Lecture 27: Transient and Steady-State Responses of LTI Differential Systems

8.5 Transient and Steady-State Responses of LTI Differential Systems

The *transient response* (also called natural response) of a causal, stable LTI differential system is the homogeneous response, i.e., with the input set to zero.

The *steady-state response* (or forced response) is the particular solution corresponding to a constant or periodic input. We say that a stable system is in steady-state when the transient component of the output has practically disappeared. For example, consider the step response

$$s(t) = u(t) - e^{-5t}u(t) . \tag{8.35}$$

The transient part of this response is the term $e^{-5t}u(t)$, and the steady-state part is $u(t)$.

As another example, assume that a causal LTI differential system is subjected to the sinusoidal input signal $x(t) = \sin(\omega_0 t)u(t)$. Suppose that the resulting output is

$$y(t) = 2\sin(\omega_0 t - \phi)u(t) + e^{-2t}\cos(2t+\theta)u(t) . \tag{8.36}$$

Then the transient response of the system to the input is $e^{-2t}\cos(2t+\theta)u(t)$ while $2\sin(\omega_0 t - \phi)u(t)$ is the steady-state response.

It is a fact that the steady-state response of a causal, stable LTI system to a sinusoidal input of frequency ω_0 is also sinusoidal input of frequency ω_0, although in general with a different amplitude and phase.

8.5.1 Transient and Steady-State Analysis Using the Laplace Transform

For a causal, stable LTI system, a partial fraction expansion of the transfer function allows us to determine which terms correspond to transients (the terms with the system poles) and which correspond to the steady-state response (terms with the input poles).

Example: Consider the step response

$$\begin{aligned} Y(s) &= \frac{s+3}{\left(s^2+3s+2\right)} X(s), \quad \mathrm{Re}\{s\} > -1 \\ &= \frac{s+3}{\left(s^2+3s+2\right)s}, \quad \mathrm{Re}\{s\} > -1 \\ &= \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s}, \quad \mathrm{Re}\{s\} > -1 \end{aligned} \tag{8.37}$$

The steady-state response corresponds to the last term $\frac{C}{s}$, which in the time-domain is $Cu(t)$.

The other two terms correspond to the transient response $Ae^{-t}u(t) + Be^{-2t}u(t)$.

For the steady-state response of an LTI system, there is actually no need to do a partial fraction expansion. The transfer function and the frequency response of the system (which is the transfer function evaluated at $s = j\omega$) directly give us the answer.

<u>Step response:</u> We can use the final value theorem to determine the steady-state component of a step response. In general, this component is a step function $Au(t)$. The "gain" A is given by

$$A = \lim_{s\to 0} sH(s)\frac{1}{s} = H(0) \qquad (8.38)$$

<u>Response to a sinusoid or a periodic exponential:</u>

The frequency response of the system directly gives us the steady-state response to a sinusoid or a periodic complex exponential signal. For the latter, $x(t) = Ae^{j\omega_0 t}$, the steady-state response is

$$\begin{aligned} y_{ss}(t) &= H(j\omega_0)Ae^{j\omega_0 t} \\ &= \left|H(j\omega_0)\right|Ae^{j(\omega_0 t + \angle H(j\omega_0))} \end{aligned} \qquad (8.39)$$

For a sinusoidal input, say $x(t) = A\sin(\omega_0 t)$, it is easy to show that the steady-state response is (same thing for a cosine)

$$\begin{aligned} y_{ss}(t) &= H(j\omega_0)A\sin(\omega_0 t) \\ &= \left|H(j\omega_0)\right|A\sin(\omega_0 t + \angle H(j\omega_0)) \end{aligned} \qquad (8.40)$$

An important application is the steady-state analysis of circuits at a fixed frequency, e.g., 60 Hertz. For example, if a circuit is described by its impedance $Z(s)$, then its steady-state response to a 60 Hertz sinusoidal current is characterized by the complex number $Z(j2\pi 60)$.

<u>Response to a periodic signal:</u>

Again, the frequency response of the system gives us the steady state response to a periodic signal admitting a Fourier series representation. For $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$, the steady-state response is

$$y_{ss}(t) = \sum_{k=-\infty}^{+\infty} H(jk\omega_0)a_k e^{jk\omega_0 t}. \qquad (8.41)$$