Coupler Curves of Planar Four-Bar Linkages

The trajectory traced by a point P of the couple link of a four-bar linkage, like the one shown in Fig. 1, is called the *coupler curve* traced by that point.

Synthesis of the Coupler Curve

We start by proving a basic result in planar kinematics regarding the nature of the coupler curve of a planar four-bar linkage, namely,

Theorem *The curve traced by any point of the coupler link of a planar four-bar linkage, as the one depicted in Fig. 1, is algebraic, of sixth degree.*

In general, a curve can be either *algebraic* or *non-algebraic*. A planar curve is algebraic if it is given by an implicit function $F(x,\, y) = 0$, with $F(x,\, y)$ being the sum of products of integer powers of x and y. The *degree* of the curve is the highest degree of the various terms making up $F(x,\, y)$. Moreover, a curve of degree n intersects a line at a maximum of n points. Thus, the coupler curve of a four-bar linkage intersects a line at a maximum of six points. As a consequence, the coupler curve under discussion *cannot have straight segments of finite length.*

Shown in Fig. 1 is a four-bar linkage in a Cartesian frame $\mathcal{F}$ with origin at O_1 and axes X and Y. The coupler link carries a point $P(x, y)$, which serves as origin of a second Cartesian frame, $\mathcal{G}$, with origin at P and axes U and V, fixed to this link. What we need now is an implicit function $F(x, y) = 0$, free of any linkage variable, and having as parameters the link lengths.

The desired function is obtained by first noticing that, irrespective of the linkage posture,

$$\|\overrightarrow{O_1O_2}\|^2 = a_2^2\,, \quad \|\overrightarrow{O_4O_3}\|^2 = a_4^2 \tag{1}$$

Henceforth, we shall use subscripted brackets to indicate the Cartesian frame in which vector components are represented. Thus,

$$[\,\overrightarrow{PO_2}\,]_{\mathcal{G}} = \begin{bmatrix} -u \\ -v \end{bmatrix} = \text{const},\ [\,\overrightarrow{PO_3}\,]_{\mathcal{G}} = \begin{bmatrix} w \\ -v \end{bmatrix} = \text{const},\ [\,\overrightarrow{O_1P}\,]_{\mathcal{F}} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{2}$$

Note that

$$\overrightarrow{O_1O_2} = \overrightarrow{O_1P} + \overrightarrow{PO_2} \tag{3}$$

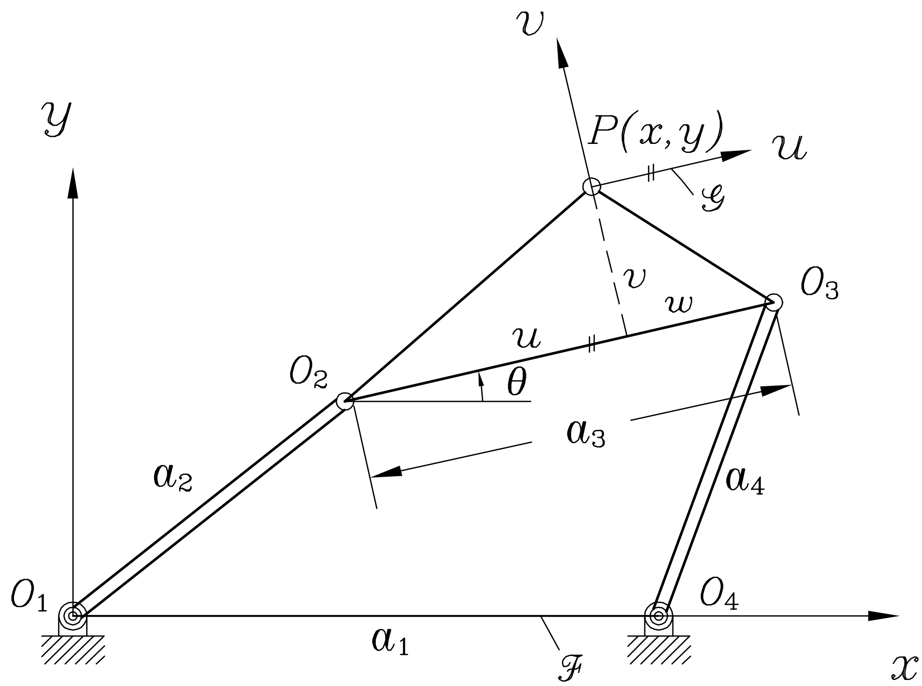

Figure 1: A point on the coupler link of a four-bar linkage

We have $\overrightarrow{O_1P}$ in $\mathcal{F}$, but $\overrightarrow{PO_2}$ in $\mathcal{G}$. In order to be able to add the two vectors in the above equation, we transform first the components of the second into $\mathcal{F}$, which is done via the matrix $\mathbf{Q}$ rotating $\mathcal{F}$ into $\mathcal{G}$, namely,

$$\mathbf{Q} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{4}$$

Hence,

$$[\,\overrightarrow{PO_2}\,]_{\mathcal{F}} = \mathbf{Q}[\,\overrightarrow{PO_2}\,]_{\mathcal{G}} = \begin{bmatrix} -u\cos\theta + v\sin\theta \\ -u\sin\theta - v\cos\theta \end{bmatrix} \tag{5}$$

Therefore,

$$[\,\overrightarrow{O_1O_2}\,]_{\mathcal{F}} = \begin{bmatrix} x - u\cos\theta + v\sin\theta \\ y - u\sin\theta - v\cos\theta \end{bmatrix} \tag{6}$$

On the other hand,

$$\overrightarrow{O_4O_3} = \overrightarrow{O_1O_3} - \overrightarrow{O_1O_4} = \overrightarrow{O_1P} + \overrightarrow{PO_3} - \overrightarrow{O_1O_4} \tag{7}$$

where

$$[\,\overrightarrow{O_1P}-\overrightarrow{O_1O_4}\,]_{\mathcal{F}} = \begin{bmatrix} x-a_1 \\ y \end{bmatrix},\ [\,\overrightarrow{PO_3}\,]_{\mathcal{F}} = \mathbf{Q}[\,\overrightarrow{PO_3}\,]_{\mathcal{G}} = \begin{bmatrix} w\cos\theta + v\sin\theta \\ w\sin\theta - v\cos\theta \end{bmatrix} \tag{8}$$

Upon substitution of eqs.(8) into eq.(7), we obtain

$$[\,\overrightarrow{O_4O_3}\,]_{\mathcal{F}} = \begin{bmatrix} x-a_1+w\cos\theta+v\sin\theta \\ y+w\sin\theta-v\cos\theta \end{bmatrix} \tag{9}$$

Now, let us substitute eqs.(6) and (9) into eqs.(1), to obtain, respectively,

$$x^2+y^2-2(ux+vy)\cos\theta+2(vx-uy)\sin\theta+u^2+v^2-a_2^2=0 \tag{10a}$$

$$x^2+y^2+2[w(x-a_1)-vy]\cos\theta+[v(x-a_1)+wy]\sin\theta \\ -2a_1x+a_1^2+v^2+w^2-a_4^2=0 \tag{10b}$$

The above two equations yield the desired implicit function $F(x,\,y)$, upon elimination of θ from the two of them. While we can do this at this stage, we risk ending up with a resultant equation of too high a degree, for notice that those two equations are *quadratic* in x and y, and linear in $\cos\theta$ and $\sin\theta$. In order to reduce the degree of the resultant equation, let us subtract eq.(10b) from eq.(10a):

$$2[(u+w)x-a_1w]\cos\theta+2[2a_1v+(w+u)y]\sin\theta-2a_1x+a_1^2+a_2^2-a_4^2+u^2+w^2=0 \tag{10c}$$

thereby obtaining an alternative equation that is linear in x and y as well as in $\cos\theta$ and $\sin\theta$. Now, we can eliminate θ from any of the two eqs.(10a) or (10b) and (10c). We do this by a) choosing eqs.(10b) and (10c), and b) using dialytic elimination: First, we introduce the familiar tan-half trigonometric identities, which we reproduce below for θ:

$$\cos\theta \equiv \frac{1-T^2}{1+T^2}, \quad \sin\theta \equiv \frac{2T}{1+T^2}, \quad T \equiv \tan\left(\frac{\theta}{2}\right) \tag{11}$$

Now we substitute the above expressions for $\cos\theta$ and $\sin\theta$ into eqs.(10b) and (10c), thereby obtaining

$$A_1T^2-2B_1T+C_1=0 \tag{12a}$$

$$A_2T^2-2B_2T+C_2=0 \tag{12b}$$

with

$$
\begin{aligned}
A_1 &\equiv x^2 + y^2 + 2(ux + vy) + u^2 - a_2^2 && \text{(12c)}\\
B_1 &\equiv 4(vx - uy) && \text{(12d)}\\
C_1 &\equiv x^2 + y^2 - 2(ux + vy) + u^2 - a_2^2 && \text{(12e)}\\
A_2 &\equiv -2(a_1 + u + w)x + a_1^2 + a_2^2 + a_4^2 + u^2 + w^2 - 2a_1 w && \text{(12f)}\\
B_2 &\equiv 4[-a_1 v + (w + u)y] && \text{(12g)}\\
C_2 &\equiv -2(a_1 - u - w)x + a_1^2 + a_2^2 + a_4^2 + u^2 + w^2 + 2a_1 w && \text{(12h)}
\end{aligned}
$$

Now, in order to eliminate dialytically T from eqs.(12a & b), we first multiply both sides of each of these equations by T, thereby obtaining two additional equations, both cubic in T:

$$
\begin{aligned}
A_1 T^3 - 2B_1 T^2 + C_1 T = 0 && \text{(12i)}\\
A_2 T^3 - 2B_2 T^2 + C_2 T = 0 && \text{(12j)}
\end{aligned}
$$

Equations (12a–j) now represent a system of four linear homogeneous equations in T^0, T^1, T^2 and T^3, i.e.,

$$\mathbf{M}\mathbf{x} = \mathbf{0}_4 \tag{13a}$$

where $\mathbf{0}_4$ is the four-dimensional zero vector, while $\mathbf{M}$ and $\mathbf{x}$ are given below:

$$\mathbf{M} \equiv \begin{bmatrix} A_1 & -2B_1 & C_1 & 0 \\ A_2 & -2B_2 & C_2 & 0 \\ 0 & A_1 & -2B_1 & C_1 \\ 0 & A_2 & -2B_2 & C_2 \end{bmatrix}, \quad \mathbf{x} \equiv \begin{bmatrix} T^4 \\ T^3 \\ T^2 \\ T \\ 1 \end{bmatrix} \tag{13b}$$

Apparently, the trivial solution $\mathbf{x} = \mathbf{0}$ is not admissible, and hence, $\mathbf{M}$ must be singular, i.e.,

$$F(x,\, y) \equiv \det(\mathbf{M}) = 0 \tag{14}$$

which is the desired implicit function defining the coupler curve sought. It is apparent that the first and third rows of $\mathbf{M}$ are quadratic in x and y, while the second and fourth are linear in the same variables. Consequently, $F(x,\, y)$ is *sextic* in x and y, q.e.d.