Lecture 30: Frequency Response of Second-Order Systems

9.4 Frequency Response of Second-Order Systems

A general second-order system has a transfer function of the form

$$H(s) = \frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0}. \tag{9.24}$$

It can be stable, unstable, causal or not, depending on the signs of the coefficients and the specified ROC. Let's restrict our attention to causal, stable LTI second-order systems of this type. It can be shown that a necessary condition for stability is that the coefficients a_i be all positive, or all negative (this is also true for higher order systems). Let's also assume that $b_2 = b_1 = 0$, i.e. we basically have a lowpass system. Under these conditions the transfer function can be expressed as

$$H(s) = \frac{A\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2}. \tag{9.25}$$

where ς is the damping ratio ω_n is the undamped natural frequency of the second-order system. Systems such as the mass-spring-damper system or a lowpass second-order filter can be modeled by this transfer function.

Example: Consider the second-order transfer function

$$\begin{aligned} H(s) &= \frac{1}{-2s^2 - 6s - 9} \\ &= -\frac{1}{2}\frac{1}{s^2 + 3s + 9/2} \end{aligned} \tag{9.26}$$

Its undamped natural frequency is $\omega_n = 3\sqrt{2}/2$, and its damping ratio is

$$\varsigma = \frac{3}{2\omega_n} = \frac{3}{2\,\frac{3\sqrt{2}}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = 0.707 \tag{9.27}$$

Since the damping ratio is less than one, the two poles are complex. The poles are

$$\begin{aligned} p_1 &= -\varsigma\omega_n + j\omega_n\sqrt{1-\varsigma^2} = -\frac{3\sqrt{2}}{2}\frac{\sqrt{2}}{2} + j\frac{3\sqrt{2}}{2}\sqrt{1-\frac{1}{2}} = -\frac{3}{2} + j\frac{3}{2} \\ p_2 &= -\varsigma\omega_n - j\omega_n\sqrt{1-\varsigma^2} = -\frac{3}{2} - j\frac{3}{2} \end{aligned} \tag{9.28}$$

There are three cases of interest for the damping ratio that lead to different pole patterns and frequency response types.

9.4.1 Case $\varsigma > 1$

In this case, the system is said to be *overdamped*. The step response doesn't exhibit any ringing. The two poles are real, negative and distinct:
$p_1 = -\varsigma\omega_n + \omega_n\sqrt{\varsigma^2 - 1}, \quad p_2 = -\varsigma\omega_n - \omega_n\sqrt{\varsigma^2 - 1}$. The second-order system can be seen as a cascade of two standard first-order systems (lags).

$$H(s) = \frac{A\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2} = \frac{A\omega_n^2}{p_1 p_2}\frac{1}{\frac{s}{-p_1}+1}\frac{1}{\frac{s}{-p_2}+1} = A\frac{1}{\frac{s}{-p_1}+1}\frac{1}{\frac{s}{-p_2}+1} \qquad (9.29)$$

The Bode plot of $H(j\omega) = A\frac{1}{\frac{j\omega}{-p_1}+1}\frac{1}{\frac{j\omega}{-p_2}+1}$ can then be sketched using the technique presented in the previous lectures for systems with real poles and zeros.

9.4.2 Case $\varsigma = 1$

In this case, the system is said to be *critically damped*. The two poles are negative and real, but they are the same. We say that it's a repeated pole; $p_1 = -\varsigma\omega_n + j\omega_n\sqrt{1-\varsigma^2} = -\varsigma\omega_n = p_2$. In this situation, the second-order system can also be seen as a cascade of two first-order transfer functions having the same pole.

$$H(s) = A\frac{1}{(\frac{s}{-p_1}+1)^2} \qquad (9.30)$$

9.4.3 Case $\varsigma < 1$

In this case, the system is said to be *underdamped*. The step response exhibits some ringing, although it really becomes visible only for $\varsigma < 1/\sqrt{2} = 0.707$. The two poles are distinct, complex conjugates of each other: $p_1 = -\varsigma\omega_n + j\omega_n\sqrt{1-\varsigma^2}, \quad p_2 = -\varsigma\omega_n - j\omega_n\sqrt{1-\varsigma^2}$.

The magnitude and phase of

$$H(j\omega) = \frac{A\omega_n^2}{(j\omega)^2 + 2\varsigma\omega_n(j\omega) + \omega_n^2} = \frac{A}{\frac{(j\omega)^2}{\omega_n^2} + \frac{2\varsigma}{\omega_n}(j\omega) + 1} \qquad (9.31)$$

are given by:

$$20\log_{10}\left|H(j\omega)\right| = 20\log_{10} A - 10\log_{10}\left\{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + 4\varsigma^2\frac{\omega^2}{\omega_n^2}\right\}, \qquad (9.32)$$

$$\angle H(j\omega) = -\arctan\left\{\frac{2\varsigma\frac{\omega}{\omega_n}}{\left(1-\frac{\omega^2}{\omega_n^2}\right)}\right\}. \qquad (9.33)$$

Notice that the denominator in Equation (9.31) was written in such a way that its dc gain is 1. The break frequency is simply the undamped natural frequency. At this frequency, the magnitude is

$$20\log_{10}\left|H(j\omega_n)\right| = -20\log_{10}\left\{2\varsigma\right\}. \qquad (9.34)$$

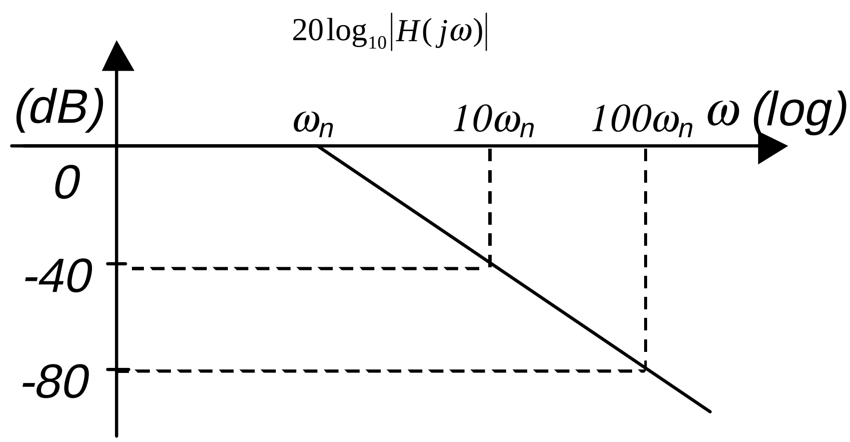
For example, with $\varsigma = 0.1$ and $A = 1$, the magnitude at ω_n is 13.98dB. Note that this is not the maximum of the magnitude as it occurs at the *resonant frequency* $\omega_{\max} = \omega_n\sqrt{1-2\varsigma^2}$ which is close to ω_n for low damping ratios. At the resonant frequency, the magnitude of the peak resonance is given by

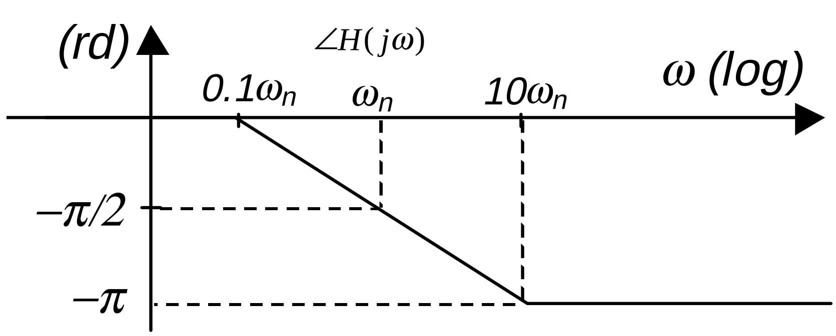
$$\begin{aligned}20\log_{10}\left|H(j\omega_{\max})\right| &= -10\log_{10}\left\{\left(1-\tfrac{\omega_n^2(1-2\varsigma^2)}{\omega_n^2}\right)^2 + 4\varsigma^2\,\tfrac{\omega_n^2(1-2\varsigma^2)}{\omega_n^2}\right\}\\ &= -10\log_{10}\left\{4\varsigma^4 + 4\varsigma^2(1-2\varsigma^2)\right\}\\ &= -10\log_{10}\left\{4\varsigma^2(1-\varsigma^2)\right\}\\ &= -20\log_{10}\left\{2\varsigma\sqrt{1-\varsigma^2}\right\}\end{aligned}$$

and thus for our example $20\log_{10}\left|H(j\omega_{\max})\right| = -20\log_{10}\left\{0.2\sqrt{1-0.4}\right\} = 16.20\,dB$.

The Bode plot for the case $\varsigma < 1$ can be sketched using the asymptotes, but at the price of an increasingly large approximation error around ω_n that increases as the damping ratio decreases.

The roll-off rate past the break frequency is -40dB/decade. The phase starts at 0 and tends to $-\pi$ at high frequencies.





The Bode plot approximation using the asymptotes does not convey the information of the resonance in the system caused by the complex poles. That is, the damping ratio was not used to draw the asymptotes for the magnitude and the phase plot. The peak resonances in the magnitude produced by different values of $\varsigma < 1$ are shown in Figure 6.23 in the textbook. This figure also shows that the phase drop around ω_n becomes steeper as the damping ratio is decreased.

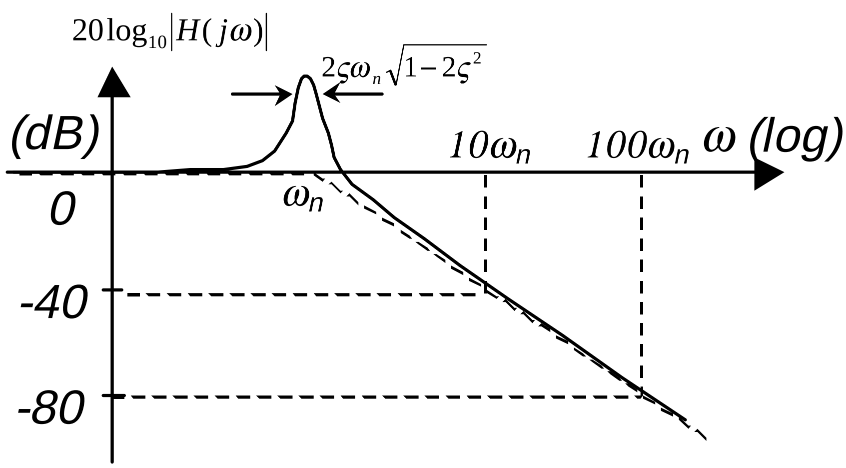
9.4.4 Quality Q

In the field of communications, the underdamped second-order filter has played an important role as a simple frequency-selective bandpass filter. When the damping ratio is very low, the filter becomes highly selective due to its high peak resonance at $\omega_{\max}$. The *quality Q* of the filter is defined as

$$Q = \frac{1}{2\varsigma}. \tag{9.35}$$

The higher the quality, the more selective the filter is. To support this claim, the $-3dB$ bandwidth (frequency band between the two frequencies where the magnitude is 3dB lower than $20\log_{10}|H(j\omega_{\max})|$) of the bandpass (not really considered as lowpass for high Q) second-order filter can be shown to be

$$\Delta\omega \approx \frac{\omega_{\max}}{Q} = 2\varsigma\omega_{\max}. \tag{9.36}$$



9.4.5 Maximal Flatness and Butterworth Filters

In the transition from a second-order system without resonance and one that starts to exhibit a peak that is higher than the dc gain, there must be an optimal damping ratio ς for which the magnitude stays flat over the widest possible bandwidth before rolling off. It turns out that this occurs for $\varsigma < 1/\sqrt{2} = 0.707$. For this damping ratio, the real part and imaginary part of the two poles all have the same absolute value, and the poles can be expressed as

$$p_1 = \omega_n e^{j\frac{3\pi}{4}}, \quad p_2 = \omega_n e^{-j\frac{3\pi}{4}}. \tag{9.37}$$

We recognize the poles of a lowpass second-order Butterworth filter with cutoff at ω_n. Thus, a Butterworth filter is optimized to be *maximally flat*.