdot s + \ddot{\varphi} k (E_i \times s_i) - \dot{\varphi}^2 (E_i \cdot s_i) = \ddot{L}_i - L_i \dot{\alpha}_i^2$$

$$\Rightarrow \boxed{\ddot{L}_i = \underbrace{\begin{bmatrix} s_{ix} & s_{iy} & E_{ix} s_{iy} - s_{ix} E_{iy} \end{bmatrix}}_{J_x} \cdot \begin{bmatrix} a_{Gx} \\ a_{Gy} \\ \ddot{\varphi} \end{bmatrix} + L_i \dot{\alpha}_i^2 - \dot{\varphi}^2 (E_i \cdot s_i)} \qquad (\mathrm{III})$$

Now cross multiply (3) to $\hat{s}_i$ to cancell $\ddot{L}_i$ and find $\ddot{\alpha}_i$

$$s_i \times a_G + \ddot{\varphi}\, \overbrace{s_i \times (k \times E_i)}^{k(E_i \cdot S_i)} - \dot{\varphi}^2 S_i \times E_i = (2\mathring{L}_i \dot{\alpha}_i + L_i \ddot{\alpha}_i)\, \overbrace{s_i \times (k \times s_i)}^{k}$$

$$s_i \times a_G + \ddot{\varphi}\, k (E_i \cdot S_i) - \dot{\varphi}^2 (S_i \times E_i) = (2\mathring{L}_i \dot{\alpha}_i + L_i \ddot{\alpha}_i)\, k$$

$$\boxed{\frac{1}{L_i}\left\{ \left[-S_{iy} \,|\, S_{ix} \,|\, E_i \cdot S_i\right] \begin{bmatrix} a_{Gx} \\ a_{Gy} \\ \ddot{\varphi} \end{bmatrix} - \dot{\varphi}^2 (S_{ix} E_{iy} - S_{iy} E_{ix}) - 2\mathring{L}_i \dot{\alpha}_i \right\} = \ddot{\alpha} \qquad (\mathrm{IV})}$$

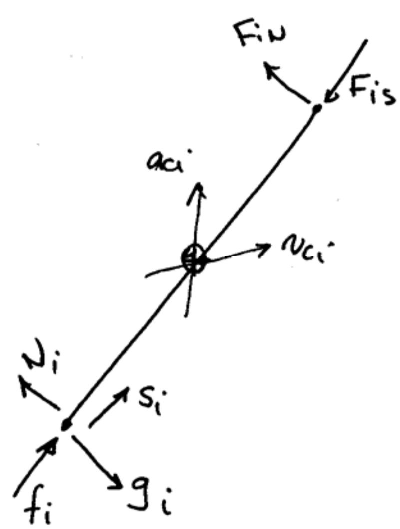
Now we have all the required component to calculate the accelerations of Center of manes of each limb:

$$x_{c_i} = \frac{1}{2} L_i \hat{s}_i \rightarrow v_{c_i} = \frac{1}{2}\left\{ \mathring{L}_i \hat{s}_i + L_i \dot{\hat{s}}_i \right\} = \frac{1}{2}\left\{ \mathring{L}_i s_i + L_i \dot{\alpha}_i (k \times \hat{s}_i) \right\}$$

$$a_{c_i} = \frac{1}{2}\left\{ \ddot{L}_i s_i + 2\mathring{L}_i \dot{\alpha}_i (k \times \hat{s}_i) + L_i \ddot{\alpha}_i (k \times \hat{s}_i) + L_i \dot{\alpha}_i^2 \overbrace{k \times (k \times \hat{s}_i)}^{-s_i} \right\}$$

$$a_{c_i} = \frac{1}{2}\left\{ (\ddot{L}_i - L_i \dot{\alpha}_i^2)\hat{s}_i + (\ddot{\alpha}_i L_i + 2\mathring{L}_i \dot{\alpha}_i)(k \times \hat{s}_i) \right\}$$

2) Dynamic Equations of each limb



Free-Body diagram:

Consider; f_i: actuator force

F_{iN} & F_{is} the forces at Point B_i

we have variable man

N-E equations: First Consider N-E

N-E $\quad (f_i - F_{is})\hat{s}_i + (F_{iN} - g_i)\hat{N}_i = m_i a_{c_i} + \dot{m}_i v_{c_i}$

E-E: $\quad \sum M_{A_i} = F_{iN} \cdot L_i = {}^{A_i}I \cdot \ddot{\alpha}_i + {}^{A_i}\dot{I} \cdot \dot{\alpha}_i$

because of change of man consider: ρ: density of limb/length

$$m_i = \rho L_i \rightarrow \dot{m}_i = \rho \mathring{L}_i \quad \& \quad {}^{A_i}I = \frac{1}{3} m L_i^2 = \frac{\rho}{3} L_i^3 \rightarrow {}^{A_i}\dot{I} = \rho L_i^2 \mathring{L}_i$$

Rewite N-E Equation componentwise:

$$
\text{N-E}\left\{\begin{array}{l} \xrightarrow{S_i} \\ \xrightarrow{N_i} \end{array}\right.
\quad
\left\{\begin{array}{l}
f_i - F_{iS} = \dfrac{\rho L_i}{2}\left\{(\ddot{L}_i - L_i\dot{\alpha}_i^2)\right\} + \dfrac{\rho \dot{L}_i^2}{2} = \rho/2\left\{L_i\ddot{L}_i - (L_i\dot{\alpha}_i)^2 + \dot{L}_i^2\right\} \\[2mm]
F_{iN} - g_i = \dfrac{\rho L_i}{2}\left\{\ddot{\alpha}_i L_i + 2\dot{L}_i\dot{\alpha}_i\right\} + \dfrac{\rho \dot{L}_i}{2}\left\{L_i\dot{\alpha}_i\right\} \\[2mm]
F_{iN} = \dfrac{1}{L_i}\left\{\dfrac{\rho}{3}L_i^3(\ddot{\alpha}_i) + \rho L_i^2\dot{L}_i(\dot{\alpha}_i)\right\} = \rho\left\{\dfrac{1}{3}L_i^2\ddot{\alpha}_i + L_i\dot{L}_i\dot{\alpha}_i\right\}
\end{array}\right.
$$

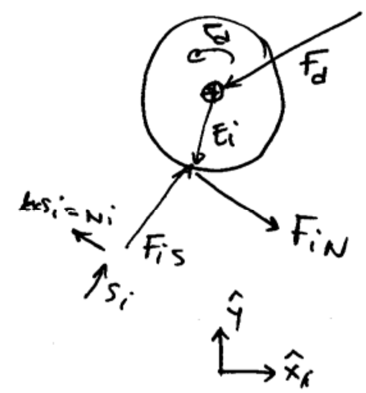
E-E

From 1^{st} and 3^{rd} Equation we solve for F_{iN} & F_{iS}

$$
\boxed{\left\{\begin{array}{l}
F_{iN} = \rho\left\{\dfrac{1}{3}L_i^2\ddot{\alpha}_i + L_i\dot{L}_i\dot{\alpha}_i\right\} \\[2mm]
F_{iS} = f_i - \rho/2\left\{L_i\ddot{L}_i - (L_i\dot{\alpha}_i)^2 + \dot{L}_i^2\right\}
\end{array}\right.}
\qquad (\text{V})
$$

3) Dynamic Equations for M.P.

Free-Body diagram:



$$
\text{N-E: } \sum F_{ext} = F_d + \sum_{i=1}^{3} F_{iS}\hat{s}_i - F_{iN}(k\times s_i) = M a_G
$$

write it componentwise

$$
M\ddot{x}_G = f_{dx} + \sum_{i=1}^{3}\left\{F_{iS}\, s_{ix} - F_{iN}\overbrace{(k\times s_i)\cdot\hat{x}}^{-s_{iy}}\right\}
$$

$$
M\ddot{y}_G = f_{dy} + \sum_{i=1}^{3}\left\{F_{iS}\, s_{iy} - F_{iN}\overbrace{(k\times s_i)\cdot\hat{y}}^{s_{ix}}\right\}
$$

$$
\rightarrow \boxed{\left\{\begin{array}{l}
M\ddot{x}_G - \sum_i\left\{F_{iS}\, s_{ix} + F_{iN}\, s_{iy}\right\} - f_{dx} = 0 \\
M\ddot{y}_G - \sum_i\left\{F_{iS}\, s_{iy} - F_{iN}\, s_{ix}\right\} - f_{dy} = 0
\end{array}\right.}
\qquad (\text{VI})
$$

$$
\text{E-E: } I\ddot{\varphi}\cdot k = \tau_d\cdot k + \sum_{i=1}^{3} E_i\times\left(F_{iS}\hat{s}_i - F_{iN}(k\times s_i)\right)
$$

$$
I\ddot{\varphi} = \tau_d + \sum_{i=1}^{3}\left\{F_{iS}(E_{ix}\, s_{iy} - E_{iy}\, s_{ix}) - F_{iN}(E_i\cdot s_i)\right\}
$$

$$
\rightarrow \boxed{I\ddot{\varphi} + \sum_{i=1}^{3} F_{iS}(E_{iy}\, s_{iy} - E_{ix}\, s_{iy}) + F_{iN}(E_i\cdot s_i) - \tau_d = 0}
\qquad (\text{VII})
$$