6.0 Review of Lyapunov Analysis:

As seen in previous chapter the Equations of Motion of a general parallel manipulator can be written as a state-space Nonlinear Eq.

$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}, u)$$

in which $\underset{\sim}{x}$ is the state vector & u is the control input, the objective of this chapter is to design the control input, u, as a function of x, such that the closed loop system becomes or retain its stability, and be able to track a prescribed trajectory $x_d(t)$ with minimum error. hence if

$$u = g(x) \quad \text{is designed}$$

then

$$\dot{x} = f(x, g(x)) = f(x)$$

the formulation $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x})$ is a general Nonlinear dynamic equation for a closed loop system or an unforced system $u=0$

to define Stability, first consider the dynamic property of system:

6.0.1. Equilibrium point. x^*

a state x^*, or we call a point x^* is an equilibrium point for

$\dot{x} = f(x)$ if

as $t \to \infty$ the trajectories $x(t) \to x^*$. The equilibrium point can be derived from

$$\dot{x}^* = f(x^*) = 0$$

Since at steady-state no more variation in $x(t)$ is seen

The stability of the Nonlinear system is defined usually locally in the vicinity of each equilibrium point, note that since the above equation can have multiple Solution, and hence we may observe multiple behavior in a nonlinear System.

Let us rigourously define different kind of stability for a nonlinear system formulated as $\dot{x} = f(x)$ (1)

Def: The equilibrium point $\underset{\sim}{x} = 0$ of (1) is

- stable if for each $\epsilon > 0$, $\exists\, \delta(\epsilon) > 0 \Rightarrow$

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon \quad \forall t \geq 0$$

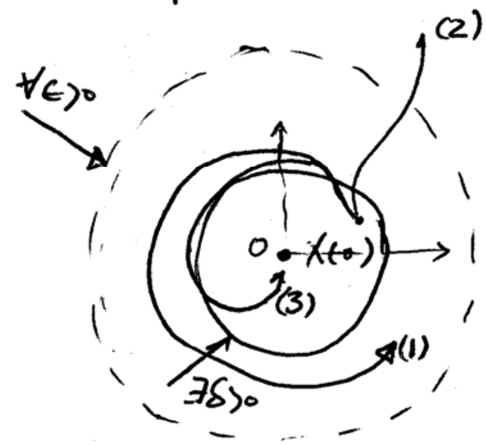
- unstable, if not stable

- asymptotically stable, if it is stable & δ can be chosen $\Rightarrow$

$$\|x(0)\| < \delta \Rightarrow \lim x(t) = 0$$

Note1: the assumption that x^* is at origin $x^*=0$ provides no limitation to the problem, since if $x^* \neq 0$ by use of a change of variable $y = x - x^*$, y^* will become at equilibrium @ origin.

Note2: Geometrical Representation



Note 3: The most useful stability closer to the definition of stability for Linear System is asymptotic stability, and we are seeking asy. stab. for Robotic applications.

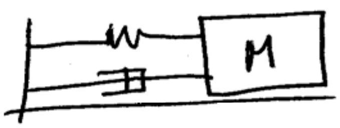
Note 4:

Lyapunov Direct Method

The philosophy of Lyapunov's direct method for stability analysis of a nonlinear System is the mathematical extension of a fund. physical observation:

If the total energy of a system is continuously dissipating then the System must eventually settle down to an eq. point

Start with an example: M-Spring-damper with Nonlinear stiffnen $F_S = K_o x + K_1 x^3$ and " damping $F_D = b\dot{x}|\dot{x}|$



$$m\ddot{x} + b\dot{x}|\dot{x}| + k_o x + k_1 x^3 = 0$$

Suppose we have large distances from origin $(x=\dot{x}=0)$, will the resulting Motion becomes Stable?!!

It is very hard to use the definition of the stability, Examine the Energy (total) of System

Total Energy $V(x) = K_E + P_E = \frac{1}{2}m\dot{x}^2 + \int_0^x F_S \, dx$

$$V(x) = \frac{1}{2}m\dot{x}^2 + \int_0^x (k_o x + k_1 x^3)\,dx$$

$$V(x) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}K_o x^2 + \frac{1}{4}k_1 x^4$$

Compare the definition of stability corresponding to total Energy

- zero energy corresponds to the equilibrium point ($x = \dot{x} = 0$).
- asymptotic stability implies the convergence of total energy to zero.
- instability is related to the growth of energy.

Hence the stability is related to the variation of the total energy

$$\dot{V}(x) = m\dot{x}\ddot{x} + (k_0 x + k_1 x^3)\dot{x}$$

→ here the Equation of motion of system comes into picture.

$$\dot{V}(x) = \dot{x}(m\ddot{x} + k_0 x + k_1 x^3)$$

$$= \dot{x}(-b\dot{x}|\dot{x}|) = -b|\dot{x}|^3 < 0$$

The energy of the system is continuously dissipating until $\dot{x} = 0$.

To generalize this approach, look that the energy function has the following properties:

1) $V(x)$ is strictly positive $V(x) > 0$ except at origin $V(x)\big|_{\substack{x=0 \\ \dot{x}=0}} = 0$

2) $\dot{V}(x)$ is monotonically decreasing.

Def: A scalar continuous function $V(x)$ is locally positive definite if $V(0)=0$ and in a Ball B_{R_0} $V(x)>0$ if $x \neq 0$.

Note: if the above holds for the whole space $\rightarrow$ globally positive definite.

Related concepts:

$V(x)$ is negative definite: if $-V(x)$ is pos. def.

$V(x)$ is positive semi definite if $V(0)=0$ and $V(x) \geq 0$ for $\underset{\sim}{x} \neq \underset{\sim}{0}$

Theorem: (Lyapunov Local Stability)

If in a ball B_{R_0}, $\exists$ a function $V(x)$ which is pos. def, and has continuous partial derivatives, and if its time derivative $\left(\dot{V}(x) = \frac{\partial V}{\partial x} \cdot \dot{x}\right)$ along any state trajectory is

- negative semi definite then eq. point $\underset{\sim}{0}$ is stable.
- negative definite then eq. point 0 is asymptotically stable.

Theorem: (Lyapunov Global Stability)

If $\forall x \in \mathbb{R}^n$, $\exists$ a function $V(x)$ with continuous first order derivatives, $\ni$

- $V(x)$ is pos. def
- $\dot{V}(x)$ is neg. def
- $V(x) \longrightarrow \infty$ as $\|x\| \longrightarrow \infty$ (Radial unboundedness)

then the eq. point $\underset{\sim}{0}$ is "Globally Asymptotically Stable"

Example: Consider a class of nonlinear system, (1st order)

$$\dot{x} + f(x) = 0 \quad \text{where} \quad x f(x) > 0 \text{ for } x \neq 0$$

Lyapunov Candidate
$$\begin{cases} V = x^2 > 0 \\ \dot{V} = 2x\dot{x} = -2x f(x) < 0 \\ V \to \infty \text{ as } \|x\| \to \infty \end{cases}$$

$\Rightarrow$ the System is Globally asymptotically stable at $x=0$.

this example includes systems as $\dot{x} = \sin^2 x - x$
or $\dot{x} = -x^3$

Note 1: All Theorems are sufficiency Theorem meaning that

if $\exists\ V(x) > 0 \Rightarrow \dot{V}(x) < 0 \Rightarrow$ eq. point is asy. stable

but if for $V(x) > 0 \rightarrow \dot{V}(x)$ is indefinite $\Rightarrow$ No conclusion
Look for another $V(x)$

Note 2: How to find a Lyapunov function:

Start with $V(x)$ as the total energy of the System

$$V(x) = K_E + P_E$$

and check wheather $\dot{V}(x)$ is dissipative (passivity)

As we explained before we are looking forward to asy. Stability what if $\dot{V}(x) \leq 0$?

Theorem (Lasalle's Theorem)

Consider the system $\dot{x} = f(x)$ with f continuous, and let $V(x)$ be a scalar function with continuous partial derivatives, Assume that in a certain neighborhood Ω of the origin

- $V(x)$ is locally positive definite
- $\dot{V}(x)$ is negative semi-definite
- Let R defined by $\dot{V}(x) = 0$ the set of points where $\dot{V}(x)$ becomes zero. Consider R contains no trajectories of the system other than the trivial one $x \equiv 0$.

Then the eq. point $\underline{0}$ is locally asymptotically stable.

Example: pendulum with viscous damping

$$\ddot{\theta} + \dot{\theta} + \sin\theta = 0$$

equilibrium point $x^* = \begin{bmatrix} \theta^* = 0 \\ \dot{\theta}^* = 0 \end{bmatrix}$ = origin $\quad x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$

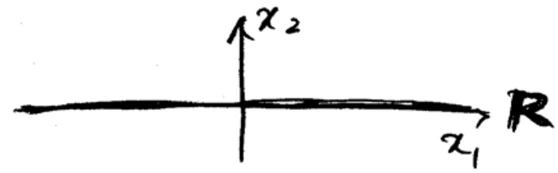
Lyapunov Candidate: $V(x) = (1 - \cos\theta) + \dfrac{\dot{\theta}^2}{2}$ (How? Total energy)

$$\dot{V}(x) = \dot{\theta}\sin\theta + \dot{\theta}\ddot{\theta} = \dot{\theta}(\ddot{\theta} + \sin\theta) = \dot{\theta}(-\dot{\theta}) = -\dot{\theta}^2$$

$$\dot{V}(x) = -\dot{\theta}^2 \leq 0$$ neg. semi definite, why?

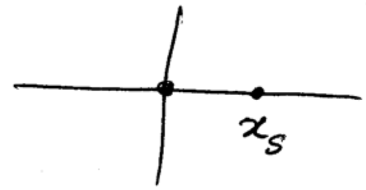
because $\dot{V}(x) = 0$ at $x = \begin{bmatrix} \text{any } \theta \\ 0 \end{bmatrix}$ and not only at $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R = \{ x \in \mathbb{R}^2 \mid x_1 = \text{anything},\ x_2 = 0 \}$



Can any trajectory be in R?

Consider a point on R

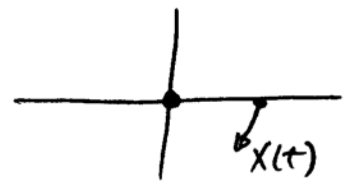


$\underset{\sim}{x}_s = \begin{bmatrix} x_s \\ 0 \end{bmatrix} \in R$ @ x_s we have $\begin{cases} \theta_s \neq 0 \\ \dot{\theta}_s = 0 \end{cases}$

$$\Rightarrow \ddot{\theta}_s + \cancel{\dot{\theta}_s}^{\,0} + \sin\theta_s = 0 \Rightarrow \ddot{\theta}_s = -\sin\theta_s \neq 0$$

hence at an incremental time Δt $\quad \dot{\theta} = -\sin\theta_s \cdot \Delta t < 0$

the trajectory have the shape of



hence No trajectory other than the equilibrium point is included in R

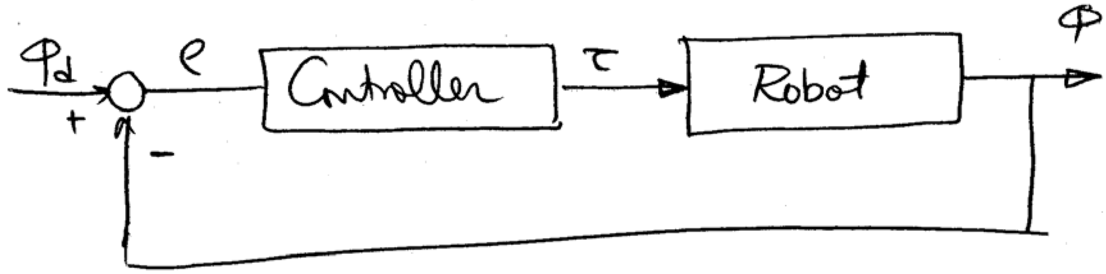
→ Lasalle's Theorem conditions holds

$\Rightarrow x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an asymptotically stable point.

6. Control of Parallel Manipulators

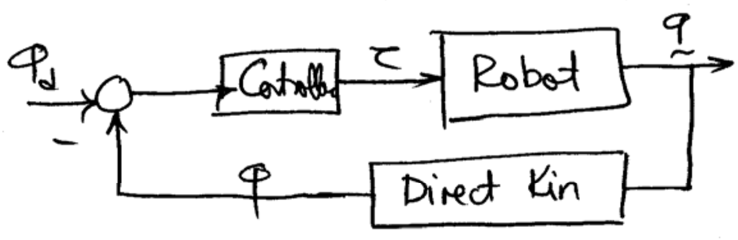
6.1 position Control

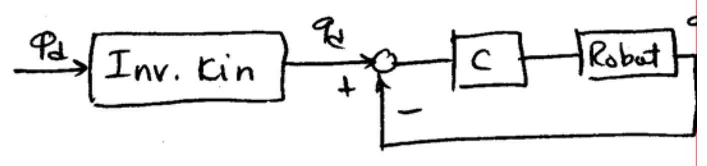
Let us consider the position Control of a robot manipulator, whose structure is a parallel mechanism. In position Control, we assume that the Moving platform position & orientation must follow a desired trajectory, and the controller computes the actuator forces required to produce such motion. Let us use the Cartesian Coordinate $\varphi = [x \;\; y \;\; z \;\; \alpha \;\; \beta \;\; \gamma]$ of the manipulator, as used in Dynamic analysis as the instantaneous position/orientation of the moving platform and $\varphi_d(t)$ as the desired trajectory to be followed, hence in general the topology of Control is depicted in the following Block diagram: where φ is measured and used in the feedback



However, in general measuring the position/orientation of the M.P., is not directly plausible. For this to be implemented an inertial "Imu" which includes gyroscopes & accelerameters are required, that could be implemented in applications, where the accuracy of the P/O of M.P., is of importance, such as in military applications.

However, the usual parameter that can be measured directly is usually the limb's length (and if needed the forces exerted to the M.P by the actuators). Let us denote $q = [q_1, \cdots, q_6]$ as the joint-space coordinate or the limb length in a 6DOF SGP, which we have examined in detail, hence it is required to translate the measured variables q into $\mathcal{P}$ or the converse translate the desired $\mathcal{P}_d$ into q_d, hence two platforms for Control is applicable



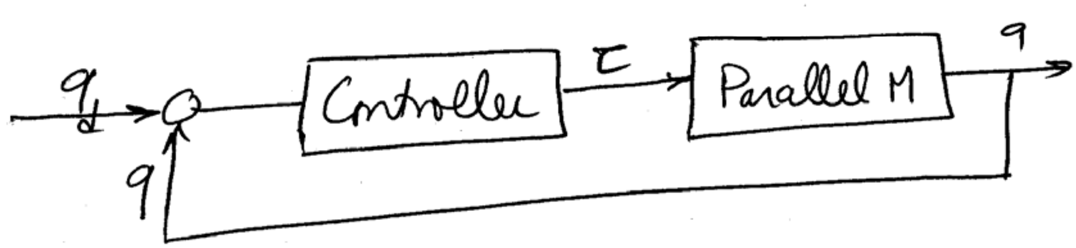


a) Use of Direct Kinematics

b) use of Inverse Kinematics

As it is seen in Kinematics analysis the Forward Kinematics of parallel manipulators are more complicated than Inv. Kin. Hence usually the diagram (b) is used. Unless a closed form Solution for D. Kinematics is available; in that case both diagrams are applicable.

The generation of $q_d(t)$ from $\varphi_d(t)$ is out of loop and can be done offline, if $\varphi_d(t)$ is known apriori, usually the function who does this task either online or offline is called trajectory planner or path planner, in which special attention to generate smooth functions in time, that minimizes motion jerks, but optimizes motion travelling time is done in this function, which is a topic of current research itself, taking this task out of Controller design task we reduce the problem into the following structure



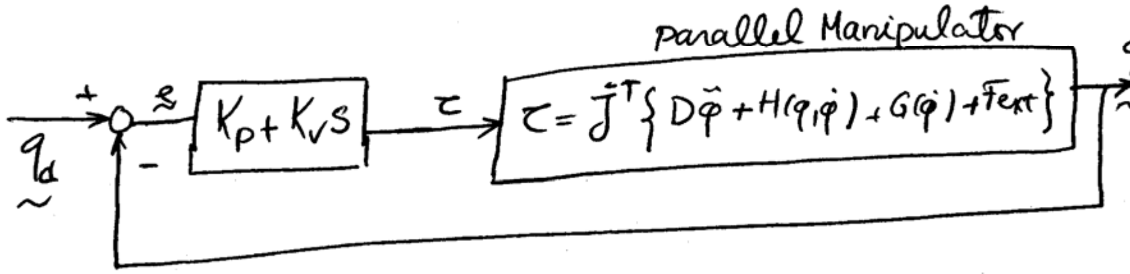
To design a controller, we may start with the simplest topology of using decoupled-linear controller for each actuator.

6.1.1 Decentralized PD Control

In this primitive method, linear PD Controller are used to determine the actuator forces required in each limb

$$\underset{\sim}{\tau} = K_p(\underset{\sim}{q}_d - \underset{\sim}{q}) + K_v(\dot{q}_d - \dot{q})$$

in which K_p & K_v are constant and diagonal matrices.



Although using this Structure is very easy, and No knowledge of system parameters are needed, such controller even with high gain PD parameters result in poor tracking, static errors, configuration and trajectory dependent and usually not sufficient for the system.

Note that for Serial manipulators having DC-motor + gear box as their actuators, it can be shown that the effect of gear box with high gear ratio results in dominating the linear actuator dynamics to the Nonlinear robot dynamics, and such topologies are practically implemented in industrial Serial manipulators with success. But this is Not true for a general parallel manipulator

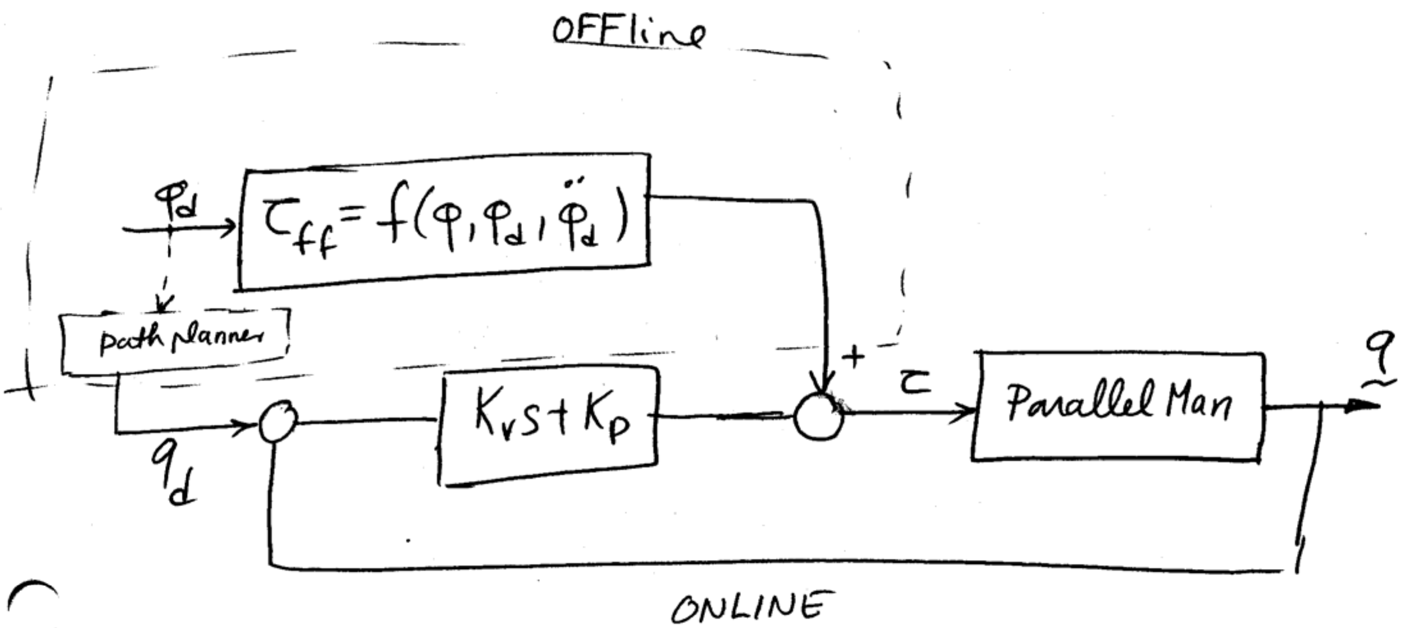
6.1.2 Feed Forward Control

In this structure we assume that the inverse Dynamics of the System is in hand, and also the trajectory of the manipulator is known apriori; then the feed forward actuator torque component can be calculated from

$$\tau_{ff} = \hat{J}^{-T} \{ \hat{D}\, \ddot{\varphi}_d + \hat{H}(\varphi_d, \dot{\varphi}_d) + \hat{G}(\varphi_d) + \hat{F}_{ext} \}$$

and the controller torque consists of a ff. + a feedback PD such as:

$$\tau = \tau_{ff} + K_v (\dot{\underset{\sim}{q}}_d - \dot{\underset{\sim}{q}}) + K_p (\underset{\sim}{q}_d - \underset{\sim}{q})$$



Note that in offline path it is considered that $\underset{\sim}{q}_d \simeq \underset{\sim}{q}$

and hence online measurements of $\underset{\sim}{q}$ is not used for τ_{ff} computat

rather, an offline computation is performed on q_d instead of

actual q.

(I) If $\hat{D} = D$ & $\hat{H} = H$ & $\hat{G} = G$ and $\underset{\sim}{q}_d = q$ then the

Error dynamics can be derived as

$$\hat{J}^{-T}\{\hat{D}\,\ddot{\varphi}_d + \hat{H}(\varphi_d, \dot{\varphi}_d) + \hat{G}(\varphi_d) + \hat{F}_{ext}\} + K_v(\dot{q}_d - \dot{q}) + K_p(q_d - q)$$

$$= J^{-T}\{D\ddot{\varphi} + H(\varphi, \dot{\varphi}) + G(\varphi)\}$$

$$\Rightarrow \quad J^{-T} D(\ddot{\varphi}_d - \ddot{\varphi}) + K_v(\dot{q}_d - \dot{q}) + K_p(q_d - q) = 0$$

$$\underbrace{J^{-T} D J^{-1}}_{M}\,\underbrace{(\ddot{q}_d - \ddot{q})}_{\ddot{E}} + K_v\underbrace{(\dot{q}_d - \dot{q})}_{\dot{E}} + K_p\underbrace{(q_d - q)}_{E} = 0$$

$$\Rightarrow \quad M\ddot{E} + K_v\dot{E} + K_p E = 0$$

By choosing appropriate K_v & K_p diagonal members good tracking

performance can be obtained IF assumptions (I) holds!

6.1.3. Inverse Dynamics Control

(Compute torque or Feedback linearization)

If the inverse Dynamics of the manipulator is solved and can b be numerically (digitally) implemented by controller, we may use online calculation of I.D. based on measured variables.

Consider the inverse dynamics of the manipulator

$$\tau = J^{-T}\{ D(\varphi)\ddot{\varphi} + H(\varphi,\dot{\varphi}) + G(\varphi) + F_{ext} \}$$

$$\tau = \tau_1 + \cancel{\tau_2} = J^{-T} D(\varphi)\ddot{\varphi} + \underbrace{J^{-T}\{H \cancel{+} G + F_{ext}\}}_{\tau_2}$$

here $\tau_2 = J^{-T}\{H + G - F_{ext}\}$ for feedback linearization

and

$$\tau_1 = J^{-T} D(\varphi)\ddot{\varphi}_d \quad \text{OR} \quad \tau_1 = J^{-T} D(\varphi) J^{-1} \ddot{q}_d$$

+ PD Controller

hence

$$\tau_1 = J^{-T} D(\varphi) J^{-1} \{ \ddot{q}_d + K_p(q_d - q) + K_v(\dot{q}_d - q) \}$$

resulting into

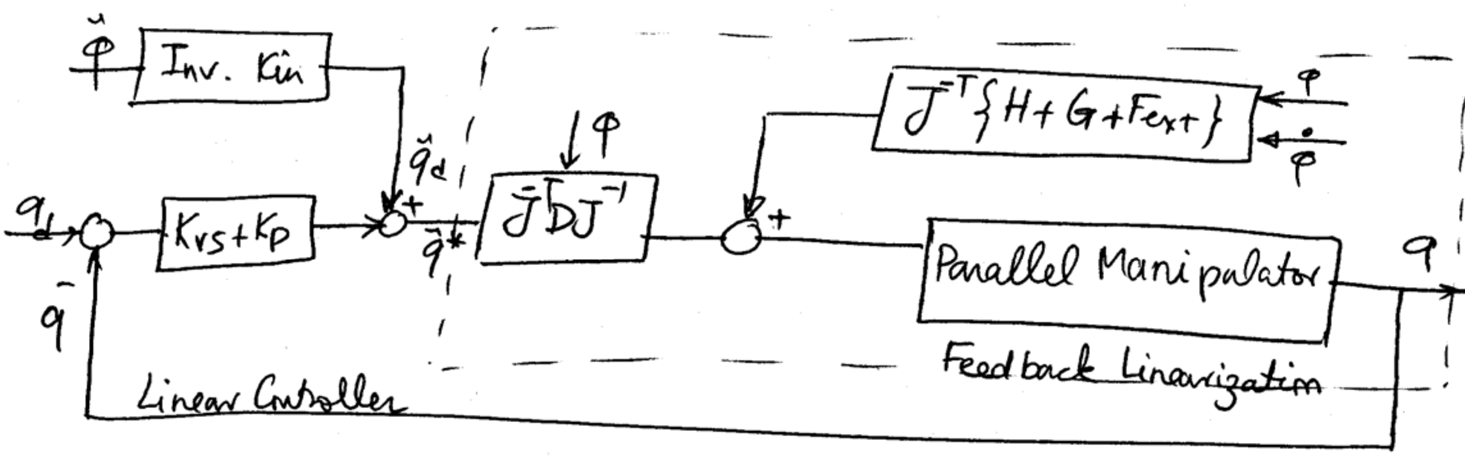
$$J^{-T} D J^{-1} \{ \ddot{q}_d - q) + K_D(\dot{q}_d - \dot{q}) + K_p(q_d - q) \} = 0$$

OR $\ddot{E} + K_D \dot{E} + K_p E = 0$ iff $\underset{\sim}{J^{-T} D J^{-1}}$ non singular

Hence in this case the actuator torques/forces are equal to:

$$\begin{cases} \tau = J^{-T} D J^{-1} \ddot{q}^{*} + J^{-T}\{H + G + F_{ext}\} \\ \ddot{q}^{*} = \ddot{q}_d + K_v(\dot{q}_d - \dot{q}) + K_p(q_d - q) \end{cases}$$

in which



In this method the full inverse dynamics of the system is used to linearize the governing Dynamics of the System, and a linear Controller is used to guarantee the required performance requirements such as steady-state error, rise time, bandwidth and more important robustness properties.

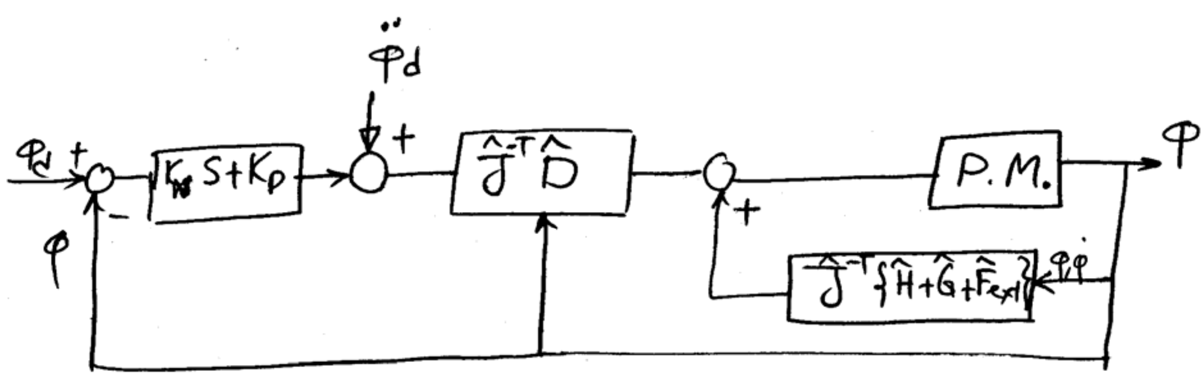
This is a general topology used for Control of Robot manipulators, and many variation of this structure is used in practice.

A Inverse Dynamics Control : φ measured

Consider the case where φ & its time derivatives are directly measured in a System, in that case a more straight forward and easier implementable IDC can be advised :

$$\begin{cases} \tau = \hat{J}^{-T}\hat{D}\,\ddot{\varphi}^{*} + J^{-T}\{\hat{H}+\hat{G}+\hat{F}_{ext}\} \\ \ddot{\varphi}^{*} = \ddot{\varphi}_d + K_v\,(\dot{\varphi}_d-\varphi) + K_p(\varphi_d-\varphi) \end{cases}$$

OR in terms of block diagram



The error dynamics in this case is

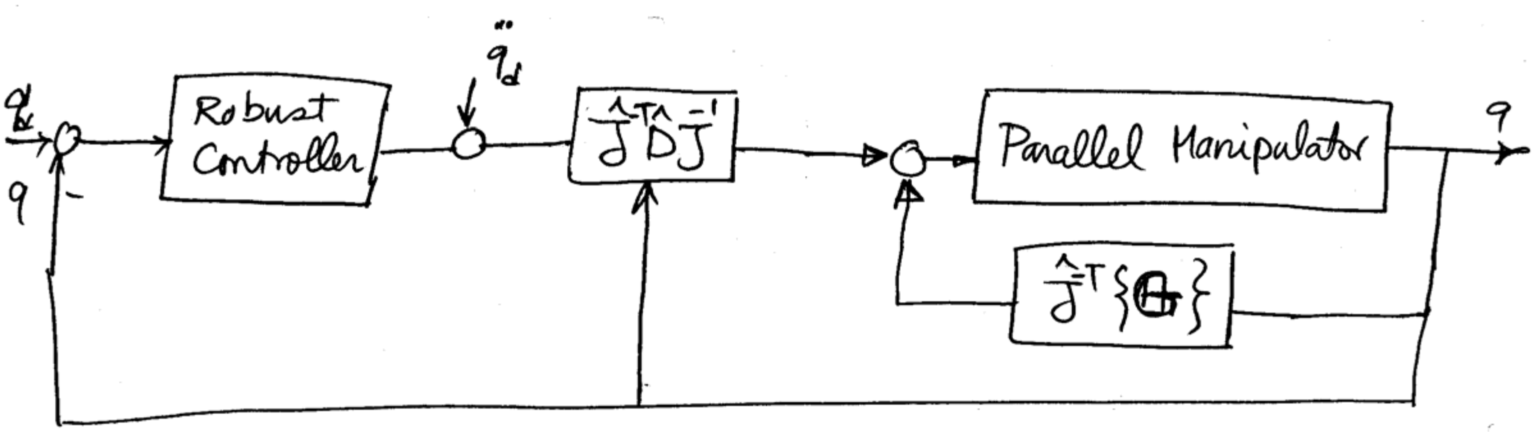
$$\hat{J}^{T}\hat{D}\{(\ddot{\varphi}_d-\varphi) + K_v(\dot{\varphi}_d-\dot{\varphi}) + K_p(\varphi_d-\varphi)\} = 0$$

which results directly into tracking performance on φ

B Inverse Dynamic Control : partial linearization

As explained in last chapter the effect of $H(q,\dot{q})$ terms in the dynamics of System is usually very limited ($<1\%$!) Hence H & F_{ext} terms can be taken out of the F.L. loop But it is always important to include G & $\boxed{\Theta}$ terms to avoid constant disterbace & coupling of equations.

Now that the System is partially linearized, a robust Controller (either H_∞ linear, or Lyapunov based Nonlinear) Controllers can be used to strengthen the robustness properties of the overall controller, the Block diagram of such topology is given below



To design the Robust Controller, the dynamics behavior of the manipulator including the Feedback linearizing loop is examined through identification experiments. If H_∞ Controller are to be designed the linearized System is identified by a linear Nominal plant and an uncertainty profile. having determined upper bounds for uncertainty description, tractible Controller design approaches such as H_∞ Controller, μ-synthesis or Sliding mode Controller can be implemented for the System, below are a No. of research papers including such algorithms in practice:

[LSCH03]: Position Control of a SP using inverse dynamics control with approximate dynamics, Mechatronics 13, 605-619, 2003

In this paper inverse Kinematics and Jacobian analysis is reviewed for a 6DOF SP, where the Moving platform coordinates are chosen as

$$\dot{\underset{\sim}{x}} = \begin{bmatrix} v_p \\ \omega_p \end{bmatrix} \quad \ddot{\underset{\sim}{x}} = \begin{bmatrix} a_p \\ a_p \end{bmatrix} \longrightarrow \underset{\sim}{x} = \begin{bmatrix} x_p \\ \int \omega_p \, dt \end{bmatrix} \longrightarrow \text{not the euler angles}$$

The inverse Dynamics is written in form of

$$M(x)\ddot{x} + H(x,\dot{x}) + G(x) = J^T \tau = \mathcal{F} \qquad (9)$$

and Jacobian is as before

$$J = \begin{bmatrix} s_1 & s_2 & \cdots & s_6 \\ {}^{o}b_1 \times s_1 & {}^{o}b_2 \times s_2 & & {}^{o}b_6 \times s_6 \end{bmatrix}^T \qquad (10)$$