McGill University
Department of Electrical and Computer Engineering

ECSE 506 - Stochastic Control

Project Report

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Team Decision Theory and Information Structures in Optimal Control Problems – Part I
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1 Introduction

1.1 Team Decision Problem

- A team is an organization in which there is a single goal or payoff common to all the members. Each member receives certain information z_i and controls the decision variable u_i.
- The payoff of the decision problem (J) is defined to be a function of the control action taken by each member.

$$J = J(\gamma_1, \gamma_2, ..., \gamma_N) \tag{1}$$

- The control action u it is a policy function γ that depends on the observation of each member.

$$u_i = \gamma_i(z_i) \;\; and \;\; \gamma_i \in \Gamma_i \tag{2}$$

where Γ_i represents the set of admissible control law.
The objective of the team decision problem is to find $\gamma_i^* \in \Gamma_i \forall i$ such that, $J(\gamma_1, \gamma_2, ..., \gamma_N)$ is minimized.

1.2 Types of a Team Decision Problem

Team decision problem is of two types: static, and dynamics. In a static team decision problem, the information state depends only on some random variable ξ which represents the uncertainty of the external world which are not controlled by any member. While in dynamic team decision problem, the present information also depends on the past control actions.
Prior to this paper, only static team decision problems have been considered (Marschak and Radner). This paper by Yu–chi and Chu is a start point for tackling dynamics team decision problem, where they proved that if the information has a special structure (called partially nested), the dynamic problem can be converted into a static one and hence, the previously developed methods can still be employed for solving these particular problems.

1.3 Information Structure

The information z_i is assumed to be a known linear combination of ξ and some of the control actions of the predecessor members, i.e.

$$z_i = H_i \xi + \sum_j D_{ij} u_j \qquad \forall i$$

where $\xi \in \mathbb{R}^n$ defined on the probability space $(\mathbb{R}^n, \mathbf{F}, P)$ represents all the uncertainty that cannot be controlled by any member.
Assume that:

- ξ is Gaussian $N(0, X)$ with $X > 0$ and this probability distribution is know to all the members,
- H_i and D_{ij} are known to all members, and
- The system is causal, i.e. if $D_{ij} \neq 0$ then $D_{ji} = 0, \quad \forall i, j \in I$

Definition 1. *We say member j is related to member i i.e. jRi, if $D_{ij} \neq 0$*

Definition 2. *We say member j is precedent to member i, denoted by $j\{i$, iff*

(a) jRi or

(b) $\exists\ r, s, t, ..., k \in I$ such that $jRr, rRs, ..., kRi$

In graphical representation $j\{i$ is equivalent to the existence of a path from member i to member j .

Example 1. *In a **static** team decision problem with N members, no precedence exists among members.*

$$D_{ij} = 0 \qquad \forall i, j \in I$$
$$z_i = H_i \xi \qquad \forall i$$

Figure 1 represent the precedence diagram for the static case.

Figure 1: Precedence Diagram for a Static Team Decision Problem

Example 2. *The Classical multi-stage stochastic control problem.*

$$x_{i+1} = Fx_i + Gu_i + \omega_i, \qquad i = 1, .., N$$

with the observation at each stage:

$$y_i = Hx_i + v_i \quad i = 1, ..., N$$

The distribution of x_1, w_i, and v_i are independent, zero-mean and Gaussian. The system is assumed to posses perfect memory i.e. $x_i =$ linear in $(x_1, u_1, ..., u_{i-1}, \omega_1, ..., \omega_{i-1})$ $\forall i$

Thus, the information state can be expressed as:

$$z_i = H_i \xi + \sum_j D_{ij} u_j \tag{3}$$

where $\xi = (x_1, u_1, ..., u_{i-1}, \omega_1, ..., \omega_{i-1})$. Figure 2 illustrate the fact that z_j is included in z_i if $i < j$. This fact is

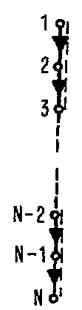

Figure 2: Precedence Diagram for Multi-stage stochastic control

stressed by the doted line in Figure 2.

Example 3. *Consider the following partially nested Linear-Gaussian information structure*

$z_1 = H_1 \xi$
$z_2 = H_2 \xi$
$z_3 = H_3 \xi + D_{31} u_1 + D_{32} u_2$

$$z_4 = \begin{bmatrix} H_4' \\ H_3 \end{bmatrix} \xi + \begin{bmatrix} 0 & 0 & D_{43} \\ D_{31} & D_{32} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$
$$z_5 = D_{54} u_4$$
$$z_6 = H_6 \xi + D_{64}$$
$$z_7 = H_7 \xi$$
$$z_8 = D_{87} u_7$$

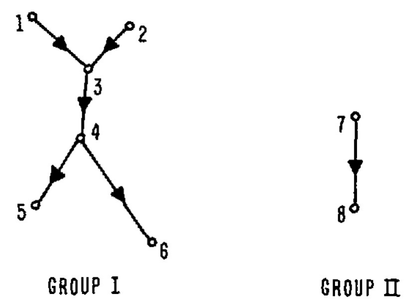

Figure 3: Precedence Diagram for Example 3

Figure 3 represents the precedence diagram of example 3.

1.4 Control Laws

The set of admissible control laws for the i^{th} decision maker, Γ_i, is defined as the the set of all Borel-measurable functions $\gamma_i : \mathbb{R}^{q_i} \to \mathbb{R}^{k_i}$ [1]. If we take $u_i \in U_i = \mathbb{R}^{k_i}$, then we have a σ-algebra $\mathbb{F}_i$ on U_i such that $\gamma_i^{-1}(\mathbb{F}_i) = \mathbb{Z}_i$. It is important to note that except for example 1, $\mathbb{F}_i$ and $\mathbb{Z}_i$ $\forall i$, are generally dependent on the choice of $\gamma = [\gamma_1, ..., \gamma_N]$.

1.5 Payoff Function

The goal is to minimize $J(\gamma_1, ..., \gamma_N)$. This goal is the same for all the members in the team. Here we are only considering quadratic utility functions i.e.

$$J(\gamma_1, ..., \gamma_N) = E[\jmath] = E_\xi[\frac{1}{2} u^T Q u + u^T S \xi + u^T c], \tag{4}$$

where $u = \begin{bmatrix} u_1 \\ . \\ . \\ . \\ u_N \end{bmatrix}$, $Q > 0$, $Q^T = Q$ and the expectation is taken with respect to a priori ξ. Matrices, Q, S, and vector c are known to all the members

2 Necessary Conditions for optimality

A necessary condition for optimality is the so-called "member-by-member optimal" condition which is defined by finding $\gamma_i(z_i) \in \Gamma_i$ such that $u_i = \gamma_i(z_i)$ satisfies:

$$\min_{u_i} E_\xi\{J(\gamma_1^*(z_1), ..., \gamma_{i-1}^*(z_{i-1}), \gamma_i(z_i), \gamma_{i+1}^*(z_{i+1}), ..., \gamma_N^*(z_N)|z_i\} \tag{5}$$

For the considered quadratic utility function (4), this is equivalent to:

$$\min_{u_i} E\{\frac{1}{2}u^T Qu + u^T S\xi + u^T c|z_i\} = \min_{u_i} J_i \tag{6}$$

where, $u^T = [\gamma_1^*(z_1), ..., \gamma_{i-1}^*(z_{i-1}), \gamma_i(z_i), \gamma_{i+1}^*(z_{i+1}), ..., \gamma_N^*(z_N)]$ for fixed/$\gamma_j^*(z_j)$. Taking partial derivatives of J_i with respect to u_i and keeping in mind that for $j\}i$ the terms involving u_j are dependent on u_i due to causality, the necessary condition for optimality becomes:

$$Q_{ii}\gamma_i + \sum_{i\neq j} Q_{ii}E(\gamma_j|z_i) + S_iE(\xi|z_i) + c_i + \sum_{j\}i}[u_i^T Q_{ij}\frac{\partial}{\partial u_i}E(\gamma_j|z_i) + \frac{\partial}{\partial u_i}E(\gamma_j^T S_j\xi|z_i) + \frac{\partial}{\partial u_i}E(\gamma_j^T|z_i)c_j + \sum_{k\neq i}\frac{\partial}{\partial u_i}E(\gamma_k^T Q_{kj}\gamma_j|z_i)] = 0 \tag{7}$$

The last four partial derivative terms of (7) depend explicitly on the form of controls of the following members. This is rather unsatisfactory in as much as we are attempting to solve for those control laws through the consideration of (7).

3 Static Team

As shown by Radner [2], in the case of static teams the quadratic utility function is strictly convex in all γ_i, and hence the necessary condition for member-by-member optimization is sufficient for optimality.
In static teams, each member's information is a linear function of ξ only i.e.

$$D_{ij} = 0 \quad in \quad (3)$$

$$z_i = H_i\xi \quad \forall i$$

where H_i is $q_i \times n$ with $n > q_i$ and it is of maximal rank i.e. rank(H_i)=q_i.
Thus, the necessary condition for optimality (7) simplifies to:

$$Q_{ii}\gamma_i(z_i) + \sum_{j\neq i} Q_{ij}E(\gamma_j(z_j)|z_i) + S_iE(\xi|z_i) + c_i = 0 \quad \forall z_i \quad and \quad \forall i \tag{8}$$

since all partial derivatives in (7) vanish. Also, the optimal solution for (4) is unique in a static team decision problem.
Because foe a jointly Gaussian random variables the best estimator is linear, we seek control laws of the form:

$$u_i = \gamma_i(z_i) = A_iz_i + b_i\forall i \tag{9}$$

Solving (8) using (9) we get the following for b and A:

$$(b_1^T, ..., b_N^T) = c^T Q^{-1} \tag{10}$$

$$\sum_j Q_{ij}A_j(H_jXH_i^T) = -S_iXH_i^T, \forall i \tag{11}$$

The coefficient of the elements of A_i in (11) form a positive definite matrix, hence each matrix A_i is uniquely solved from (11).
As a conclusion (Radner's Theorem) [2], the control law(9)with (10) and (11) is optimal for static-team optimization problem.

4 Dynamic Team with Partially Nested Information Structure

Based on Eq.(7), in a general case of dynamic teams neither the payoff function is necessarily convex in γ_i even though it is quadratic in u with $Q > 0$, nor are z_i Gaussian even though ξ is given as Gaussian. However, Ho and Chu [1] proved that for a particular information structure called *partially nested*, there exists a transformed information structure $\widehat{z}_i$ for which the utility function is strictly convex. Thus, the unique optimal control law is linear in $\widehat{z}_i$.

Definition 3: An information structure (3) is called partially nested if $j\{i$ implies $\mathbf{Z}_j \subset \mathbf{Z_i}$ for all i, j, and $\gamma \in \Gamma$.

Theorem 1. *In a dynamic team with the partially nested information structure,*

$$z_i = H_i\xi + \sum_{j\{i} D_{ij}u_j \tag{12}$$

is equivalent to an information structure in static form for any fixed set of control laws

$$\widehat{z}_i = [H_j\xi \ \mid \ j\{i \ \ or \ \ j = i]. \tag{13}$$

Proof. We partition the N-member into the following disjoint sets:
N_1 = {set of starting members}
N_2 = {set of members having i as precedent, where $i \in N_1$ }

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N_j = {set of members having i as a precedent, where $i \in N_{j-1}$ }.

It is clear that $\widehat{z}_i = z_i = H_i\xi$ for all $i \in N_1$. Now let

$$\widehat{z}_i = z_i - \sum_{j\{i} D_{ij}\gamma_j(z_i) = H_i\xi \qquad \forall i \in N_2 \qquad and \qquad j \in N_1 \tag{14}$$

since $Z_i \supset Z_j$, then $\widehat{z}_i$ will be Z_i measurable. Conversely, knowing $\widehat{z}_i$, we compute

$$z_i = \widehat{z}_i + \sum_{j\{i} D_{ij}\gamma_j(\widehat{z}_i) \qquad \forall i \in N_2 \qquad and \qquad j \in N_i \tag{15}$$

Now, by recursion we can calculate z_i from $\widehat{z}_i$ or vice versa for $i \in N_3, N_4, ..., etc.$ [1] □

Remark 1. *The reduction of (26) to (27) is possible only when we are considering pure strategy solution exclusively.*

Remark 2. *Note that the validity of Theorem 1 is independent of the nature of the criterion function J. Furthermore, z_i need not be linear function of ξ and u_j for $j\{i$; nor does ξ have to be Gaussian.*

Example 4. *A partially nested information structure*

$z_1 = H_1\xi$
$z_2 = H_2\xi$

$$z_3 = \begin{bmatrix} H_1 \\ H_2 \\ H_3' \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 0 \\ D_{31}' \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ D_{32}' \end{bmatrix} u_2$$

$$z_4 = \begin{bmatrix} H_1 \\ H_4' \end{bmatrix} \xi + \begin{bmatrix} 0 \\ D_{41}' \end{bmatrix} u_1$$

$$z_5 = \begin{bmatrix} H_1 \\ H_2 \\ H'_3 \\ H'_5 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 0 \\ D'_{31} \\ D'_{51} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ D'_{32} \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ D'_{53} \end{bmatrix} u_3$$

$$z_6 = H_6 \xi$$

$$z_7 = \begin{bmatrix} H_6 \\ H'_7 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ D'_{76} \end{bmatrix} u_6$$

$$z_8 = H_8 \xi$$

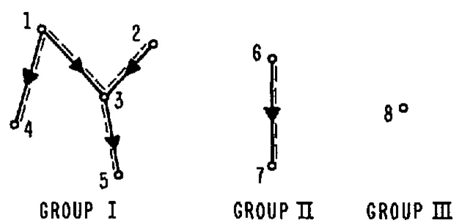

Figure 4: Precedence Diagram for Example 4

Figure 4 shows the precedence diagram of the above example. From Theorem 1 we can convert this dynamic team problem to a static one with the information structure as:

$$\widehat{z}_i = H_i \xi$$

Theorem 2. *In dynamic team with partially nested information structure, the optimal control for each member exits, is unique, and linear in z_i.*

Proof. As shown by Theorem 1, a team with information structure (12) is equivalent to one with static structure (13). Therefore, by Radner's theorem [2], the optimal control law exists, is unique and linear in $\widehat{z}_i$. Since $\widehat{z}_i = z_i$ for $i \in N_1$, γ_i is linear in z_i for $i \in N_1$ also. By (14) we deduce that γ_i is also linear in z_i for $i \in N_2$. Repeated application of Theorem 1 and (14) yields the desired result [1]. □

5 Applications

Application 1. *Linear Quadratic Gaussian Control Problem*

For an LQG control problem (example 2), the information z_i at each time t is linear and nested. Because the utility function (16):

$$J = E[\frac{1}{2} x_{N+1}^T S x_{N+1} + \frac{1}{2} \sum_{k=1}^{N} (x_k^T H^T H x_k + u_k^T b u_k)] \tag{16}$$

can be converted to the form:

$$J = E[\frac{1}{2} u^T Q u + u^T S \xi] + \quad \text{terms independent of} \quad u \tag{17}$$

By theorem 2, the optimal controls for all time instants exist, are unique, and linear in z_i i.e.

$$u_i^* = A_i z_i + b_i \qquad \forall i \tag{18}$$

Remark 3. *The conclusion concerning the linearity of the control law is independent of the correlation between the initial state, process noise, and observation noise. It only requires that they are jointly Gaussian.*

Application 2. *One-step Communication Delay Control Systems*

Suppose we have two coupled linear-discrete time-dynamic system controlled by $u_1(t)$ and $u_2(t)$, $t = 1, 2, ..., N$, with the usual Gaussian distribution and noise setup. $y_1(t)$, $y_2(t)$ are the noise observations for the system. Suppose

$$z_1(t) = \{y_1(\tau), y_2(k) | \ \tau = 1, 2, ..., t; k = 1, ..., t-1\}$$

$$z_2(t) = \{y_2(\tau), y_1(k) | \ \tau = 1, 2, ..., t; k = 1, ..., t-1\}$$

i.e., two controllers share all the past information with one step communication delay. Figure 5(a) represents the precedented diagram, which show that the information structure is partially nested. Hence, if the criterion is quadratic in the state and control variables, then by Theorem 2, the optimal solution is linear. Furthermore, if there is a third linear system coupled to the first system via the second system as shown in Figure 5b, then we may conclude that the first system can tolerate a two-step delay in sharing information with the third system.

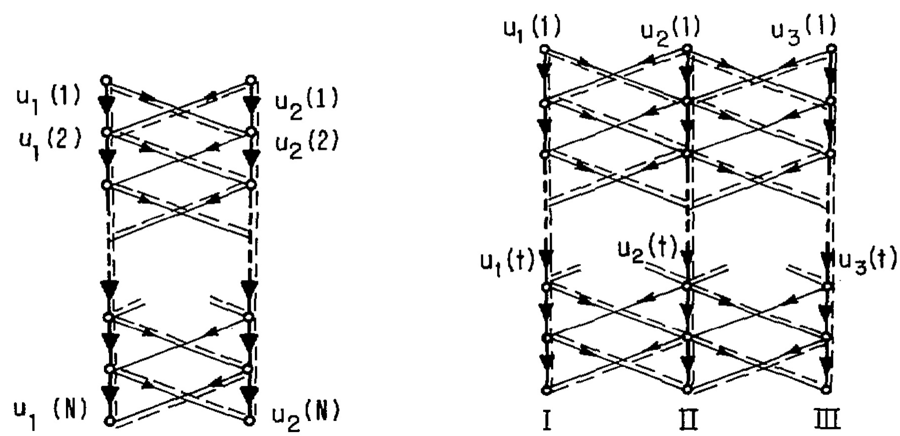

Figure 5: Communication Delay Control Systems

Application 3. *Hierarchical Control System*

Suppose an information structure diagram is that, of Figure 6, which informally represents a chain of commands. Then under linear-quadratic-Gaussian assumptions, the optimal solution is a gain linear without the need for lateral communication.
Roughly speaking, the implication of Theorem 2, is that, if a *DMS action affects our information, then knowing what he knows will yield linear optimal solutions.*

Application 4. *Two Person Zero-Sum Multi-stage Games.*

The result of Theorem 2 can be employed to show that in LQG zero-sum decision games with perfect memory for each player and no cooperation, linear strategy for both players is optimal.

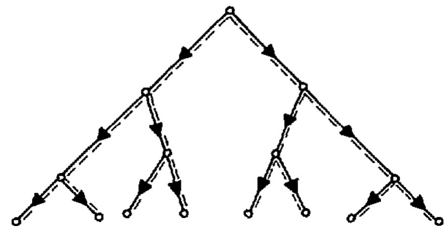

Figure 6: Hierarchical Control System

6 Conclusion

1. A dynamic team problem can be reduced to a static one if the information structure is partially nested.

2. In addition, if

   - the uncertainty in the system ξ is Gaussian
   - the information structure z_i is linear in ξ and u_j for $j\{i$ and it has a partially nested structure
   - the payoff function is quadratic

   then the optimal control exists, is unique, and linear in z_i.

Although these results provide an approach for solving some team problems, the following limitations still hold:

- Partially nested information structure is not always the case for many team decision problems. In the part II of this paper Chu [4] showed that even if an information structure is not partially nested, under some conditions the optimal solution can be found from an auxiliary problem with partially nested information structure. However this is still not general.
- Generally, uncertainty may not be Gaussian/ may not be normally distributed.
- The result is limited to quadratic utility functions.
- For a decision maker with a large number of precedent the size of information is very large which makes the calculation too heavy.

As we learned in the course, in the case of having Markov property (which holds for many problems) there are more efficient ways to calculate the optimal decision. For instance in the LQG problem in *application 1* dynamic programming using Kalman Filter and for *application 2* using a coordinator can reduce the amount of calculations required to find the optimal strategy.

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