Stochastic control viewpoint in coding and information theory for communications

ECSE 506 Stochastic Control and Decision Theory (Winter 2012)

Project Report

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1. Introduction

Information theory and coding are two important pillars in the research of communication systems. In this project, we are going to cast some problems in information theory and coding from a stochastic control viewpoint.

Information theory was established by Claude Shannon in 1948, which was considered as one of the most influential research results in the last century and led to revolutionary developments in digital communication theory.

Information theory aims to find the fundamental limits on the processing of information and operations of signals, and it attempts to find the thresholds for reliable data compression and data communication. One of the most important quantities in information is the channel capacity (*C*). Consider the communication channel in Figure 1, which has input *X* and output *Y*.

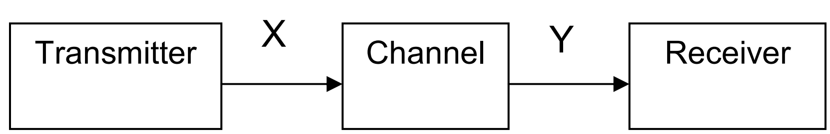

Figure 1 Communication channel model block diagram

The channel capacity is defined as

$$C = \max_{P_X} I(X;Y), \qquad (1.1)$$

where $I(X;\ Y)$ denotes the mutual information between the channel input X and channel output Y. Channel capacity is a sharp upper limit to the transmission rate (R) used for reliable transmissions. That is to say, if $R < C$, one can have reliable transmission such that the probability of transmission error can be made arbitrarily small. However, if $R > C$, the probability of transmission error converges to 1 for long packets.

Coding theory can be divided into two sub-areas, namely source coding and channel coding. Source coding is commonly known as compression, with the objective to remove redundant data. On the other hand, channel coding tries to add some redundancy to the data in order to protect the data from noise corruption during the transmission process. Channel coding is often realized by error-correcting codes, e.g., convolutional codes. Viterbi decoding algorithm is one option to decode the coded information.

Recently, it has been shown that multiple coding/information theory research problems can be formulated using a *stochastic control viewpoint* and solved using standard stochastic control techniques, such as dynamic programming (DP).

For instance, communication systems with feedback [1]-[5] have been studied by viewing the encoding function as an optimal control action (during encoding process) and by using other theoretical concepts and tools provided by control theory, such as Robbins-Monro stochastic approximation procedure, model of imperfect and delayed observations, belief update, least squares estimation and prediction, and dynamic programming decomposition. The stochastic control viewpoint on posterior matching-style feedback communication schemes is a recent result which considers the feedback encoder design problem from a stochastic control perspective, and it has been further generalized into the concept of information-theoretic viewpoint on optimal control [6].

Another example in coding theory is the interpretation of the Viterbi decoding algorithm based on state-space approach to dynamical system [7]. It has been pointed out by Omura that Viterbi algorithm can be formulated as a dynamic programming solution to a generalized regulator control problem. By viewing the trellis-encoder as a discrete time dynamical system driven by an information source, Viterbi algorithm can be derived using dynamic programming/principle of optimality.

The rest of this report is arranged as follows: Section 2 presents a stochastic control viewpoint in information theory. In this section, I will review the feedback schemes proposed by Schalkwijk and Kailath [2] and Horstein [3][4] using the tools provided by control theory. Then, Coleman's stochastic viewpoint on posterior matching (PM) scheme [1] will be presented. Lastly, a brief summary of the extensions of PM scheme [6] will be provided. Section 3 provides an example of the use of stochastic control tools in coding/decoding algorithms. The Viterbi decoding algorithm will be reviewed from a control viewpoint and derived using dynamic programming. Section 4 concludes the project with relevant discussions.

2. Stochastic control viewpoint in information theory

Feedback structure has been adopted in many communication systems since it can provide various merits in term of improving the system performance, e.g., reducing decoding complexity in point-to-point communication and enlarging the achievable rate region/capacity region of multi-terminal communication system.

Following milestone work done by various researchers, this section reviews the communication system with feedback from a stochastic control viewpoint, with the focus on the concepts and techniques provided by control theory.

2.1 Feedback coding scheme by Schalkwijk and Kailath

Schalkwijk and Kailath's scheme [2] considers the communication under additive white Gaussian noise (AWGN) by exploiting the merit of an immediate and noise-free feedback link. This capacity-achieving scheme recursively transmits the *mean* of the message interval which eventually leads to *vanishing* probability of error. The scheme was inspired by the *Robbins-Monro stochastic approximation procedure* as presented in [2]

The objective of Robbins-Monro stochastic approximation is to determine a zero of a function *F(x)* by measuring the values of the function *F(x)* at any desired point. However, the exact shape of *F(x)* cannot be accessed.

Moreover, one can only get a *partial observation* of the measurement, i.e. the measurement *Y(x)* is corrupted by noise, that is

$$Y(x) = F(x) + Z, \tag{2.1}$$

where *Z* is an independent and identically distributed (i.i.d) AWGN with zero mean and σ^2 variance.

Let $a_n, n = 1,2,...,$ be a sequence satisfying the following conditions

$$\begin{aligned} &a_n \geq 0; \\ &\sum a_n = \infty; \\ &\sum a_n{}^2 < \infty, \end{aligned} \tag{2.2}$$

and *F(x)* and *Z* satisfy

1. $F(x) > 0$ if $x > \theta$; $F(x) < 0$ if $x < \theta$;
2. $\inf\{|F(x)|; \varepsilon < |x-\theta| < 1/\varepsilon\} > 0,$ for all $\varepsilon > 0$;
3. $|F(x)| \leq K_1 |x-\theta| + K_2$;
4. If $\sigma^2(x) = E\left[Y(x) - F(x)\right]^2$, then $\sup_x \sigma^2(x) = \sigma^2$ is finite.

Start with an arbitrary initial guess X_1 and update future guesses according to

$$X_{n+1} = X_n - a_n Y_n(X_n), \tag{2.3}$$

and if $E|X_1|^2 < \infty$, then

$$E|X_n - \theta|^2 \longrightarrow 0. \tag{2.4}$$

Furthermore, it the following conditions are met,

5. $\sigma^2(x) \to \sigma^2(\theta)$ as $x \to \theta$;

6. $F(x) = \alpha(x-\theta) + \delta(x)$, for $\alpha > 0$ and $\delta(x) = O(|x-\theta|^{1+\rho}), \rho > 0$;

7. There exist $t > 0, \delta > 0$ and $\sup\{E|Z(x)|^{2+\delta}; |x-\theta| \le t\} < \infty$;

8. $a_n = 1/\alpha n$, $2\alpha > a$.

Then,

$$\sqrt{n}(X_{n+1} - \theta) \sim N\left(0, \frac{\sigma^2}{a(2\alpha - a)}\right). \tag{2.5}$$

Now the *coding scheme* in [2] can be established using the above result.

First divide the unit interval into M message intervals with equal length. Then, select one message (to be transmitted) and find the middle point θ of that message interval. Put a straight line $F(x) = \alpha(x-\theta), \alpha > 0$ through θ. Start with $X_1 = 0.5$ and send $F(X_1) = \alpha(X_1 - \theta)$.

At the receiving end, the receiver observes $Y_1(X_1) = \alpha(X_1 - \theta) + Z_1$, computes $X_2 = X_1 - (a/1)Y_1(X_1)$, with $a = 1/\alpha$, and feeds X_2 back to the transmitter.

At the next time, the transmitter sends $F(X_2) = \alpha(X_2 - \theta)$. The process will be executed recursively, and in this case $\sqrt{n}(X_{n+1} - \theta) \sim N\left(0, \sigma^2 / \alpha^2\right)$.

In the Gaussian case, with $a_n = 1/\alpha n$, from

$$\begin{aligned} X_{n+1} &= X_n - 1/\alpha n \cdot Y_n(X_n); \\ Y_n(X_n) &= \alpha(X_n - \theta) + Z_n, \end{aligned} \tag{2.6}$$

one can get

$$X_{n+1} = \theta - (1/\alpha n)\sum_{i=1}^{n} Z_i \,. \tag{2.7}$$

It is clear that X_{n+1} can be viewed as the *Maximum Likelihood Estimation* of θ, and with distribution $N(\theta, \sigma^2 / (n\alpha^2))$.

Therefore, after N iterations, the probability of error is given by

$$P_e = 2erfc\left(\frac{\frac{1}{2} M^{-1}}{\sigma / (\alpha\sqrt{N})} \right), \tag{2.8}$$

which can be made arbitrarily small after enough iterations.

Finally, one last remark for this scheme is that in order to ensure appropriate signaling rate, *1/M* need to be made to decrease at a rate lightly less than $1/\sqrt{N}$ to prevent the rate $R = \ln M / T$ goes to zero. A proper selection is given by $M(N) = N^{1/2(1-\varepsilon)}$ which lead to error probability to be

$$P_e = 2erfc\left(\frac{\alpha N^{\varepsilon/2}}{2\sigma} \right). \tag{2.9}$$

2.2 Feedback coding scheme by Horstein

Horstein's feedback coding scheme considers an optimal strategy to transmit over a binary symmetric channel (BSC) [4].

This coding falls in the category of *median* feedback, which has been summarized in [3].

First, a simple example of median feedback was considered in [3] which is a noiseless binary forward channel. Similar to the Schalkwijk and Kailath's scheme, one can divide the unit interval into M message intervals with equal length where $M = 2^N$, start with a uniform prior distribution at the receiver, with a priori mean $m_0 = 1/2$.

If the selected message point θ (to be transmitted) is to the left of m_0, the transmitter sends zero(send 1 otherwise). Thus, if one is received, it tells the receiver that θ is on the right hand side m_0, and the new receiver distribution is then uniform over *(1/2, 1)* with the new median $m_0 = 3/4$. In the next step, the transmitter sends zero if $\theta \in (1/2, 3/4)$ and sends one if $\theta \in (3/4, 1)$. This process will be repeated.

It is clear that this idea related to a binary search algorithm, or bi-section method, which can determine θ with probability 1. The corresponding rate is $R = \frac{1}{N}\log(M) = 1$ bit/channel use.

Horstein's scheme [4] is a generalized version of the above procedure. In this case (BSC), the difference/complication is due to the crossover probabilities of BSC. However, the above method can be modified to accommodate the transmission over BSC, which was explained in [3] using Figure 2.

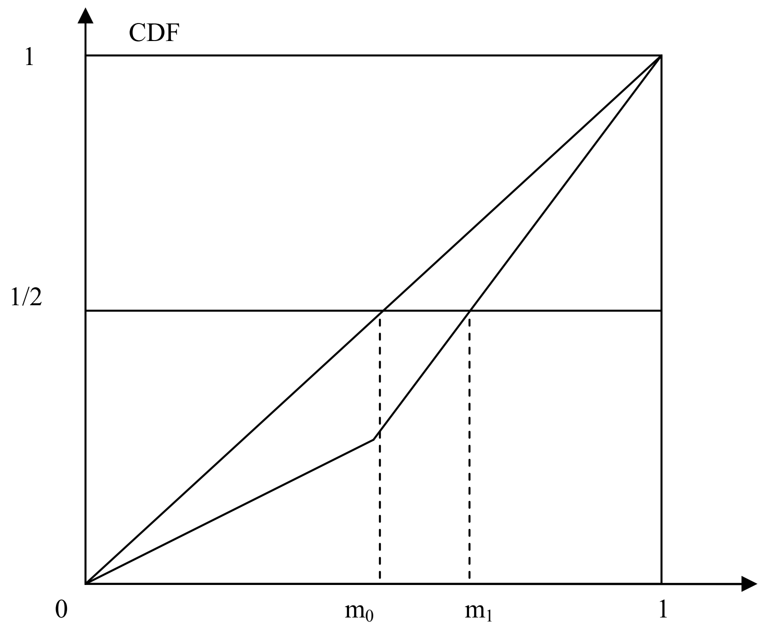

Figure 2 Distribution of the message at the receiver (Horstein's coding scheme).

The uniform distribution at the receiver is given by the diagonal line from (0, 0) to (1, 1) with median $m_0 = 1/2$. If the first received bit is one, then the new distribution at the receiver can be presented by the other bent line in Figure 2 with the new median m_1 which will fed back to the transmitter in the next iteration. Thus, the recursion can be described as follows:

Start with the diagonal line (representing the uniform distribution over $[0,1)$) .

If zero is received, the slopes of the distribution that are on left hand side of the median are scaled by *2(1-p)*, and those on the right hand side are scaled by *2p* , where *p* denotes the crossover probability of the BSC. If one is received, do the opposite.

Horstein has demonstrated that a sequential version of this scheme will lead to a distribution such that most of the probability mass is concentrated close to one of the possible message point.[1] Furthermore, this scheme can be shown to be capacity achieving and with vanishing small error after enough iterations. Please see [3][4] for references.

2.3 Posterior matching feedback coding scheme

In 2007, Shayevitz and Feder developed the idea of posterior matching (PM), and shown that the schemes of Schalkwijk/Kailath and Horstein can be derived as special cases of the PM scheme [5]. Later, Coleman concluded that the above feedback schemes can be viewed from a stochastic control angle, i.e. the encoding functions (for message and the previous feedback) can be considered as stochastic control actions. He further provided the detailed formulation for PM under stochastic viewpoint in his work [1].

Follow the footstep of Coleman [1], we will now look at PM scheme from a stochastic control viewpoint.

[1] Please note that Horstein's feedback scheme is related to and can be used to solve the ***aviation control problem*** that has been mentioned during the presentation, i.e. one wants to stabilize an aviation system which the control actions are sometimes inverted.

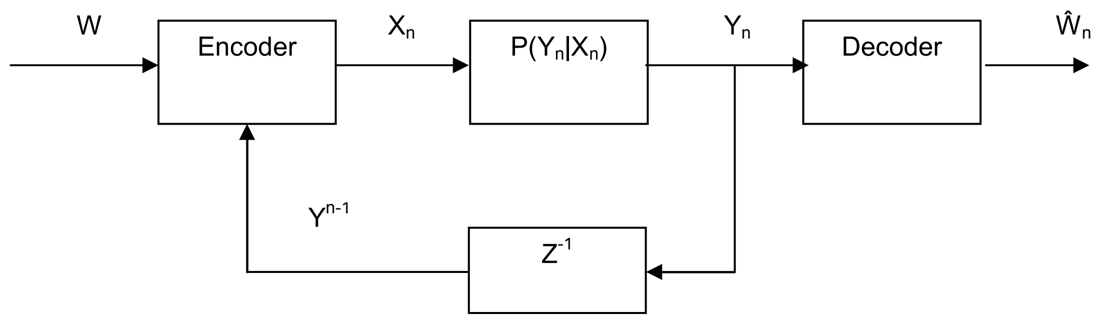

Figure 3 Communication system with feedback

Consider the communication system in Figure 3. Let W be a message point which is uniformly distributed on [0,1), it is chosen and encoded into channel input X_n at time n, and the corresponding feedback is Y^{n-1}.

The channel capacity C is can be determined by the transition probability function $P_{Y|X}(\bullet|X)$ and the capacity-achieving input probability distribution $P_X(\bullet)$, with the corresponding cumulative density function (CDF) given by $F_X(\bullet)$. A transmission scheme with feedback is now $\{g_n : [0,1) \times \mathscr{Y}^{n-1} \to \mathscr{X}\}$, so the output at time $n+1$ is

$$X_{n+1} = g_{n+1}(W, Y^n). \tag{2.10}$$

Denote $\mathscr{F}$ as the space of all CDF functions on [0,1).

Denote $\mathscr{F}^*$ as the set of all unit step functions on [0,1).

Denote $F_w^* \in \mathscr{F}^*$ as the CDF that is a step function which the value of the function changes from zero to one at w.

Denote the posterior CDF of W at time n as $F_n(\bullet) = F_{W|Y^n}(\bullet|Y^n)$.

Then, the PM scheme is given by

$$X_{n+1} = g_{n+1}(W, Y^n) = F_X^{-1}(F_n(W)) \,. \tag{2.11}$$

The receiver calculates the a-posteriori probability of the message point starting with a uniform distribution on the unit interval [0,1). The transmitter also shares this information with the help of feedback. The objective is to select encoding functions which will lead to a fast convergence of the posterior probability around the desired message point *W*. A good option is to match the posterior probability according to the channel. This is because the posterior probability best describes the receiver's knowledge, so it is feasible to match the posterior probability into the desired input distribution. Furthermore, $F_n(W)$ is uniformly distributed on the unit interval, and X_{n+1} is independent of Y^n since the channel is memoryless. Thus, Y_{n+1} is also independent of Y^n.

Now, we apply the Kullback-Leibler (KL) distance, the reduction in distance from the desired step function (CDF) at time *n* and *n+1* is given by the log-likelihood ratio:

$$D(F_w^* \| F_n) - D(F_w^* \| F_{n+1}) \tag{2.12}$$

$$= \int_{x=0}^{1} \delta(x-w) \log \frac{\delta(x-w)}{dF_n(x)} - \int_{x=0}^{1} \delta(x-w) \log \frac{\delta(x-w)}{dF_{n+1}(x)} \tag{2.13}$$

$$= \log \frac{dF_{n+1}(w)}{dF_n(w)} \tag{2.14}$$

Intuitively, with the help of feedback, the encoder is recursively pushing the posterior distribution function F_n towards the distribution of F_w^*.

At this stage, we are ready to formulate this communication problem into a stochastic control problem. From the feedback, the posterior CDF F_{i-1} can be used along with the message *W* to determine the next input symbol

$$X_i = u_i(W, F_{i-1}) \tag{2.15}$$

with $u_i : [0,1) \times \mathscr{F} \to \mathscr{X}$ being the control action at time *i*. We want to always choose a control action u_i such that the expected reduction in KL distance is maximized.

<u>*Stochastic control problem formulation*</u>:

1. Define the <u>state</u> of at time *i* as the posterior CDF,

$$S_i = F_i \in \mathscr{F} \tag{2.16}$$

2. Defined the <u>state transition</u>

$$dF(\bullet) = \frac{f_{Y|X}(y_i \mid u_i(\bullet, F_{i-1}))dF_{i-1}(\bullet)}{\int\limits_{w''=0}^{1} f_{Y|X}(y_i \mid u_i(w'', F_{i-1}))dF_{i-1}(w'')} \tag{2.17}$$

3. Define the <u>control action</u>

$$u_i = \gamma_i(u^{i-1}, s^{i-1}), u_i : [0,1) \times \mathscr{F} \to \mathscr{X} \tag{2.18}$$

4. Define the <u>reward</u> at State *s* with control *u* as the expected reduction in KL distance.

$$\begin{aligned} g(u,s) &= E\left[D(F_W^* \,\|\, F_n) - D(F_W^* \,\|\, F_{n+1}) \mid S_n = f, U_n = u \right] \\ &= E\left[\log \frac{dF_{n+1}(W)}{dF_n(W)} \middle| F_n = f, U_n = u \right] \end{aligned} \tag{2.19}$$

Then, various techniques offered by control theory can be used to solve this communication problem. Furthermore, it can be shown that PM scheme is an *optimal iterative solution* to this problem if we were to solve it.

This is proven by using a neat property of the above stochastic control problem that maximizing the expected reduction in KL distance is equivalent to maximizing the mutual information $\frac{1}{n}I(W;Y^n)$.

Proof: Under the problem setup given by (2.15)-(2.19). The time average reward is given by

$$\begin{aligned}\frac{1}{n}E\left(\sum_{i=1}^{n}g(U_i,S_i)\right)&=\frac{1}{n}\sum_{i=1}^{n}I(W;Y_i\,|\,F_{i-1})\\&=\frac{1}{n}\sum_{i=1}^{n}I(W;Y_i\,|\,Y^{i-1})\\&=\frac{1}{n}I(W;Y^n).\end{aligned}\tag{2.20}$$

On the other hand, by using Shannon's channel coding theorem

$$\frac{1}{n}I(W;Y^n)=\frac{1}{n}H(Y^n)-\frac{1}{n}H(Y^n\,|W)\tag{2.21}$$

$$=\frac{1}{n}H(Y^n)-\frac{1}{n}\sum_{i=1}^{n}H(Y_i\,|\,Y^{i-1},W)\tag{2.22}$$

$$\leq\frac{1}{n}\sum_{i=1}^{n}H(Y_i)-\frac{1}{n}\sum_{i=1}^{n}H(Y_i\,|\,Y^{i-1},W)\tag{2.23}$$

$$=\frac{1}{n}\sum_{i=1}^{n}H(Y_i)-\frac{1}{n}\sum_{i=1}^{n}H(Y_i\,|\,X_i,Y^{i-1},W)\tag{2.24}$$

$$=\frac{1}{n}\sum_{i=1}^{n}H(Y_i)-\frac{1}{n}\sum_{i=1}^{n}H(Y_i\mid X_i) \tag{2.25}$$

$$=\frac{1}{n}\sum_{i=1}^{n}H(Y_i)-H(Y_i\mid X_i) \tag{2.26}$$

$$\leq C\,. \tag{2.27}$$

The above derivation used definition of mutual information (2.21), chain rule of entropy(2.22), $H(Y^n)\leq\sum_{i=1}^{n}H(Y_i)$ in (2.23), (2.10) in (2.24) and (2.25).[2] Thus, PM scheme maximizes the mutual information $\frac{1}{n}I(W;Y^n)$ which is equivalent to maximizing the time average reward.

Therefore, PM scheme is a solution to the stochastic control problem defined by (2.15)-(2.19). Furthermore, Shannon's coding theorem ensures that PM scheme is capacity-achieving due to the fact that X_{i+1}, Y^i are independent and X_i has a capacity-achieving distribution, which validate the optimality of the PM solution for any n. **QED**

Finally, it can be noted that the posterior probability evolves in a controlled-Markovian manner which is governed by the stochastic control problem defined by (2.16)-(2.19), as shown in Figure 4.

[2] Please note that in [1], there seems to be some mismatch between the explanation and the corresponding equation number.

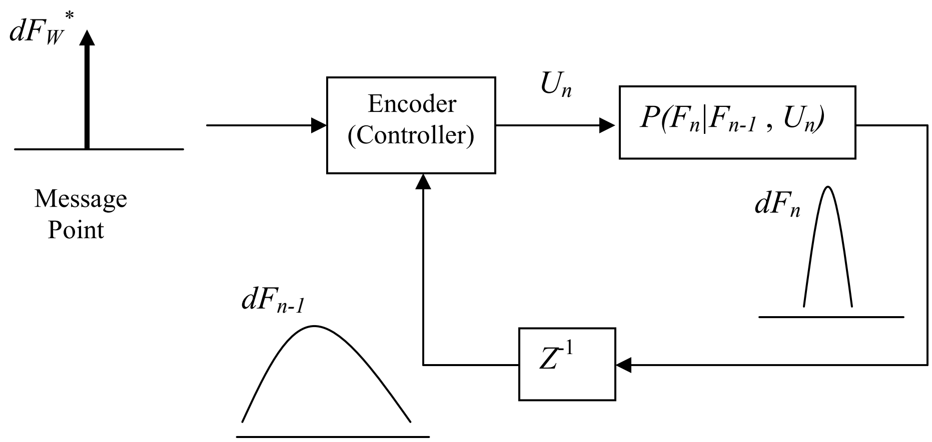

Figure 4 Posterior matching scheme as a stochastic control problem

2.4 Extensions of PM: ideas of information-theoretic viewpoint on optimal control

In the same spirit discussed in section 2.3, Gorantla and Coleman further extended their idea and established a general model which provides information-theoretic viewpoints on optimal control in coding and coding problems [6]. They have also demonstrated the existence of an optimal scheme that operates with sufficient statistics in this two-agent sequential decision making problem by using dynamic programming decomposition, and showed that the decoded information stays in a space of beliefs on the source symbols.

The cost function has also been generalized from

$$g(w_i, \hat{w}_i) \tag{2.28}$$

into

$$g(w_i, x_i, \hat{w}_{i-1}, \hat{w}_i) = \rho(w_i, \hat{w}_{i-1}, \hat{w}_i) + \alpha\eta(x_i) \tag{2.29}$$

to accommodate different system structures, i.e. a cost function corresponding to the current and previous symbols.

Now, we will briefly review a main result in [6], which has been developed from the PM scheme as we discussed in section 2.3. Consider the following causal encoding/decoding optimal control problem as depicted in Figure 5.

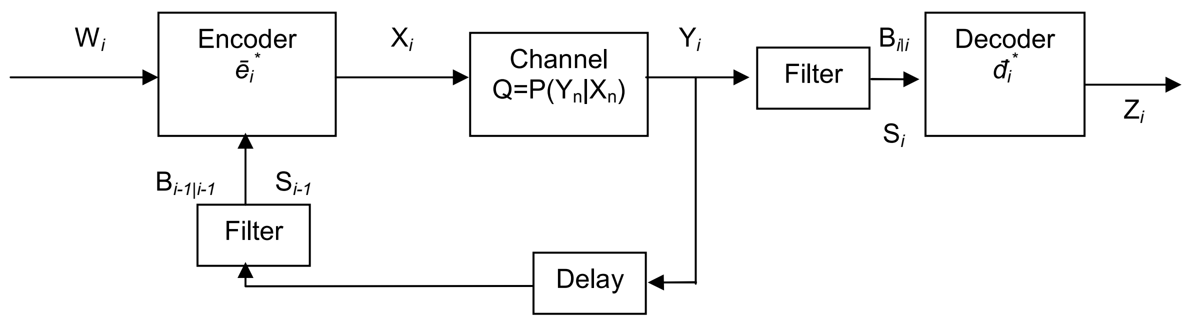

Figure 5 Causal coding and decoding optimal control problem

Following (2.17), define the transition

$$b_{i|i} = \Lambda\left(b_{i-1|i-1}, y_i, \overline{e}_i^*(\bullet, b_{i-1|i-1})\right). \tag{2.30}$$

Then, there exists an optimal policy pair (e^*, d^*) of the form

$$\begin{aligned} &e_{i+1}^*(w^{i+1}, y^i) = \overline{e}_{i+1}^*(w_{i+1}, b_{i|i}); \\ &d_i^*(y^i) = d_i^*(b_{i|i}) = b_{i|i}, \end{aligned} \tag{2.31}$$

and the corresponding cost is given by

$$J^{\alpha}_{n,\pi^*} = \min_{e}\left(-I(W^n;Y^n) + \alpha E\left(\sum_{i=1}^{n} \eta(X_i) \right) \right). \tag{2.32}$$

The idea of the proof is as follows: Dynamic programming has been used to investigate the KL divergence which determines the stability of the non-linear filter in the causal encoder and decoder via a Lyapunov function. Furthermore, due to the optimality of the posterior belief, the idea of using belief as decision variables has also been applied here.

The exact proof is supported by various lemmas which can be found in [6], and I will provide an outline of the proof using dynamic programming:

1. Define the state $s_n = (z_{n-1}, b_{n|n})$ and control action $u_n = (\overline{e}_{n+1}, z_n)$.

2. Consider the value function at last stage $V_n(s_n) = \inf_{u_n} \overline{g}_n(s_n, u_n)$ and show it is equal to the negative KL distance of $b_{n|n}$ and $\Phi(z_{n-1})$, where $\Phi(b)dw = \int_{w' \in W} Q_W(dw \mid w')b(dw')$.

3. For *k* = *n-1*, find $V_{n-1}(s_{n-1})$. Then, show for any fixed encoder policy $\overline{e}_n$, the optimal choice is $z_{n-1} = b_{n-1|n-1}$.

4. By induction, show the optimal choice at any time *k*, $z_{k-1} = b_{k-1|k-1}$.

5. By using a similar argument as (2.19)-(2.27), i.e. $I(W_i, Y_i \mid Y^{i-1}) = E\left(D\left(B_{i|i} \parallel \Phi(B_{i-1|i-1})\right)\right)$, the proof can be concluded.

In order to show the consistency of this generalized model to the communication feedback problem with PM scheme that we have considered earlier, one just need to view

F_{n+1} as $B_{i|i}$ and view F_n as $B_{i-1|i-1}$ to establish the system setup in section 2.3. In fact, the PM scheme falls in the category of *likelihood ratio cost and information gain problem* due to the fact that the reduction in KL distance (reward) is in the form of log-likelihood ratio of the posterior probabilities.

It has also been shown in [6] that the above model can be used in various optimal control problems including:

- Hidden Markov model and non-linear filter;
- Brain-machine interfaces;
- Gauss-Markov inverse optimal control;
- M/M/1 queue;
- Blackwell's trapdoor channel with inverse optimal control, etc.

3. Stochastic control viewpoint in coding/decoding: Viterbi decoding algorithm

In this section, I will provide an example of the application of stochastic control which has been used in another pillar of communication system, namely coding/decoding theory. The Viterbi algorithm will be formulated into a dynamic programming solution to a generalized regulator control problem and derived using dynamic programming decomposition [7]. The detailed problem formulation and derivations are presented next.

An information source generates an output ($u_{k\text{-}1}$) at each time index k. The encoder encodes the information symbol into b channel symbols, say $\mathbf{y}_k$. Assume the trellis[3] of this encoder is terminated at stage L, so the corresponding information symbols are given by $\{u_0, u_1, \ldots, u_{L-1}\}$.

Assume the encoder has memory size v, so the encoded output at time k only depends on the last v symbols $\{u_{k\text{-}1},\ u_{k\text{-}2},\ \ldots,\ u_{k\text{-}v}\}$. In addition, the state of the encoder at time k is also given by $\mathbf{x}_k = \{u_{k\text{-}1},\ u_{k\text{-}2},\ \ldots,\ u_{k\text{-}v}\}$. The output is generated with the parity check matrix $\mathbf{G}_k$ and is given by

$$\mathbf{y}_k = \mathbf{G}_k(\mathbf{x}_k), k = 1, 2, \ldots, L. \tag{3.1}$$

Furthermore, this system can be model as a *general regulator problem* with

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \tag{3.2}$$

where $\mathbf{A}$ shifts $\mathbf{x}_k$ by one unit and $\mathbf{B}$ is a vector with one at the last entry and zero otherwise.

An example of convolutional encoder with rate 1/2 is shown in Figure 6 for demonstration proposes. In this encoder, the memory size $v = 7$. The information source is fed into the encoder from the left, and for every input bit, two output coded bits are generated.

[3] The trellis of an encoder can be constructed with the finite state machine diagram of the corresponding encoder.

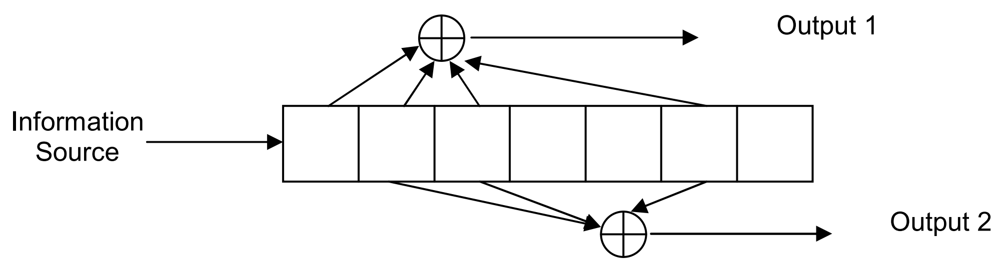

Figure 6 An example of convolutional encoder

The output channel symbols $\mathbf{r} = \{\mathbf{r_1}, \mathbf{r_2}, ..., \mathbf{r_L}\}$ are the noise corrupted version of the input channel input symbols $\mathbf{y} = \{\mathbf{y_1}, \mathbf{y_2}, ..., \mathbf{y_L}\}$. Now, consider the negative log-likelihood posterior probability

$$\begin{aligned} J(\mathbf{y}) &= -\ln P(\mathbf{y} \mid \mathbf{r}) \\ &= -\sum_{k=1}^{L} \ln P(\mathbf{y_k} \mid \mathbf{r_k}). \end{aligned} \tag{3.3}$$

The objective now is to choose the sequence of symbols $\{\hat{u}_0, \hat{u}_1, ..., \hat{u}_{L-1}\}$ such that $J(\mathbf{y})$ is minimized. Or equivalently, $P(\mathbf{y} \mid \mathbf{r})$ is maximized (maximum a-posteriori decoder).

Please note that Viterbi decoding algorithm can be used for maximum likelihood decoding (ML) and extended for maximum a-posteriori decoding (MAP).

In the maximum a-posteriori decoding (MAP), the objective of the decoder is to maximize

$$\begin{aligned} P(\mathbf{y} \mid \mathbf{r}) &= P(\mathbf{r} \mid \mathbf{y}) P(\mathbf{y}) / P(\mathbf{r}) \\ &= P(\mathbf{r} \mid \mathbf{y}) P(\mathbf{y}) \cdot C. \end{aligned} \tag{3.4}$$

In MAP, the prior probability $P(\mathbf{y})$ may be different and needs to be considered when minimizing $-\ln P(\mathbf{y}\,|\,\mathbf{r})$. Therefore, minimizing $-\ln P(\mathbf{y}\,|\,\mathbf{r})$ and $-\ln P(\mathbf{r}\,|\,\mathbf{y})$ are different.

However, in the maximum likelihood (ML) setup, maximizing $P(\mathbf{y}\,|\,\mathbf{r})$ and $P(\mathbf{r}\,|\,\mathbf{y})$ are equivalent since the prior probability $P(\mathbf{y})$ is assumed to be the same. Omura's work [7] was actually based on maximum-likelihood (ML) decoding. In ML, the decoder has no prior knowledge of $\mathbf{y}$, so the best it can do is to assume that the inputs are equi-probable, i.e. $P(\mathbf{y}_\mathrm{k} = 0) = P(\mathbf{y}_\mathrm{k} = 1) = 1/2$ at every entry $\mathbf{y}$. Thus, $P(\mathbf{y})$ is the same for all input vector $\mathbf{y}$ and can be eliminated during the maximization/minimization. Consequently, minimizations of $-\ln P(\mathbf{y}\,|\,\mathbf{r})$ and $-\ln P(\mathbf{r}\,|\,\mathbf{y})$ are the same (and inter-changeable) in the ML setup. In this report, I will continue with the MAP setup.

Now, from a stochastic control viewpoint, we can view the sequence $\{\hat{u}_0, \hat{u}_1, ..., \hat{u}_{L-1}\}$ as the control actions at each time, and the goal is to minimize the total cost $J(\mathbf{y})$. Apply a standard dynamic programming decomposition approach, define the intermediate cost (cost-to-go) as

$$V_{L-k}(\mathbf{x_k}) = \min_{u_j, j=1,..,k}\left(-\sum_{j=k+1}^{L} \ln P(\mathbf{y_k}\,|\,\mathbf{r_k})\right), \text{for } k = 0,1,2,...L-1, \tag{3.5}$$

$$V_0(\mathbf{x}_L) = 0\,, \tag{3.6}$$

where $P(\mathbf{y_k}\,|\,\mathbf{r_k})$ can be found using Bays' rule as shown in (3.4). Then,

$$V_{L-k}(\mathbf{x_k}) = \min_{u_k}\left(-\ln P(\mathbf{y}_{\mathbf{k}+1}\,|\,\mathbf{r}_{\mathbf{k}+1}) + V_{L-(k+1)}(\mathbf{x}_{k+1})\right). \tag{3.7}$$

To be more precise, start with the initial state $\mathbf{x}_0$. Find $\hat{u}_0$ by searching for the state in the next time slot which has the smallest value of $V_{L-1}(\mathbf{x}_1)$ and is reachable under the condition (3.2). This guarantees that $V_{L-1}(\hat{\mathbf{x}}_1) = V_{L-1}(\mathbf{A}\mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}_0)$ is the smallest in the next step under the condition of (3.2). This process is recursively executed till $\mathbf{x}_L$ is reached.

It can be noted that instead of the performing a brute-force search for the minimum which has *exponential complexity*, Viterbi algorithm utilized dynamic programming to realize a *step-by-step minimization*. One can efficiently eliminate the non-survivor paths [8] during the step-step minimization and a low decoding complexity can be reached.

4. Conclusions

In this project, we have reviewed some communication problems from a control prospective. By looking at the communication system with feedback and Viterbi algorithm from a stochastic viewpoint, multiple concepts and tools offered by control theory became available which greatly facilitated the study of this project, including Robbins-Monro stochastic approximation procedure, model of imperfect and delayed observations, belief update, least squares estimation and prediction, model of general regulator problem, and dynamic programming decomposition.

In fact, many other research problems in communications can also be cast from a different angle and viewed as stochastic control problems, i.e. shortest path problems in networking, e.g., Dijkstra's algorithm, capacity regions of multi-terminal systems with feedback, e.g., multiple access channel with feedback, speech communication problems involving hidden Markov models, etc.

By applying the techniques of the stochastic control theory, some of these problems may be simplified, and others may be solved with more efficient methods, e.g. dynamic programming techniques. Furthermore, by viewing the communication problems from the stochastic control viewpoint, some useful insights may be obtained and eventually lead to new research ideas and topics, e.g. interdisciplinary research.

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