304-303B Signals and Systems I
Solutions to Final Exam

Friday April 30, 1999
17:30-19:30, McConnell building, rooms 12 and 13
Examiner: Prof. Benoit Boulet
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- This is a closed-book examination
- Faculty-approved calculators only are allowed
- Attempt all six problems (total of 100 marks)
- Good luck

Problem 1 (10 marks)

True or False?

(a) The Fourier series coefficients a_k of a real and periodic signal $x(t)$ have the following property: $a_k^* = a_{-k}$.
TRUE.
(b) The Fourier transform $X(j\omega)$ of the convolution of two real and even signals is purely imaginary and odd.
FALSE.
(c) The system defined by $y(t) = x(t-2)$ is time-invariant.
TRUE.
(d) The signal $y[n] = e^{j2n}$ is not periodic
TRUE.
(e) The homogeneous response for $t > 0$ of $\dfrac{d^2y(t)}{dt^2} + 2\dfrac{dy(t)}{dt} + y(t) = 0$ has the form $y_h(t) = Ate^{-t}$.
FALSE.

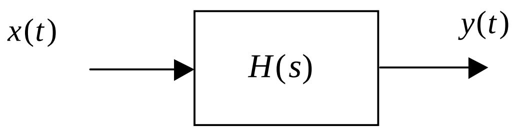
Problem 2 (20 marks)

Suppose we know that the input of a differential LTI system is

$$x(t) = te^{-2t}u(t),$$

and we measured the output to be

$$y(t) = e^{-t}\left[\cos t + \sin t\right]u(t) - e^{-2t}u(t).$$



(a) [10 marks] Find the transfer function $H(s)$ of the system and its region of convergence. Is the system causal? Is it stable? Justify your answers.

First take the Laplace transforms of the input and output signals using the table:

$$X(s) = \frac{1}{(s+2)^2}, \quad \mathrm{Re}\{s\} > -2$$

$$\begin{aligned}
Y(s) &= \underbrace{\frac{s+1}{(s+1)^2+1^2}}_{\mathrm{Re}\{s\}>-1} + \underbrace{\frac{1}{(s+1)^2+1^2}}_{\mathrm{Re}\{s\}>-1} - \underbrace{\frac{1}{s+2}}_{\mathrm{Re}\{s\}>-2} \\
&= \underbrace{\frac{s+2}{(s+1)^2+1^2}}_{\mathrm{Re}\{s\}>-1} - \underbrace{\frac{1}{s+2}}_{\mathrm{Re}\{s\}>-2} \\
&= \frac{(s^2+4s+4)-(s^2+2s+2)}{(s^2+2s+2)(s+2)}, \ \mathrm{Re}\{s\} > -1 \\
&= \frac{2(s+1)}{(s^2+2s+2)(s+2)}, \ \mathrm{Re}\{s\} > -1
\end{aligned}$$

Then, the transfer function is simply

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\dfrac{2(s+1)}{(s^2+2s+2)(s+2)}}{\dfrac{1}{(s+2)^2}} = \frac{2(s+1)(s+2)}{(s^2+2s+2)} = 2\frac{s^2+3s+2}{s^2+2s+2}$$

To determine the ROC, first note that the ROC of $Y(s)$ should contain the intersection of the ROC's of $H(s)$ and $X(s)$. There are two possible ROC's for $H(s)$: (a) an open left half-plane to the left of $\mathrm{Re}\{s\} = -1$, (b) an open right half-plane to the right of $\mathrm{Re}\{s\} = -1$. But since the ROC of $X(s)$ is an open right half-plane to the right of $s = -2$, the only possible choice is (b). Hence, the ROC of $H(s)$ is $\mathrm{Re}\{s\} > -1$.

The system is causal as the transfer function is rational and the ROC is a right half-plane. It is also stable as both complex poles $p_{1,2} = -1 \pm j$ are in the open left half-plane.

(b) [5 marks] Find an LTI differential equation representing the system.

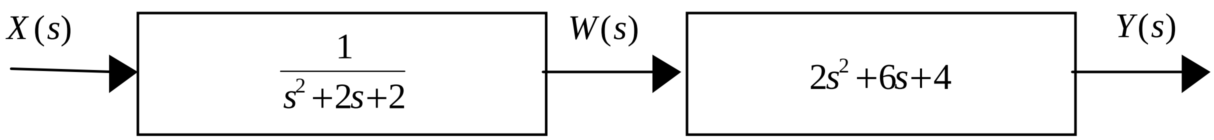
It can be derived from the transfer function obtained in a:

$$\frac{d^2y(t)}{dt^2}+2\frac{dy(t)}{dt}+2y(t)=2\frac{d^2x(t)}{dt^2}+6\frac{dx(t)}{dt}+4x(t)$$

Causality is then equivalent to the initial rest condition.

(c) [5 marks] Find the direct form realization of the transfer function $H(s)$.

The transfer function can be split up into two systems as follows:



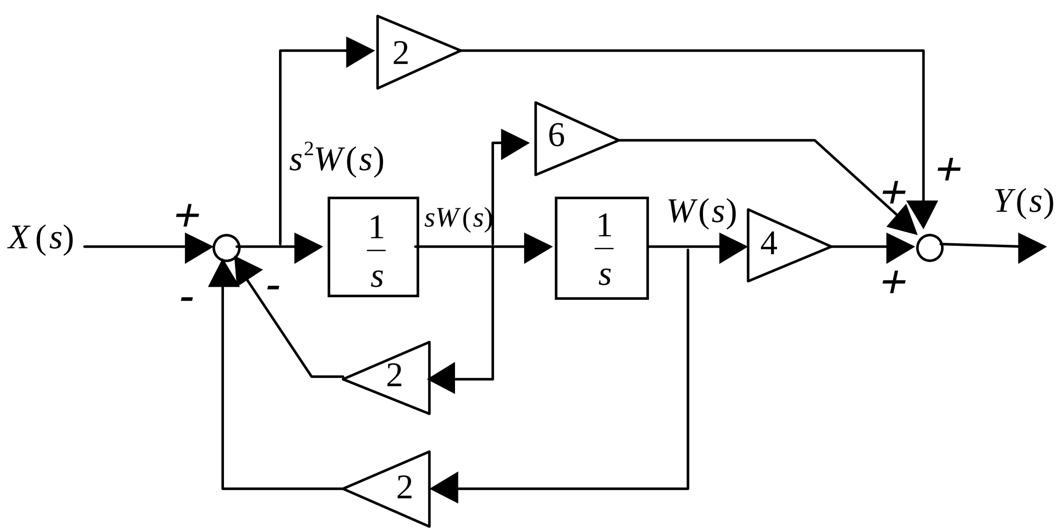
The input-output system equation of the first subsystem is

$$s^2W(s)=-2sW(s)-2W(s)+X(s)\ ,$$

and for the second subsystem we have

$$Y(s)=2s^2W(s)+6sW(s)+4W(s)\ .$$

The direct form realization of the system is given below:



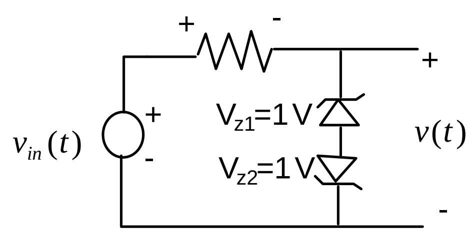
Problem 3 (15 marks)

The following circuit with two Zener diodes is an ideal clamping circuit.

The input voltage is

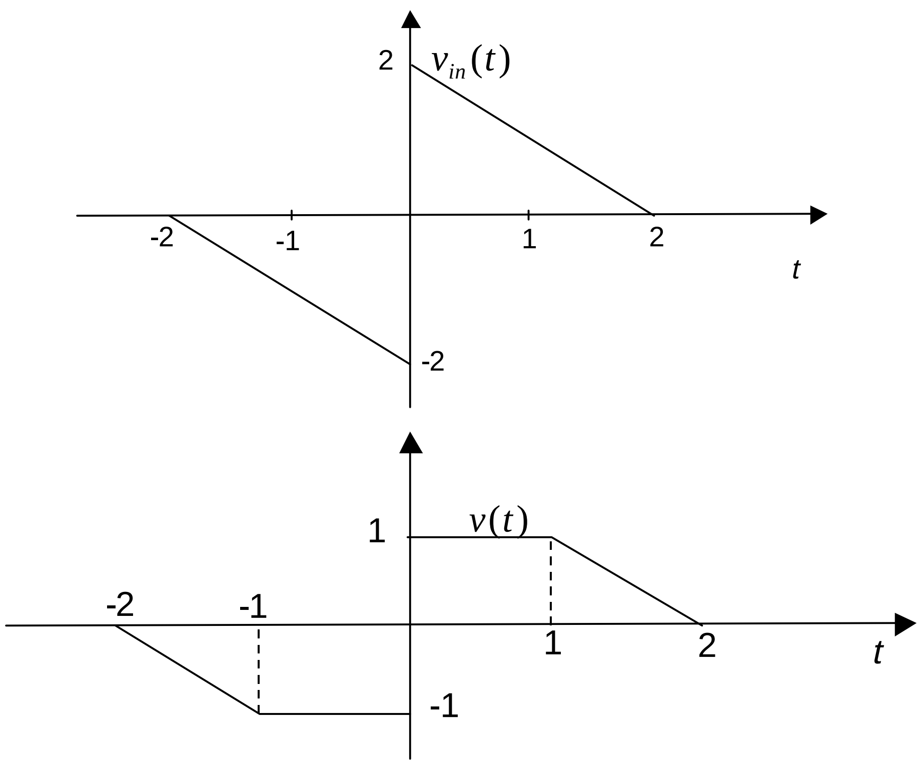
$$v_{in}(t) = (-t-2)u(t+2) + 4u(t) + (t-2)u(t-2) \text{ Volts,}$$

and the output voltage is $v(t) = \begin{cases} v_{in}(t), & -1 < v_{in}(t) < 1 \\ 1, & v_{in}(t) \geq 1 \\ -1, & v_{in}(t) \leq -1 \end{cases}$.



(a) [5 marks] Sketch the output voltage $v(t)$.

Input and output voltage:



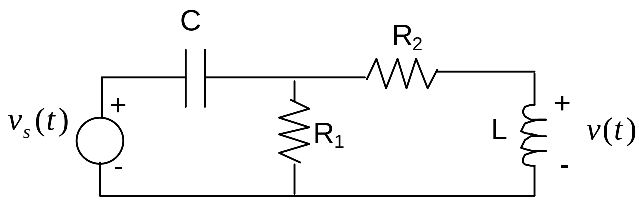
(b) [10 marks] Compute the Fourier transform of the output voltage $v(t)$.

$$
\begin{aligned}
V(j\omega) &= \int_{-\infty}^{\infty} v(t)e^{-j\omega t}dt \\
&= \int_{-2}^{-1}(-2-t)e^{-j\omega t}dt + \int_{1}^{2}(2-t)e^{-j\omega t}dt - \int_{-1}^{0}e^{-j\omega t}dt + \int_{0}^{1}e^{-j\omega t}dt \\
&= -\int_{1}^{2}(2-t)(e^{j\omega t}-e^{-j\omega t})dt - \int_{0}^{1}(e^{j\omega t}-e^{-j\omega t})dt \\
&= -2j\int_{1}^{2}(2-t)\sin(\omega t)dt - 2j\int_{0}^{1}\sin(\omega t)dt \\
&= -\frac{2j}{\omega}\big[(t-2)\cos(\omega t)\big]_{1}^{2} + \frac{2j}{\omega}\int_{1}^{2}\cos(\omega t)dt + \frac{2j}{\omega}\big[\cos(\omega t)\big]_{0}^{1} \\
&= -\frac{2j\cos\omega}{\omega} + \frac{2j}{\omega^2}(\sin 2\omega - \sin\omega) + \frac{2j}{\omega}\big[\cos\omega - 1\big] \\
&= \frac{2j}{\omega^2}(\sin 2\omega - \sin\omega - \omega)
\end{aligned}
$$

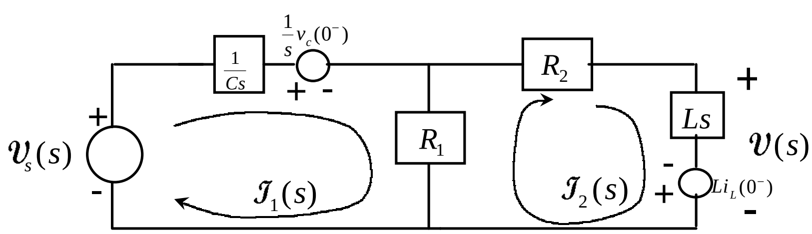
Problem 4 (25 marks)

Transformed Circuit + Bode plot

The following circuit has initial conditions on the capacitor $v_C(0^-)$ and inductor $i_L(0^-)$.



(a) Transform the circuit using the unilateral Laplace transform.



(b) Find the unilateral Laplace transform of $v(t)$.

Let's use mesh analysis.
For mesh 1:

$$\mathcal{V}_s(s)-\frac{1}{Cs}\mathcal{I}_1(s)-\frac{1}{s}v_c(0^-)-R_1[\mathcal{I}_1(s)-\mathcal{I}_2(s)]=0$$

$$\Rightarrow \quad \mathcal{I}_2(s)=-\frac{1}{R_1}\mathcal{V}_s(s)+\frac{1}{R_1 s}v_c(0^-)+(1+\frac{1}{R_1Cs})\mathcal{I}_1(s)$$

For mesh 2:

$$R_1[\mathcal{I}_1(s)-\mathcal{I}_2(s)]-(R_2+Ls)\mathcal{I}_2(s)+Li_L(0^-)=0$$

$$\Rightarrow \quad \mathcal{I}_1(s)=\frac{1}{R_1}(R_1+R_2+Ls)\mathcal{I}_2(s)-\frac{L}{R_1}i_L(0^-)$$

Substituting, we obtain

$$\mathcal{I}_2(s)=-\frac{1}{R_1}\mathcal{V}_s(s)+\frac{1}{R_1 s}v_c(0^-)+(1+\frac{1}{R_1Cs})\left[\frac{1}{R_1}(R_1+R_2+Ls)\mathcal{I}_2(s)-\frac{L}{R_1}i_L(0^-)\right]$$

$$[R_1^2Cs-(1+R_1Cs)(R_1+R_2+Ls)]\mathcal{I}_2(s)=-R_1Cs\mathcal{V}_s(s)+R_1Cv_c(0^-)-(1+R_1Cs)Li_L(0^-)$$

$$-[LR_1Cs^2+(L+R_1R_2C)s+R_1+R_2]\mathcal{I}_2(s)=-R_1Cs\mathcal{V}_s(s)+R_1Cv_c(0^-)-(1+R_1Cs)Li_L(0^-)$$

Solving for $\mathcal{I}_2(s)$, we get

$$\mathcal{I}_2(s)=\frac{R_1Cs\mathcal{V}_s(s)}{LR_1Cs^2+(L+R_1R_2C)s+R_1+R_2}+\frac{(1+R_1Cs)Li_L(0^-)-R_1Cv_c(0^-)}{LR_1Cs^2+(L+R_1R_2C)s+R_1+R_2}$$

And finally the output voltage is

$$\mathcal{V}(s)=Ls\mathcal{I}_2(s)-Li_L(0^-)=\frac{LR_1Cs^2\mathcal{V}_s(s)}{LR_1Cs^2+(L+R_1R_2C)s+R_1+R_2}+\frac{(1+R_1Cs)L^2si_L(0^-)-R_1CLsv_c(0^-)}{LR_1Cs^2+(L+R_1R_2C)s+R_1+R_2}-Li_L(0^-)$$

$$=\frac{LR_1Cs^2\mathcal{V}_s(s)}{LR_1Cs^2+(L+R_1R_2C)s+R_1+R_2}+\frac{\{(1+R_1Cs)L^2s-[L^2R_1Cs^2+L(L+R_1R_2C)s+L(R_1+R_2)]\}i_L(0^-)-R_1CLsv_c(0^-)}{LR_1Cs^2+(L+R_1R_2C)s+R_1+R_2}$$

$$=\frac{LR_1Cs^2\mathcal{V}_s(s)}{LR_1Cs^2+(L+R_1R_2C)s+R_1+R_2}+\frac{[LR_1R_2Cs+L(R_1+R_2)]i_L(0^-)-R_1CLsv_c(0^-)}{LR_1Cs^2+(L+R_1R_2C)s+R_1+R_2}$$

(c) Draw the Bode plot of the frequency response from the input voltage $\mathcal{V}_s(j\omega)$ to the output voltage $\mathcal{V}(j\omega)$ (assuming that the initial conditions on the capacitor and the inductor are 0).
Use the numerical values:

Problem 5 (20 marks)

Consider the transfer function

$$H(s) = \frac{s+5}{(s^2+2s+10)(s-3)}$$

(a) [5 marks] Give all possible regions of convergence of the transfer function $H(s)$.

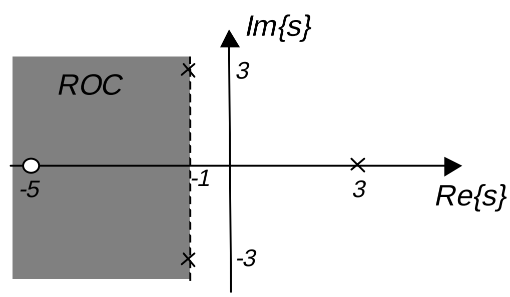
The complex poles are found by identifying the damping ratio and undamped natural frequency of the second-order denominator factor with the standard second-order polynomial $s^2 + 2\varsigma\omega_n + \omega_n^2$. Here $\omega_n = \sqrt{10}$, $\varsigma = \frac{1}{\sqrt{10}}$. Thus the poles are

$$p_1 = 3$$
$$p_2 = -\varsigma\omega_n + j\omega_n\sqrt{1-\varsigma^2} = -1+3j$$
$$p_3 = p_2^* = -1-3j$$

There are three possible ROCs: $\mathrm{Re}\{s\} < -1, \quad -1 < \mathrm{Re}\{s\} < 3, \quad \mathrm{Re}\{s\} > 3$.

(b) [10 marks] Give the region of convergence that corresponds to a left-sided impulse response. Compute the left-sided impulse response.

The region of convergence corresponding to a left-sided impulse response is $\mathrm{Re}\{s\} < -1$



The partial fraction expansion of $H(s)$ yields:

$$H(s)=\frac{s+5}{(s^2+2s+10)(s-3)}=\underbrace{\frac{-24+j7}{150}}_{A}\frac{1}{s+1-3j}+\underbrace{\frac{-24-j7}{150}}_{B}\frac{1}{s+1+3j}+\underbrace{\frac{8}{25}}_{C}\frac{1}{s-3}$$

$$A=\left.\frac{s+5}{(s+1+3j)(s-3)}\right|_{s=-1+3j}=\frac{4+3j}{(6j)(-4+3j)}=\frac{(4+3j)(-4-j3)}{150j}=\frac{-7-j24}{150j}=\frac{-24+7j}{150}$$

$$B=A^*=\frac{-24-j7}{150}$$

$$C=\left.\frac{s+5}{(s+1+3j)(s+1-3j)}\right|_{s=3}=\frac{8}{(4+3j)(4-3j)}=\frac{8}{25}$$

Using the table and simplifying, we find the following impulse response:

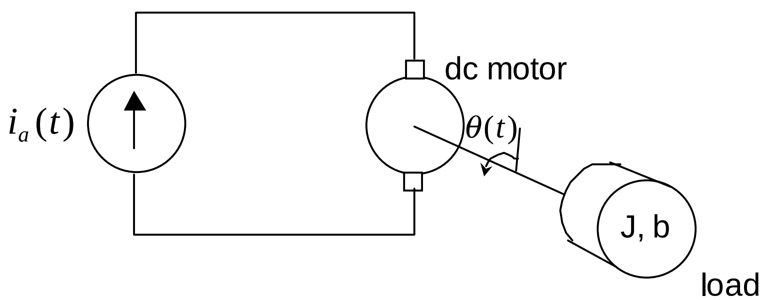
ROC $\mathrm{Re}\{s\}<-1$:

$$\begin{aligned}
h(t)&=-\frac{-24+j7}{150}e^{(-1+j3)t}u(-t)-\frac{-24-j7}{150}e^{(-1-j3)t}u(-t)-\frac{8}{25}e^{3t}u(-t)\\
&=\frac{24-j7}{150}e^{(-1+j3)t}u(-t)+\frac{24+j7}{150}e^{(-1-j3)t}u(-t)-\frac{8}{25}e^{3t}u(-t)\\
&=\frac{1}{150}2e^{-t}\,\mathrm{Re}\left\{(24-j7)e^{j3t}\right\}u(-t)-\frac{8}{25}e^{3t}u(-t)\\
&=\frac{2*25}{150}e^{-t}\cos(3t-0.283794)u(-t)-\frac{8}{25}e^{3t}u(-t)\\
&=\frac{1}{3}e^{-t}\cos(3t-0.283794)u(-t)-\frac{8}{25}e^{3t}u(-t)\\
&=\frac{1}{75}e^{-t}[24\cos(3t)+7\sin(3t)]u(-t)-\frac{8}{25}e^{3t}u(-t)
\end{aligned}$$

Problem 6 (20 marks)

DC motor control system

A typical method to control the position of an inertial load driven by a permanent magnet DC motor is to vary the armature current based on a feedback measurement of the load angle. Before closing the feedback loop, we look at the open-loop dynamics of this system.



The torque $\tau(t)$ in Newton-metres applied to the load by the motor is proportional to the armature current in Amps:

$$\tau(t) = Ai_a(t)\,.$$

The load is assumed to be an inertia with viscous friction. The equation of movement for the load is

$$J\frac{d^2\theta(t)}{dt^2} + b\frac{d\theta(t)}{dt} = \tau(t)\,.$$

(a) Assume that the system is initially at rest. Find the transfer function $G(s) := \Theta(s)/I_a(s)$ relating the load angle $\Theta(s)$ to the motor's armature current input $I_a(s)$.

$$G(s) = \frac{\Theta(s)}{I_a(s)} = \frac{A}{s(Js+b)}$$

(b) Assume that $A = 1\,\mathrm{Nm/A}$, $J = 1\,\mathrm{Nm/}^{\mathrm{rd}}\!/_{\mathrm{s^2}}$ and $b = 2\,\mathrm{Nm/}^{\mathrm{rd}}\!/_{\mathrm{s}}$ Compute the unit step response $s_\omega(t)$ of the load's *angular velocity* $\omega(t) = \dfrac{d\theta(t)}{dt}$ for a unit step in armature current. What is the ±5% settling time t_s (an approximation is OK)?

$$G_\omega(s) = \frac{s\Theta(s)}{I_a(s)} = \frac{1}{s+2}$$

The unit step response $s_\omega(t)$ is given by

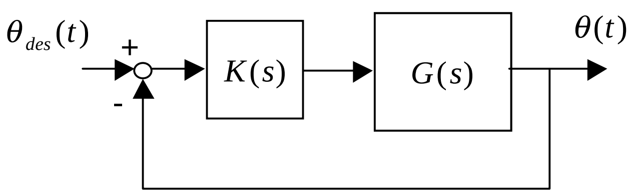
$$\frac{1}{s}G_\omega(s)=\frac{\Theta(s)}{I_a(s)}=\frac{1}{s(s+2)}=\frac{0.5}{s}-\frac{0.5}{s+2}$$

Taking the inverse Laplace transform, we get

$$s_\omega(t)=\frac{1}{2}(1-e^{-2t})u(t)$$

The ±5% settling time is equal to three time constants: $t_s=\frac{3}{2}s$.

(c) Assume that we can measure the load angle $\theta(t)$ perfectly. We want to test the feedback controller $K(s)=\lambda(s+\alpha)$ to control the load angle.



Find the closed-loop transfer function $H(s):=\Theta(s)/\Theta_{des}(s)$. Use Routh's criterion to determine the ranges of λ and α for which the closed-loop system is stable.

Closed-loop transfer function:

$$\begin{aligned}H(s)&=\frac{K(s)G(s)}{1+K(s)G(s)}\\&=\frac{\frac{\lambda(s+\alpha)}{s(s+2)}}{1+\frac{\lambda(s+\alpha)}{s(s+2)}}\\&=\frac{\lambda(s+\alpha)}{s(s+2)+\lambda(s+\alpha)}\\&=\frac{\lambda(s+\alpha)}{s^2+(2+\lambda)s+\alpha\lambda}\end{aligned}$$

We compute the Routh array:

$$\begin{array}{c|cc} s^2 & 1 & \alpha\lambda \\ s^1 & 2+\lambda & \\ s^0 & \alpha\lambda & \end{array}$$

The system is stable for $2+\lambda > 0$ and $\alpha\lambda > 0$ which gives us the stability ranges

$-2 < \lambda < 0$ and $\alpha < 0$

$\lambda > 0$ and $\alpha > 0$

(d) Compute the closed-loop load angle response to a 1-radian step in desired angle with $\lambda = 10$ and $\alpha = 2$.

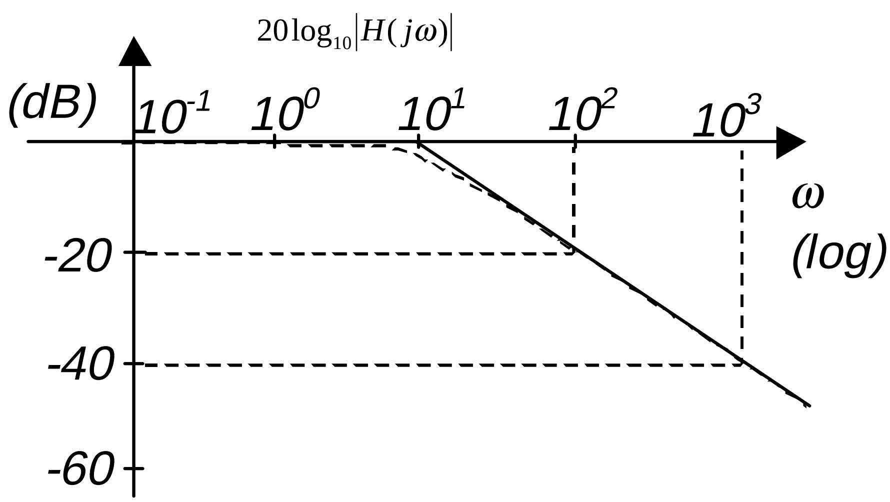
The closed-loop transfer function is

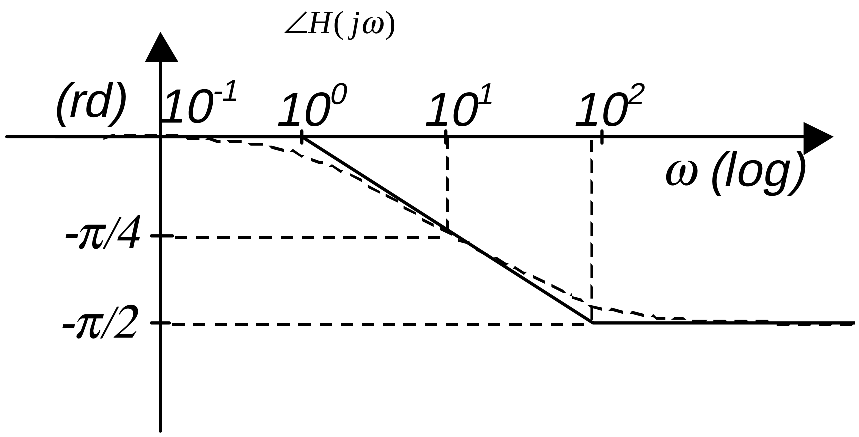
$$H(s) = \frac{10(s+2)}{s^2+12s+20} = \frac{10(s+2)}{(s+2)(s+10)} = \frac{10}{(s+10)}$$

The step response is simply

$$s_\theta(t) = (1-e^{-10t})u(t)$$

(e) Draw the Bode plot (magnitude and phase) of the closed-loop frequency response $H(j\omega)$ for $\lambda = 10$ and $\alpha = 2$.




∠H(jω)
(rd)
10^-1
10^0
10^1
10^2
ω (log)
-π/4
-π/2