Lecture 12: Differential Equations and Dynamical Systems

4 Differential Equations and Dynamical Systems

Why should we study dynamical systems?

The fact is that virtually all macroscopic physical phenomena follow the classical laws of physics (Newton's laws, Maxwell's equations, etc.) which are differential equations defining dynamical systems. The concept of *state* of a dynamical system becomes important to predict its future behavior.

Physical phenomena are generally described by nonlinear mappings between input signals, states, and output signals, and thus one often comes across nonlinear differential equations during the process of modeling a system.

However, it is often possible to describe the behavior of a system by a linear model, such as linear differential equations. This is for example the case if the system operates close to its nominal operating point.

4.1 Introduction

A commonly used model for a physical phenomenon is the nonlinear state-space differential equation:

$$\dot{x}(t) = f\left(t, x(t), u(t)\right), \quad x(0) = x_0 \in \mathbb{R}^n, \tag{4.1}$$

where $f : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $x(t)$ is the *state* of the system at time t, $x(t)$ is the *input* of the system at time t.

Note that a system described by a single N^{th}-order scalar differential equation can normally be expressed as a set of N first-order state differential equations.

$$\frac{d^N y(t)}{dt^N} = f\left(t, y(t), \frac{dy(t)}{dt}, \ldots, \frac{dy^{N-1}(t)}{dt^{N-1}}, u(t)\right) \tag{4.2}$$

In discrete time, we have the similar nonlinear state-space difference equation:

$$x(k+1) = f\left(k, x(k), u(k)\right), \quad x(0) = x_0 \in \mathbb{R}^n, \; k = 0,1,2,\ldots \tag{4.3}$$

In studying the system of (4.1), one should make a distinction between two aspects generally referred to as analysis and synthesis.

Analysis: This is the case if the input function u in (4.1) is specified, and one would like to study the evolution of the corresponding state as a function of time, e.g., stability analysis problems.

Synthesis: This is the case if the desired behavior or trajectory of the state in (4.1) is specified, and one would like to find a suitable input signal that produces the desired state trajectory. This is a control problem.

The system of (4.1) is called *autonomous* (time-invariant) if the function f does not explicitly depends on its first argument (time t); it is called *nonautonomous* otherwise.

Finally, there remains one small point to be cleared up before we proceed to the analysis.

By a "solution" of (4.1) over an interval $[0,T]$, we mean an element $x \in C^n[0,T]$, such that

(i) x is differentiable everywhere,

(ii) Equation (4.1) holds at all $t \in [0,T]$

Notice that if x is a solution to (4.1) over $[0,T]$ and f is continuous, then x also satisfies the integral equation:

$$x(t) = x_0 + \int_0^t f\left(s, x(s), u(s)\right) ds, \quad t \in [0,T]. \tag{4.4}$$

On the other hand, if $x \in C^n[0,T]$ satisfies the integral equation (4.4), then clearly x is differentiable everywhere, and satisfies (4.1). Hence (4.1) and (4.4) are equivalent.

Next, consider the *unforced* differential equation:

$$\dot{x}(t) = f\left(t, x(t)\right), \quad x(0) = x_0 \in \mathbb{R}^n, \tag{4.5}$$

Many properties that are taken for granted in the case of linear systems do not necessarily hold for nonlinear systems. In order for (4.5) to represent a legitimate physical system, it should satisfy at least one of the four following conditions, and preferably all of them:

(1) Equation (4.5) has at least one solution (existence of a solution),

(2) Equation (4.5) has exactly one solution for all sufficiently small values of t (local existence and uniqueness of the solution),

(3) Equation (4.5) has exactly one solution for all $t \in [0,T]$ (global existence and uniqueness of the solution),

(4) Equation (4.5) has exactly one solution for all $t \in [0,T]$, and this solution depends continuously on the initial state $x(0) = x_0$ (well-posedness).

Unfortunately, without some restrictions on the nature of f, none of the above statements may be true, as we illustrate next.

Examples:

(a) $\dot{x}(t) = f\left(x(t)\right), \quad x(0) = 0$,

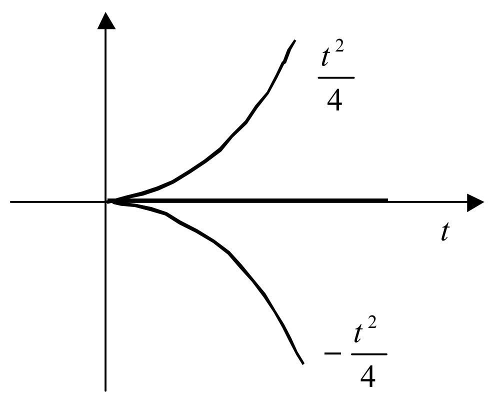
$$f\left(x\right) = \sqrt{|x|}\,\mathrm{sgn}(x) = \begin{cases} \sqrt{|x|}, & x > 0 \\ -\sqrt{|x|}, & x \le 0 \end{cases}$$

Direct substitution shows that the following three state trajectories are solutions:

1) $x(t) = 0$

2) $x(t) = \frac{t^2}{4}$

3) $x(t) = -\frac{t^2}{4}$



All three have the same initial condition, which therefore does not uniquely determine future solutions, and does not conform to the classical physics idea of a state. (statements 2, 3 fail)

(b) $\dot{x}(t) = f\left(x(t)\right), \quad x(0) = 0$,

$$f\left(x\right) = \mathrm{sgn}(x) = \begin{cases} 1, & x \ge 0 \\ -1, & x < 0 \end{cases}$$

It is easy to check that no continuous differentiable solution exists (statement (1) fails)

(c) $\dot{x}(t) = f\left(x(t)\right), \quad x(0) = 0$, $f\left(x\right) = \frac{1}{2x}$.

This equation admits two solutions $x(t) = \pm\sqrt{t}$ (statement (1) holds but (2) fails)

(d) $\dot{x}(t) = f\left(x(t)\right), \quad x(0) = 0$, $f\left(x\right) = 1 + x^2$.

Over the interval $[0,1)$, this equation has a unique solution $x(t) = \tan(t)$, but there is no continuously differentiable function x defined over the entire interval $[0,+\infty)$. This is because as $t \to \frac{\pi}{2}$, $x(t) \to \infty$, a phenomenon called *finite escape time*. (statements (1) and (2) hold but (3) does not.)

It is therefore clear that the starting point for a state-space theory of differential equations is to answer the question:

> Under what conditions does a differential equation subject to a given initial condition have a solution, and one which is unique, and one which is continuously dependent on the initial conditions?

We shall provide an answer to statement (3); the answer will involve the use of a norm on $\mathcal{V}$ and the so-called Lipschitz condition for the function f.

4.1.1 Lipschitz Functions

Let $\left(\mathcal{V}, \|\bullet\|\right)$ be a normed linear space. A function $g: \mathcal{V} \to \mathcal{V}$ is said to be *Lipschitz* if there is a constant $K \geq 0$ for which the Lipschitz condition

$$\frac{\|g(x_1) - g(x_2)\|}{\|x_1 - x_2\|} \leq K \tag{4.6}$$

is satisfied for all pairs $x_1, x_2 \in \mathcal{V},\ x_1 \neq x_2$. The bound K is called a Lipschitz constant for g.

Note:

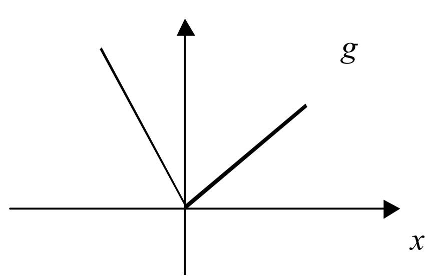
The Lipschitz property is a generalization of the idea of a function with bounded derivative. In fact,

(a) If $g: \mathbb{R} \to \mathbb{R}$ and $\frac{dg}{dx}$ exists and satisfies $\left|\frac{dg}{dx}\right| \leq K$, then g is Lipschitz and K is a Lipschitz constant for it.

(b) If $g: \mathbb{R}^n \to \mathbb{R}^n$ and $\left|\frac{dg_i}{dx_j}\right| \leq K$ exists $\forall i, j$ and satisfies $\left|\frac{dg_i}{dx_j}\right| \leq K$, then g is Lipschitz.

However, a function may be Lipschitz even if it is not everywhere differentiable. For example, the function

$$g(x) = \begin{cases} x, & x \geq 0 \\ -2x, & x < 0 \end{cases}$$



Has a Lipschitz constant 2 even though $\frac{dg}{dx}$ is not defined at $x=0$. In view of (4.6), a Lipschitz constant can be viewed as an upper bound on the incremental amplification of norm produced by the function.

In the special case of a linear transformation

$$g : \mathcal{V} \to \mathcal{V},$$

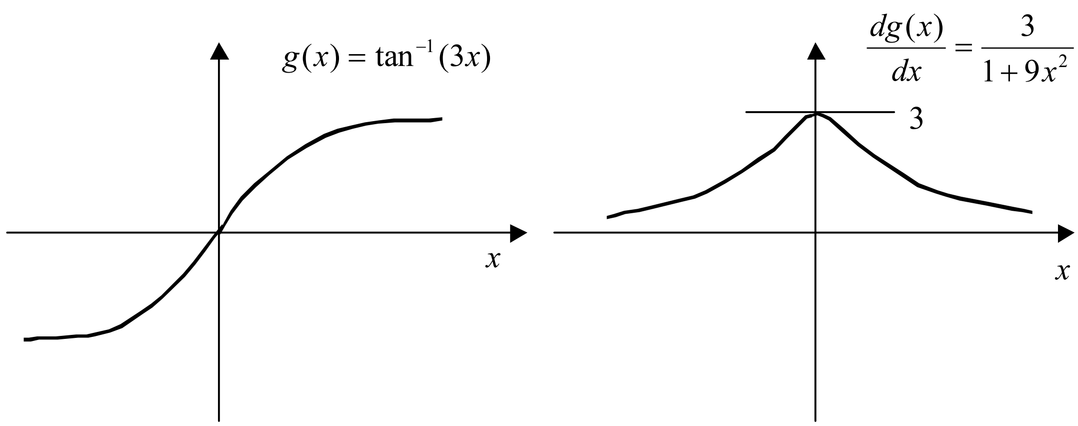
the increment $g(x_1) - g(x_2)$ in (4.6) can be replaced by $g(x_1 - x_2)$ and we can dispense with differences to obtain the simpler, equivalent inequality

$$\frac{\|g(x)\|}{\|x\|} \le K, \quad \forall x \in \mathcal{V}, \quad x \ne 0.$$

Thus, from a previous result of Chapter 3, every linear transformation $\mathcal{T} : \mathcal{V} \to \mathcal{V}$ in a finite-dimensional linear space $\mathcal{V}$ is Lipschitz.

Note that:

$$\sup_{x \in \mathcal{V} = \mathbb{R}} \left| \frac{dg}{dx} \right| = \text{smallest Lipschitz constant}$$



4.2 Existence Theorem for State-Space Differential Equations

In this section, we discuss differential equations of the form:

$$\dot{x}(t) = f\left(t, x(t)\right), \quad x(0) = x_0 \in \mathbb{R}^n, \tag{4.7}$$

where:

$$f : T \times E^n \to E^n,$$
$$T:\ \text{finite interval } [t_0, t_1] \text{ or semi-infinite interval } [t_0, +\infty)$$

<u>Theorem:</u> (Cauchy-Peano)

Suppose for each $t \in T$, $f(t,x)$ is a Lipschitz function of x and the Lipschitz constant does not depend on t, i.e., $\exists$ a finite constant K such that

$$\frac{\|f(t,x_1) - f(t,x_2)\|}{\|x_1 - x_2\|} \le K_T(t)$$

and $\sup_{t \in T} K_T(t) = K < \infty$. Then:

(a) For each $(t_0, x_0) \in T \times E^n$, there exists a function $x \in C^n[t_0, t_1]$ satisfying (4.7),

(b) Furthermore, the function $x \in C^n[t_0, t_1]$ is unique.

<u>Notes:</u>

1) The function $x \in C^n[t_0, t_1]$ is called a trajectory.

2) The function $f(x) = \sqrt{|x|}\,\mathrm{sgn}(x) = \begin{cases} \sqrt{|x|}, & x > 0 \\ -\sqrt{|x|}, & x \le 0 \end{cases}$ is not Lipschitz because it has infinite derivative at $x = 0$.

3) Linear differential equations of the type:

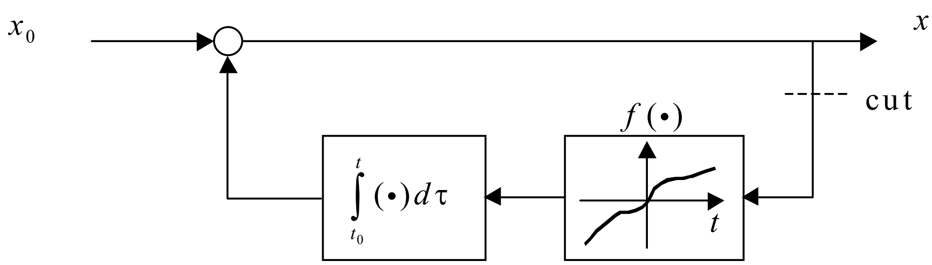
$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$
$$x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p$$

are considered as having their input $u(\bullet)$ absorbed into the time-varying $f(t, x(t))$.

4) The differential equation (4.7) is equivalent to

$$x(t) = x_0 + \int_{t_0}^{t_1} f(s, x(s))\,ds, \quad t \in [t_0, t_1],$$

which is described by the feedback system shown below.



This feedback analog suggests a method for solving (4.7) by an iteration, in which the loop is cut, and successive approximations of the trajectory over the time interval T are obtained by calculating the function obtained by going from the cut, through the loop, and back to the cut.

Thus, over $T = [t_0, t_1]$, we have

$$\begin{aligned}
x_{-1}(t) &= 0 \\
x_0(t) &= x_0 \\
x_1(t) &= x_0 + \int_{t_0}^{t_1} f\big(s, x_0(s)\big)\, ds \\
&\vdots \\
x_i(t) &= x_0 + \int_{t_0}^{t_1} f\big(s, x_{i-1}(s)\big)\, ds \\
&\vdots
\end{aligned}$$