MECH541 Kinematic Synthesis

Planar Path Generation With Prescribed Timing

November 25th, 2003

Problem Definition

Synthesize a planar four-bar linkage, like the one shown in Fig. 4.1, whose coupler point R attains a set of positions $\{R_j\}_0^m$ for corresponding values $\{\psi_j\}_0^m$ of the input angle ψ.

Problem Formulation

The general method of linkage synthesis for path generation is based on the synthesis equations derived for motion generation, which are reproduced below for quick reference:

$$\mathbf{b}^T(\mathbf{1}-\mathbf{Q}_j)\mathbf{a}_0+\mathbf{r}_j^T\mathbf{Q}_j\mathbf{a}_0-\mathbf{r}_j^T\mathbf{b}+\frac{1}{2}\mathbf{r}_j^T\mathbf{r}_j=0\,,\quad \text{for}\quad j=1,\ldots,m \tag{1a}$$

and

$$(\mathbf{b}^*)^T(\mathbf{1}-\mathbf{Q}_j)\mathbf{a}_0^*+\mathbf{r}_j^T\mathbf{Q}_j\mathbf{a}_0^*-\mathbf{r}_j^T\mathbf{b}^*+\frac{1}{2}\mathbf{r}_j^T\mathbf{r}_j=0\,,\ \text{for}\quad j=1,\ldots,m \tag{1b}$$

Notice that, now the synthesis equations of the two linkage dyads are coupled via the unknown angles $\{\phi_j\}_0^m$, given by $\phi_j=\theta_j-\theta_0$, and occuring in $\mathbf{Q}_j$. Hence, we will consider simultaneously the two sets of dyad equations.

Expression for $\mathbf{Q}_j$

Since the input link $\overline{BA_0}$ undergoes rotations about B, we can write

$$\mathbf{a}_j-\mathbf{b}=\mathbf{R}_j(\mathbf{a}_0-\mathbf{b})\,,\quad \text{for}\quad j=1,\ldots,m \tag{2}$$

where $\mathbf{R}_j$ is the rotation matrix carrying $\overline{BA_0}$ into $\overline{BA_j}$ through angle $\beta_j=\psi_j-\psi_0$. Moreover, matrix $\mathbf{R}_j$ can be represented using eq.(1.6) as

$$\mathbf{R}_j=\cos\beta_j\mathbf{1}+\sin\beta_j\mathbf{E}\,,\quad \text{for}\quad j=1,\ldots,m \tag{3}$$

where $\mathbf{1}$ is the 2×2 identity matrix and $\mathbf{E}$ is the 90°-ccw rotation matrix, introduced in eq.(1.1a). Referring to Fig. 4.2, we can write

$$\mathbf{a}_j=\mathbf{r}_j+\mathbf{Q}_j\mathbf{a}_0\,,\quad \text{for}\quad j=1,\ldots,m$$

Hence,

$$\mathbf{Q}_j\mathbf{a}_0=\mathbf{a}_j-\mathbf{r}_j\,,\quad \text{for}\quad j=1,\ldots,m \tag{4}$$

Upon substituting eq.(2) into the above equation, we have

$$\mathbf{Q}_j\mathbf{a}_0=\mathbf{R}_j\mathbf{a}_0+(\mathbf{1}-\mathbf{R}_j)\mathbf{b}-\mathbf{r}_j\,,\quad \text{for}\quad j=1,\ldots,m \tag{5}$$

Now, if we substitute eq.(1.6) into the above equation, we end up with

$$c\phi_j \mathbf{a}_0 + s\phi_j \mathbf{E}\mathbf{a}_0 = \mathbf{R}_j \mathbf{a}_0 + (\mathbf{1} - \mathbf{R}_j)\mathbf{b} - \mathbf{r}_j\,, \quad \text{for} \quad j = 1, \ldots, m \tag{6}$$

which can be cast in the form

$$[\ \mathbf{a}_0 \quad \mathbf{E}\mathbf{a}_0\ ] \begin{bmatrix} c\phi_j \\ s\phi_j \end{bmatrix} = \underbrace{\mathbf{R}_j \mathbf{a}_0 + (\mathbf{1} - \mathbf{R}_j)\mathbf{b} - \mathbf{r}_j}_{\mathbf{c}_j}\,, \quad \text{for} \quad j = 1, \ldots, m \tag{7}$$

Consequently, we can readily solve for $c\phi_j$ and $s\phi_j$ as

$$\begin{bmatrix} c\phi_j \\ s\phi_j \end{bmatrix} = [\ \mathbf{a}_0 \quad \mathbf{E}\mathbf{a}_0\ ]^{-1}\mathbf{c}_j = \frac{1}{\|\mathbf{a}_0\|^2}\begin{bmatrix} \mathbf{a}_0^T\mathbf{E} \\ -\mathbf{a}_0^T \end{bmatrix}\mathbf{E}\mathbf{c}_j = \frac{1}{\|\mathbf{a}_0\|^2}\begin{bmatrix} \mathbf{a}_0^T\mathbf{c}_j \\ -\mathbf{a}_0^T\mathbf{E}\mathbf{c}_j \end{bmatrix}, \ j = 1, \ldots, m \tag{8}$$

where we have recalled the formula for the inverse of a 2×2 matrix given in Fact 1.4.2.

The BA_0R Dyad

When the expression for $\mathbf{Q}_j\mathbf{a}_0$ of eq.(5) is substituted into the synthesis equations (1a), we obtain

$$\mathbf{b}^T\mathbf{a}_0 - \mathbf{b}^T\mathbf{R}_j\mathbf{a}_0 - \mathbf{b}^T(\mathbf{1} - \mathbf{R}_j)\mathbf{b} + \mathbf{r}_j^T\mathbf{R}_j\mathbf{a}_0 - \mathbf{r}_j^T\mathbf{R}_j\mathbf{b} - \mathbf{r}_j^T\mathbf{b} - \frac{1}{2}\mathbf{r}_j^T\mathbf{r}_j = 0\,, \quad j = 1, \ldots, m$$

which simplifies to

$$\mathbf{b}^T(\mathbf{1} - \mathbf{R}_j)\mathbf{b} + \mathbf{b}^T(\mathbf{R}_j - \mathbf{1})\mathbf{a}_0 + \mathbf{r}_j^T(\mathbf{R}_j - \mathbf{1})\mathbf{b} - \mathbf{r}_j^T\mathbf{R}_j\mathbf{a}_0 + \frac{1}{2}\mathbf{r}_j^T\mathbf{r}_j = 0\,, \quad j = 1, \ldots, m \tag{9}$$

thereby deriving the synthesis equations for the left-hand dyad of Fig. 4.1 for the problem at hand. Apparently, these m equations are quadratic in $\mathbf{b}$ and linear in $\mathbf{a}_0$, their degree being 2.

The $B^*A_0^*R$ Dyad

Vector $\mathbf{Q}_j\mathbf{a}_0^*$ appearing in eq.(1b) can be expressed as

$$\mathbf{Q}_j\mathbf{a}_0^* = [\ \mathbf{a}_0^* \quad \mathbf{E}\mathbf{a}_0^*\ ]\begin{bmatrix} c\phi_j \\ s\phi_j \end{bmatrix} = \frac{1}{\|\mathbf{a}_0\|^2}[(\mathbf{a}_0^T\mathbf{c}_j)\mathbf{a}_0^* - (\mathbf{a}_0^T\mathbf{E}\mathbf{c}_j)\mathbf{E}\mathbf{a}_0^*]\,, \quad \text{for} \quad j = 1, \ldots, m$$

which reduces to

$$\mathbf{Q}_j\mathbf{a}_0^* = \frac{1}{\|\mathbf{a}_0\|^2}\left[(\mathbf{a}_0^T\mathbf{c}_j)\mathbf{1} - (\mathbf{a}_0^T\mathbf{E}\mathbf{c}_j)\mathbf{E}\right]\mathbf{a}_0^*\,, \quad \text{for} \quad j = 1, \ldots, m \tag{10}$$

Substituting the above expression into eq.(1b), we obtain, after clearing the denominator,

$$\begin{aligned}(\mathbf{b}^*)^T\left[(\|\mathbf{a}_0\|^2 - \mathbf{a}_0^T\mathbf{c}_j)\mathbf{1} + (\mathbf{a}_0^T\mathbf{E}\mathbf{c}_j)\mathbf{E}\right]\mathbf{a}_0^* + \mathbf{r}_j^T\left[(\mathbf{a}_0^T\mathbf{c}_j)\mathbf{1} - (\mathbf{a}_0^T\mathbf{E}\mathbf{c}_j)\mathbf{E}\right]\mathbf{a}_0^* \\ -\|\mathbf{a}_0\|^2\mathbf{r}_j^T\mathbf{b}^* + \frac{1}{2}\|\mathbf{a}_0\|^2\|\mathbf{r}_j\|^2 = 0\,, \quad \text{for} \quad j = 1, \ldots, m\end{aligned} \tag{11}$$

which are the synthesis equations for the right-hand dyad of Fig. 4.1 for the problem at hand. Apparently, these m equations are all *quartic*.

Remarks

- We have $2m$ equations, (9 & 11), to solve for eight unknowns—the components of $\mathbf{a}_0$, $\mathbf{b}$, $\mathbf{a}_0^*$, $\mathbf{b}^*$. Therefore, to have a determined system of equations, we must have

$$m = 4$$

which implies that up to *five points* can be visited in a plane using a four-bar linkage, with prescribed timing.

- Since the system of eqs.(9 & 11) involves four *quadratic* and four *quartic* equations in the unknowns $\{\mathbf{a}_0, \mathbf{b}\}$, the *Bezout number* N_B of the system, giving an upper bound for the number of roots to expect, being thus

$$N_B = 2^4 \times 4^4 = 4096$$

As Morgan and Wampler (1990) claimed, this number can be substantially reduced; in this reference, they reduced the number of roots to 36. We show below that the Morgan and Wampler number can be further reduced to one-third.

- Equations (9) are linear in $\mathbf{a}_0$ and quadratic in $\mathbf{b}$. Consequently, we can eliminate $\mathbf{a}_0$ by casting the said system in the form

$$\mathbf{B}\mathbf{x} = \mathbf{0} \tag{12}$$

in which $\mathbf{x} = [\ \mathbf{a}_0^T \quad 1\ ]^T$ and $\mathbf{B}$ is a 4×3 matrix function of $\mathbf{b}$ of the from

$$\mathbf{B} = \begin{bmatrix} \{(\mathbf{R}_1^T - \mathbf{1})\mathbf{b} - \mathbf{R}_1\mathbf{r}_1\}^T & \mathbf{b}^T(\mathbf{1} - \mathbf{R}_1)\mathbf{b} + \mathbf{r}_1^T(\mathbf{R}_1 - \mathbf{1})\mathbf{b} + \mathbf{r}_1^T\mathbf{r}_1/2 \\ \{(\mathbf{R}_2^T - \mathbf{1})\mathbf{b} - \mathbf{R}_1\mathbf{r}_2\}^T & \mathbf{b}^T(\mathbf{1} - \mathbf{R}_2)\mathbf{b} + \mathbf{r}_2^T(\mathbf{R}_2 - \mathbf{1})\mathbf{b} + \mathbf{r}_2^T\mathbf{r}_2/2 \\ \{(\mathbf{R}_3^T - \mathbf{1})\mathbf{b} - \mathbf{R}_3\mathbf{r}_3\}^T & \mathbf{b}^T(\mathbf{1} - \mathbf{R}_3)\mathbf{b} + \mathbf{r}_3^T(\mathbf{R}_3 - \mathbf{1})\mathbf{b} + \mathbf{r}_3^T\mathbf{r}_3/2 \\ \{(\mathbf{R}_4^T - \mathbf{1})\mathbf{b} - \mathbf{R}_4\mathbf{r}_4\}^T & \mathbf{b}^T(\mathbf{1} - \mathbf{R}_4)\mathbf{b} + \mathbf{r}_4^T(\mathbf{R}_4 - \mathbf{1})\mathbf{b} + \mathbf{r}_4^T\mathbf{r}_4/2 \end{bmatrix} \tag{13}$$

For the 4×3 matrix $\mathbf{B}$ to have a nonzero nullspace, which is needed in light of the form of $\mathbf{x}$, $\mathbf{B}$ must be rank-deficient. This means that every 3×3 submatrix of $\mathbf{B}$ must be singular. We can thus derive four bivariate polynomial equations in the Cartesian coordinates u and v of B, the components of $\mathbf{b}$, namely,

$$\Delta_j(u, v) = \det(\mathbf{B}_j)\,, \quad \text{for} \quad j = 1, \ldots, 4 \tag{14}$$

where Δ_j is the determinant of the jth 3×3 submatrix $\mathbf{B}_j$, obtained by deleting the jth row of $\mathbf{B}$. Notice that Δ_j can be computed by the cofactors of the third row of its associated matrix. Moreover, this row is quadratic in $\mathbf{b}$, the corresponding cofactors being determinants of 2×2 matrices whose entries are linear in $\mathbf{b}$. Such a determinant is expanded in Fact 1.4.1, Subsection 1.4.2, in which it is apparent that this determinant is a bilinear expression of its rows or, correspondingly, of its columns. Hence, each 2×2 cofactor is quadratic in $\mathbf{b}$, the result being that Δ_j is quartic in $\mathbf{b}$. Therefore, the Bezout number of any pair of those equations is

$$N_B = 4^2 = 16$$

Moreover, each eq.(14) defines a *contour* in the u-v plane. The real solutions of system (12) can be visually estimated by plotting the m contours in the same figure. Notice that, at the outset, we do not have bounds for the location of B in the u-v plane. However, we always have a region available of this plane in which we can anchor the revolute center B. Our first attempt of finding real solutions for B is thus this region.

Once $\mathbf{b}$ is known, we can solve for $\mathbf{a}_0$ from eq.(12) using a least-square approximation. To this end, we rewrite eq.(12) in the form

$$\mathbf{M}\mathbf{a}_0 = \mathbf{n} \tag{15}$$

where

$$\mathbf{M} = \begin{bmatrix} \{(\mathbf{R}_1^T - \mathbf{1})\mathbf{b} - \mathbf{R}_1\mathbf{r}_1\}^T \\ \{(\mathbf{R}_3^T - \mathbf{1})\mathbf{b} - \mathbf{R}_3\mathbf{r}_3\}^T \\ \{(\mathbf{R}_3^T - \mathbf{1})\mathbf{b} - \mathbf{R}_3\mathbf{r}_3\}^T \\ \{(\mathbf{R}_4^T - \mathbf{1})\mathbf{b} - \mathbf{R}_4\mathbf{r}_4\}^T \end{bmatrix}, \mathbf{n} = \begin{bmatrix} \mathbf{b}^T(\mathbf{1} - \mathbf{R}_1)\mathbf{b} + \mathbf{r}_1^T(\mathbf{R}_1 - \mathbf{1})\mathbf{b} + \mathbf{r}_1^T\mathbf{r}_1/2 \\ \mathbf{b}^T(\mathbf{1} - \mathbf{R}_2)\mathbf{b} + \mathbf{r}_2^T(\mathbf{R}_2 - \mathbf{1})\mathbf{b} + \mathbf{r}_2^T\mathbf{r}_2/2 \\ \mathbf{b}^T(\mathbf{1} - \mathbf{R}_3)\mathbf{b} + \mathbf{r}_3^T(\mathbf{R}_3 - \mathbf{1})\mathbf{b} + \mathbf{r}_3^T\mathbf{r}_3/2 \\ \mathbf{b}^T(\mathbf{1} - \mathbf{R}_4)\mathbf{b} + \mathbf{r}_4^T(\mathbf{R}_4 - \mathbf{1})\mathbf{b} + \mathbf{r}_4^T\mathbf{r}_4/2 \end{bmatrix} \tag{16}$$

- Equation (11) is bilinear in $\mathbf{b}^*$ and $\mathbf{a}_0^*$. Once we have $\mathbf{a}_0$ and $\mathbf{b}$ from eq.(12), we can solve eq.(11) for $\mathbf{a}_0^*$ and $\mathbf{b}^*$ using dialytic elimination, as we did in the motion-generation case. That is, computing $\mathbf{b}^*$ and $\mathbf{a}_0^*$ leads to the solution of one quartic polynomial. We need not find the roots of this polynomial numerically, if we apply the contour technique introduced in Chapter 4.

Reducing the Degree of the Synthesis Equations of the BA_0R Dyad

Using the definition of $\mathbf{Q}_j$ of eq.(1.6), the first term of eq.(9) can be further simplified to

$$\mathbf{b}^T(\mathbf{1} - \mathbf{R}_j)\mathbf{b} = \mathbf{b}^T[(1 - c\beta_j)\mathbf{1} + s\beta_j\mathbf{E}]\mathbf{b} = (1 - c\beta_j)\|\mathbf{b}\|^2, \quad j = 1, \ldots m \tag{17}$$

where we used the identity $\mathbf{b}^T\mathbf{E}\mathbf{b} \equiv 0$, because matrix $\mathbf{E}$ is *skew-symmetric*. Thus, eq.(9) reduces to

$$(1 - c\beta_j)\|\mathbf{b}\|^2 + \mathbf{b}^T(\mathbf{R}_j - \mathbf{1})\mathbf{a}_0 + \mathbf{r}_j^T(\mathbf{R}_j - \mathbf{1})\mathbf{b} - \mathbf{r}_j^T\mathbf{R}_j\mathbf{a}_0 + \frac{1}{2}\mathbf{r}_j^T\mathbf{r}_j = 0, \quad j = 1, \ldots, m \tag{18}$$

Let M be $j \in \{1, \ldots, m\}$ that maximizes $|1 - c\beta_j|$. Use now the Mth equation of eqn.(18) as a pivot, to reduce the order of the remaining equations. After a reshuffling of the equations, we let $M = 1$, so that now the pivot equation is the first one of the set. Just as in Gaussian elimination, subtract a "suitable" multiple of the first equation from the remaining ones, so as to eliminate the quadratic term of those equations, which leads to

$$(1 - c\beta_1)\|\mathbf{b}\|^2 + \mathbf{b}^T(\mathbf{R}_1 - \mathbf{1})\mathbf{a}_0 + \mathbf{r}_1^T(\mathbf{R}_1 - \mathbf{1})\mathbf{b} - \mathbf{r}_1^T\mathbf{R}_1\mathbf{a}_0 + \frac{1}{2}\mathbf{r}_1^T\mathbf{r}_1 = 0 \tag{19a}$$

$$\mathbf{b}^T[\mathbf{R}_j - \mathbf{1} - q_j(\mathbf{R}_j - \mathbf{1})]\mathbf{a}_0 + [\mathbf{r}_j^T(\mathbf{R}_j - \mathbf{1}) - q_j\mathbf{r}_j^T(\mathbf{R}_1 - \mathbf{1})]\mathbf{b} - (\mathbf{r}_j^T\mathbf{R}_j - q_j\mathbf{r}_1^T\mathbf{R}_1)\mathbf{a}_0 + \frac{1}{2}(\mathbf{r}_j^T\mathbf{r}_j - \mathbf{r}_1^T\mathbf{r}_1) = 0, \; j = 2, \ldots, m \tag{19b}$$

where

$$q_j = \frac{1 - c\beta_j}{1 - c\beta_1}, \quad j = 2, \ldots, m \tag{20}$$

System (19) can be cast in linear-homogeneous form in vector $\mathbf{x} = [\ \mathbf{a}_0^T \quad 1\ ]^T$:

$$\mathbf{B}\mathbf{x} = \mathbf{0}_4 \tag{21a}$$

with

$$\mathbf{B} = \begin{bmatrix} \{(\mathbf{R}_1^T - \mathbf{1})\mathbf{b} - \mathbf{R}_1^T\mathbf{r}_1\}^T & s_1 \\ \{\mathbf{R}_2^T - \mathbf{1} - q_2(\mathbf{R}_1^T - \mathbf{1})\}^T\mathbf{b} - (\mathbf{R}_2^T\mathbf{r}_2 - q_2\mathbf{R}_2^T) & s_2 \\ \{\mathbf{R}_3^T - \mathbf{1} - q_3(\mathbf{R}_3^T - \mathbf{1})\}^T\mathbf{b} - (\mathbf{R}_3^T\mathbf{r}_3 - q_3\mathbf{R}_3^T) & s_3 \\ \{\mathbf{R}_4^T - \mathbf{1} - q_4(\mathbf{R}_4^T - \mathbf{1})\}^T\mathbf{b} - (\mathbf{R}_4^T\mathbf{r}_4 - q_4\mathbf{R}_4^T) & s_4 \end{bmatrix} \tag{21b}$$

and

$$s_1 = (1 - c\beta_1)\|\mathbf{b}\|^2 + \mathbf{r}_1^T(\mathbf{R}_1 - \mathbf{1})\mathbf{b} + \frac{1}{2}\mathbf{r}_1^T\mathbf{r}_1 \tag{21c}$$

$$s_j = [\mathbf{r}_j^T(\mathbf{R}_j - \mathbf{1}) - q_j\mathbf{r}_j^T(\mathbf{R}_1 - \mathbf{1})]\mathbf{b} + \frac{1}{2}(\mathbf{r}_j^T\mathbf{r}_j - \mathbf{r}_1^T\mathbf{r}_1), \quad j = 2, \ldots, m \tag{21d}$$

Notice that s_1 is quadratic and $\{s_j\}_2^m$ are all linear in $\mathbf{b}$. Thus, the corresponding Δ_1 of eq.(14) for system (21) is quadratic, but $\{\Delta_j\}_2^m$ are all cubic in $\mathbf{b}$. Consequently, the Bezout number of any pair of equations $(1, j)$, for $j = 2, \ldots, m$, is

$$N_B = 3 \times 4 = 12$$

thereby reducing the Bezout number reported by Morgan and Wampler to one-third.

References

Morgan, A., and Wampler, C., 1990, "Solving a Planar Four-Bar Design Problem Using Continuation," *ASME Journal of Mechanical Design*, Vol. 112, pp. 544–550.