Lecture 17: Linear Discrete-Time Systems; Reachability and Controllability

5.3 Reachability and Controllability for Discrete-Time Linear Systems

5.3.1 Time-Varying Case

Consider the discrete-time linear time-varying state-space system:

$$\begin{aligned} x(j+1) &= A(j)x(j) + B(j)u(j), \ \ x(r) = x_r \\ y(j) &= C(j)x(j) + D(j)u(j) \end{aligned} \tag{5.12}$$

where $x(j) \in \mathbb{R}^n, \ u(j) \in \mathbb{R}^m, \ y(j) \in \mathbb{R}^p$. We found the state response to be

$$x(j) = \Phi(j,r)x_r + \sum_{i=r}^{j-1} \Phi(j,i+1)B(i)u(i). \tag{5.13}$$

and the state transition matrix to be:

$$\Phi(k,r) = \prod_{i=r}^{k-1} A(i). \tag{5.14}$$

The definitions of reachability and controllability for (5.12) are entirely similar to the continuous-time case. However, complete controllability, controllability at t_0, and reachability at t_j are not always equivalent as was the case for continuous-time state-space systems. This is because $\Phi(j,r)$ is not always invertible, e.g., $A(i)$ may be singular for some $m \le i \le j-1$.

Similar to the continuous-time case, the admissible input functions $u(\cdot)$ belong to the space $l_2^m[r,k], \ r \le j \le k$:

$$\mathcal{U}_{ad} := l_2^m[r,k], \tag{5.15}$$

i.e., the space of (finite-dimensional) square-summable sequences, which we will view as finite-dimensional vectors in $\mathbb{R}^{k-r+1}$

<u>Definition: Controllability Operator</u>

The *controllability operator* for the system (5.12) is the transformation

$$\mathcal{L} : \mathcal{U}_{ad} \to \mathbb{R}^n, \quad \mathcal{L}u := \sum_{i=r}^{k-1} \Phi(k,i+1)B(i)u(i) \tag{5.16}$$

which maps admissible inputs into state vectors.

The Controllability Equation

The input which produces a terminal state $z \in \mathbb{R}^n$ from the zero initial state θ is any solution of the *controllability equation*:

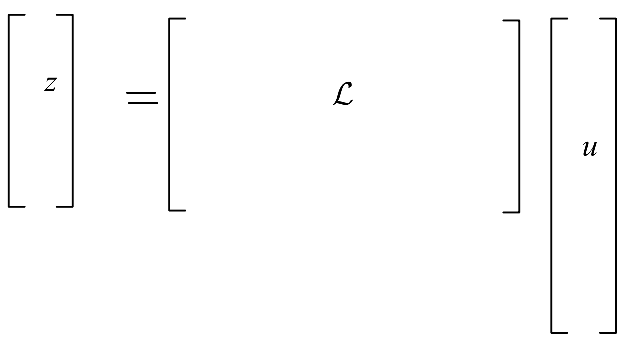
$$\mathcal{L}u = z, \quad u \in \mathbb{R}^{m(k-r+1)}, \; z \in \mathbb{R}^n \tag{5.17}$$

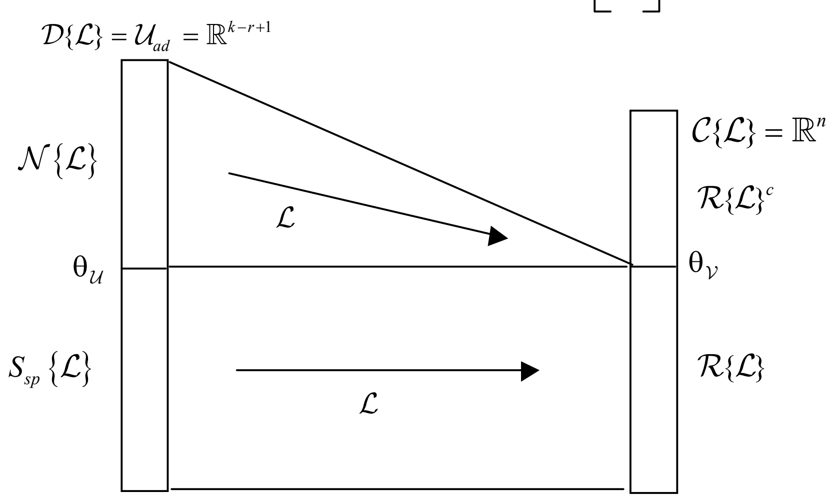
More generally, the input u produces a change in terminal state $z = x_1 - \Phi(k,r)x_r$ from what it would be as a free trajectory starting at (x_r, r).

The controllability equation can be expressed as:

$$\begin{aligned}\mathcal{L}u &:= \Phi(k,r+1)B(r)u(r) + \Phi(k,r+2)B(r+1)u(r+1) + \cdots + \Phi(k,k)B(k-1)u(k-1) \\ &= \underbrace{\begin{bmatrix}\Phi(k,r+1)B(r) & \Phi(k,r+2)B(r+1) & \cdots & \Phi(k,k)B(k-1)u(k-1)\end{bmatrix}}_{\mathcal{L}} \underbrace{\begin{bmatrix} u(r) \\ u(r+1) \\ \vdots \\ u(k-1)\end{bmatrix}}_{u} = z\end{aligned} \tag{5.18}$$

That is, the controllability operator is a matrix, with typically more columns than rows, and hence a solution to the controllability equation is composed of a component $u_1 \in S_{sp}\{\mathcal{L}\}$ and any component in the nullspace $u_{\mathcal{N}} \in \mathcal{N}\{\mathcal{L}\}$.





Proposition:

The system (5.12) is CC $\Leftrightarrow$ $\mathcal{L}$ is onto.

- If the system is CC, then $z \in \mathcal{R}\{\mathcal{L}\}$ and there are many solutions to the controllability equation.

- If the system is not CC, and $z \notin \mathcal{R}\{\mathcal{L}\}$ and there is no solution to the controllability equation.

So two problems of interest here would be:

1. Find $\mathcal{R}\{\mathcal{L}\}$, the space of attainable states, and

2. Find the smallest-norm input $u \in \mathcal{U}_{ad}$ that produces the state z.

Problem 1 can be solved by reduction of L, the matrix representation of $\mathcal{L}$, into the echelon form.

To address Problem 2, we will use the adjoint $\mathcal{L}^*$ in the next section. But first, let us look at the LTI case.

5.3.2 Time-Invariant Case

Consider the discrete-time linear time-invariant state-space system:

$$\begin{aligned} x(j+1) &= Ax(j) + Bu(j), \quad x(r) = x_r \\ y(j) &= Cx(j) + Du(j) \end{aligned} \tag{5.19}$$

The controllability equation is given by:

$$\begin{aligned} \mathcal{L}u &= \underbrace{\begin{bmatrix} \Phi(k,r+1)B(r) & \Phi(k,r+2)B(r+1) & \cdots & \Phi(k,k)B(k-1) \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} u(r) \\ u(r+1) \\ \vdots \\ u(k-1) \end{bmatrix}}_{u} \\ &= \underbrace{\begin{bmatrix} A^{k-r-1}B & A^{k-r-2}B & \cdots & B \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} u(r) \\ u(r+1) \\ \vdots \\ u(k-1) \end{bmatrix}}_{u} = z \end{aligned} \tag{5.20}$$

which we can rearrange by reversing the order of the input samples:

$$\mathcal{L}u = \begin{bmatrix} B & AB & \cdots & A^{k-r-1}B \end{bmatrix} \begin{bmatrix} u(k-1) \\ u(k-2) \\ \vdots \\ u(r) \end{bmatrix} = z \tag{5.21}$$

Starting from the initial time $r = 0$:

$$\mathcal{L}u = \underbrace{\begin{bmatrix} B & AB & \cdots & A^{k-1}B \end{bmatrix}}_{L_k} \underbrace{\begin{bmatrix} u(k-1) \\ u(k-2) \\ \vdots \\ u(0) \end{bmatrix}}_{u^k} = z \tag{5.22}$$

Definition: Controllability Matrix

The controllability matrix of system (5.19) is defined as:

$$L_n := \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times mn} \tag{5.23}$$

Proposition:

The time-invariant system is reachable (or controllable-from-the-origin) iff the controllability matrix L_n has full row rank, i.e., $\operatorname{rank}\{L_n\} = n$.

Proof:

The reason for using L_n rather than L_k, $k > n$ to define the controllability matrix is because if a transfer from the origin cannot be accomplished in n time steps of the control sequence, it cannot be accomplished by taking more than n time steps. That is, it can be shown that:

$$\mathcal{R}\{L_n\} = \mathcal{R}\{L_k\}, \ k \geq n,$$

but note that $\mathcal{R}\{L_n\} \supset \mathcal{R}\{L_k\}, \ k < n$.

It follows that it is possible to transfer the state from some vector $x(0) = x_0$ to some other vector $x(n) = x_1$ in n time steps iff there exists an n-step input sequence $\{u(0), u(1), \ldots, u(n-1)\}$ that satisfies:

$$x_1 - A^n x_0 = L_n u^n. \tag{5.24}$$

which has a solution iff $x_1 - A^n x_0 \in \mathcal{R}\{L_n\}$. Clearly, the LTI system is reachable from the origin, implying that any x_1 can be reached from $x(0) = x_0$ in finite time iff $\operatorname{rank}\{L_n\} = n$ so that $\mathcal{R}\{L_n\} = \mathbb{R}^n$.

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Note that $\mathcal{R}\{L_n\}$ is called the *reachable subspace* of the system.

Example:

$$\begin{aligned} x(j+1) &= Ax(j) + Bu(j) \\ A &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned} \tag{5.25}$$

- Controllability matrix: $L_2 = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ has $\operatorname{rank}\{L_2\} = 2$. Hence the system is reachable.

- Any state, say $x_1 = \begin{bmatrix} a \\ b \end{bmatrix}$, can be reached from the origin in $n = 2$ time steps:

$$\begin{aligned} \begin{bmatrix} a \\ b \end{bmatrix} &= L_2 u^2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} \\ \Rightarrow \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} &= \begin{bmatrix} b-a \\ a \end{bmatrix} \end{aligned}$$

  Check:

$$\begin{aligned} x(j+1) &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x(j) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(j) \\ x(1) &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(0) = \begin{bmatrix} 0 \\ a \end{bmatrix} \\ x(2) &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(1) = \begin{bmatrix} a \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ b-a \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \end{aligned}$$

- Since the system is reachable, it is possible to transfer the state from some vector $x(0) = x_0$ to some other vector $x(n) = x_1$ in $n = 2$ time steps. Let $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $x_1 = \begin{bmatrix} a \\ b \end{bmatrix}$. Then:

$$x_1 - A^2 x_0 = L_2 u^2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} u(0) \\ u(0) + u(1) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} b - a - 1 \\ a - 2 \end{bmatrix}$$

<u>Notes:</u>

- The system is controllable to the origin when $A^n x_0 \in \mathcal{R}\{L_n\},\ \forall x_0 \in \mathbb{R}^n$.
- If $\operatorname{rank}\{A\} = n$, the system is controllable when $\operatorname{rank}\{L_n\} = n$ (i.e., when the reachability condition is satisfied) because then $-A^n x_0 = L_n u^n$ can be solved for any x_0. In this case,

  $$A^{-n} L_n := \begin{bmatrix} A^{-n} B & A^{-n+1} B & \cdots & A^{-1} B \end{bmatrix} \in \mathbb{R}^{n \times mn}$$

  is of interest, and the system is controllable iff $\operatorname{rank}\{A^{-n} L_n\} = \operatorname{rank}\{L_n\} = n$. If, however, $\operatorname{rank}\{A\} < n$, then controllability does not imply reachability.
- The above example is controllable-to-the-origin. Let $x_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $x_0 = \begin{bmatrix} a \\ b \end{bmatrix}$. Then:

$$-A^2 x_0 = L_2 u^2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

$$-\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

$$-\begin{bmatrix} a + b \\ a + 2b \end{bmatrix} = \begin{bmatrix} u(0) \\ u(0) + u(1) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} -b \\ -a - b \end{bmatrix}$$

5.4 Solutions to the Discrete-Time Controllability Problem

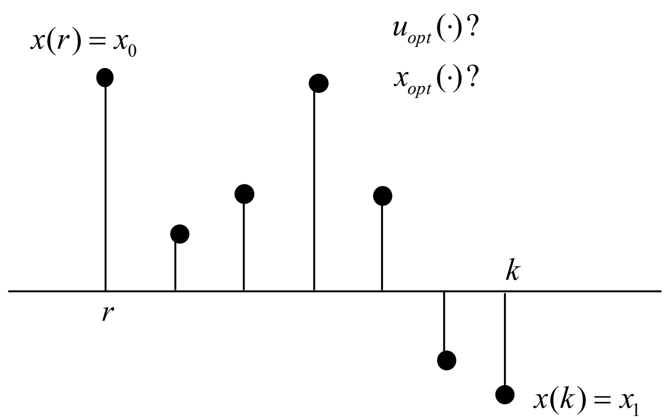
5.4.1 Linear Time-Varying Case

Since the controllability operator $\mathcal{L}$ is a map between finite-dimensional spaces, the pseudoinverse solution

$$u_{opt} = \mathcal{L}^* \left(\mathcal{L}\mathcal{L}^*\right)^{-1} z \tag{5.26}$$

is a solution of the controllability equation $\mathcal{L}u = z$.

This solution solves the problem with the minimum-norm input signal u_{opt}. Also of interest is the corresponding state trajectory u_{opt}.



We know that the system in (5.12) is CC $\Leftrightarrow \mathcal{L}$ is onto $\Leftrightarrow \left(\mathcal{L}\mathcal{L}^*\right)^{-1}$ exists. Thus, complete controllability is equivalent to the existence of the inverse operator $\left(\mathcal{L}\mathcal{L}^*\right)^{-1}$. In this case, u_{opt} as given in (5.26) exists, where

$$\mathcal{L}u = \sum_{i=r}^{k-1} \Phi(k, i+1) B(i) u(i). \tag{5.27}$$

The adjoint $\mathcal{L}^* : \mathbb{R}^n \to l_2^{\,m}[r, k-1]$ is obtained as follows:

$$\begin{aligned}
\left\langle \mathcal{L}u, z \right\rangle_{\mathbb{R}^n} &= \left(\sum_{i=r}^{k-1} \Phi(k, i+1) B(i) u(i) \right)^* z = \sum_{i=r}^{k-1} u(i)^* B(i)^* \Phi(k, i+1)^* z \\
&= \left\langle u(\cdot), B(\cdot)^* \Phi(k, \cdot+1)^* z \right\rangle_{l_2[r,k-1]} = \left\langle u(\cdot), \mathcal{L}^* z \right\rangle_{l_2[r,k-1]} \quad \forall u \in l_2[r, k-1], \forall z \in \mathbb{R}^n \\
&\Rightarrow \mathcal{L}^* = B(\cdot)^* \Phi(k, \cdot+1)^*
\end{aligned} \tag{5.28}$$

Thus,

$$\mathcal{L}\mathcal{L}^* z = \underbrace{\sum_{i=r}^{k-1} \Phi(k,i+1)B(i)B(i)^* \Phi(k,i+1)^* z}_{M(r,k)} \tag{5.29}$$
$$= M(r,k)z$$

<u>Definition: Controllability Grammian</u>

The matrix $M(r,k) \in \mathbb{R}^{n\times n}$ is the above equation is called the *controllability grammian* of the system. It depends only on the time interval $[r,k]$, but it is constant once this interval is fixed.

Finally, the optimal, minimum-norm input is given by:

$$u_{opt}(j) = \left(\mathcal{L}^* \left(\mathcal{L}\mathcal{L}^*\right)^{-1} z\right)(j) = B(j)^* \Phi(k,j+1)^* M^{-1}(r,k)z \tag{5.30}$$

Recall that the norm is $\|u\|_2 = \sqrt{\sum_{i=r}^{k-1} \|u(i)\|^2}$.

5.4.2 Linear Time-Invariant Case

Consider the system

$$x(j+1) = Ax(j) + Bu(j),\ \ x(r) = x_r\,. \tag{5.31}$$

Then,

$$x(k) = A^{k-r}x_r + \sum_{i=r}^{k-1} A^{k-i-1}Bu(i) \tag{5.32}$$

Assume the system is CC. Then, the controllability grammian is invertible:

$$M(r,k) = \sum_{i=r}^{k-1} A^{k-i-1}BB^* {A^*}^{k-i-1} \tag{5.33}$$

<u>Fact:</u>

The controllability grammian can be written in terms of the controllability matrix:

$$M(r,k) = L_k L_k^* \tag{5.34}$$

We have the adjoint:

$$\mathcal{L}^* = B^* {A^*}^{k-(\cdot)+1} \tag{5.35}$$

The optimal control input is:

$$u_{opt}(j)=B^{*}A^{*^{k-j+1}}M(r,k)(x_k-A^{k-r}x_r) \tag{5.36}$$

and finally,

$$\mathcal{R}\left\{L_n\right\}=\mathcal{R}\left\{M(r,k)\right\}. \tag{5.37}$$