Spherical Four-Bar-Linkages

October 30, 2003

Derivation of the I-O Equation

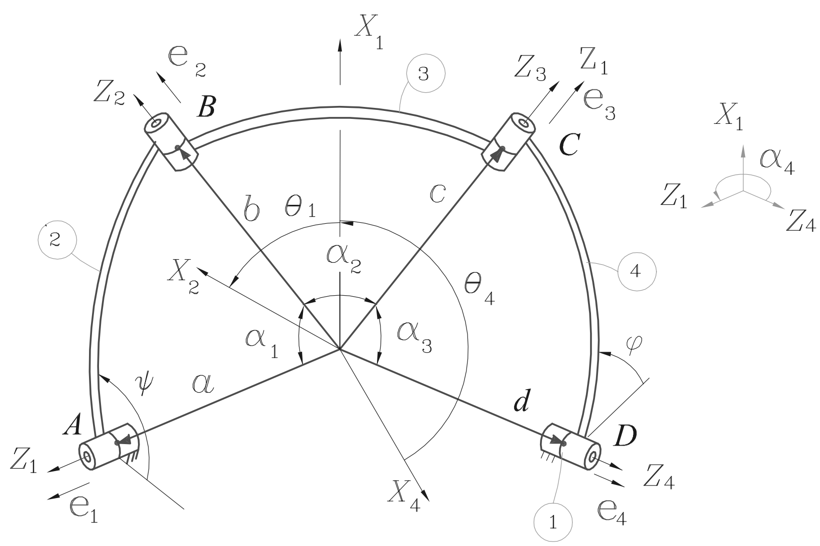

Figure 1: A spherical four-bar linkage for function generation

$$\begin{aligned} \mathbf{Q}_4 : \ \mathcal{F}_4 \ &\rightarrow \ \mathcal{F}_1 \quad \text{or} \quad [\,\cdot\,]_1 \ \rightarrow \ [\,\cdot\,]_4 \\ \Rightarrow \mathbf{Q}_4^T : \ \mathcal{F}_1 \ &\rightarrow \ \mathcal{F}_4 \quad \text{or} \quad [\,\cdot\,]_4 \ \rightarrow \ [\,\cdot\,]_1 \end{aligned}$$

where $\mathbf{Q}_i$ is the ith rotation matrix:

$$\mathbf{Q}_i \equiv \begin{bmatrix} c\theta_i & -c\alpha_i s\theta_i & s\alpha_i s\theta_i \\ s\theta_i & c\alpha_i c\theta_i & -s\alpha_i c\theta_i \\ 0 & s\alpha_i & c\alpha_i \end{bmatrix}; \qquad \begin{array}{l} c(\cdot) = \cos(\cdot) \\ s(\cdot) = \sin(\cdot) \end{array}$$

Synthesis equation:

$$\mathbf{e}_2 \cdot \mathbf{e}_3 = \cos\alpha_2 \tag{1}$$

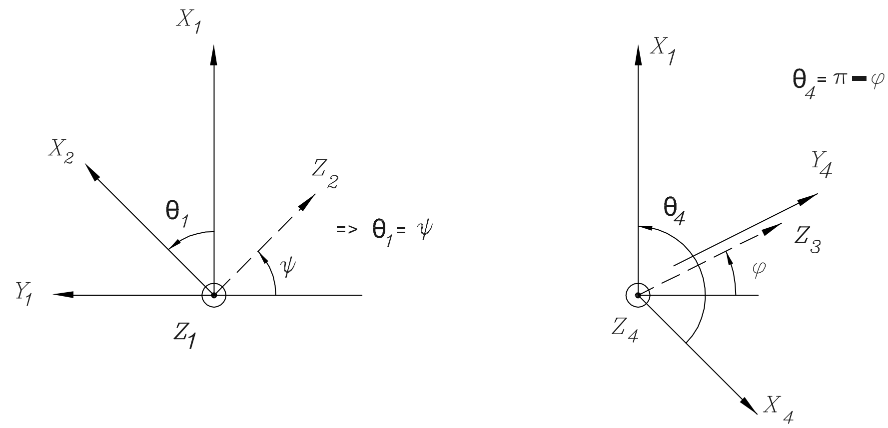

Figure 2: Equivalence between the DH notation and our I/O notation

$$[\mathbf{e}_i]_i = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

$$[\mathbf{e}_2]_1 = \mathbf{Q}_1[\mathbf{e}_2]_2 = \text{3rd column of } \mathbf{Q}_1 = \begin{bmatrix} s\alpha_1 s\theta_1 & -s\alpha_1 c\theta_1 & c\alpha_1 \end{bmatrix}^T \tag{2}$$

$$[\mathbf{e}_3]_4 = \mathbf{Q}_3^T[\mathbf{e}_3]_3 = \text{3rd column of } \mathbf{Q}_3 = \begin{bmatrix} 0 & s\alpha_3 & c\alpha_3 \end{bmatrix}^T \tag{3}$$

$$[\mathbf{e}_3]_1 = \mathbf{Q}_4^T[\mathbf{e}_3]_4 = \begin{bmatrix} c\theta_4 & s\theta_4 & 0 \\ -c\alpha_4 s\theta_4 & c\alpha_4 c\theta_4 & s\alpha_4 \\ s\alpha_4 s\theta_4 & -s\alpha_4 c\theta_4 & c\alpha_4 \end{bmatrix} \begin{bmatrix} 0 \\ s\alpha_3 \\ c\alpha_3 \end{bmatrix}$$

$$\Rightarrow \quad [\mathbf{e}_3]_1 = \begin{bmatrix} s\alpha_3 s\theta_4 \\ s\alpha_3 c\alpha_4 c\theta_4 + c\alpha_3 s\alpha_4 \\ -s\alpha_3 s\alpha_4 c\theta_4 + c\alpha_3 c\alpha_4 \end{bmatrix} \tag{4}$$

Upon substitution of eqs.(2) and (4) into eq.(1) we derive

$$s\alpha_1 s\alpha_3 s\theta_1 s\theta_4 - s\alpha_1 c\theta_1(s\alpha_3 c\alpha_4 c\theta_4 + c\alpha_3 s\alpha_4) + c\alpha_1(c\alpha_3 c\alpha_4 - s\alpha_3 s\alpha_4 c\theta_4) = c\alpha_2$$

or

$$c\alpha_1 c\alpha_3 c\alpha_4 - c\alpha_2 - s\alpha_1 c\alpha_3 s\alpha_4 c\theta_1 - c\alpha_1 s\alpha_3 s\alpha_4 c\theta_4 \\ - s\alpha_1 s\alpha_3 c\alpha_4 c\theta_1 c\theta_4 + s\alpha_1 s\alpha_3 s\theta_1 s\theta_4 = 0 \tag{5}$$

which is the input-output equation sought. Now, we introduce Freudenstein parameters upon dividing both sides of eq.(5) by $s\alpha_1 s\alpha_3$:

$$k_1 - k_2 c\theta_1 - k_3 c\theta_4 - k_4 c\theta_1 c\theta_4 + s\theta_1 s\theta_4 = 0 \tag{6a}$$

with

$$k_1 \equiv \frac{c\alpha_1 c\alpha_3 c\alpha_4 - c\alpha_2}{s\alpha_1 s\alpha_3}, \quad k_2 \equiv \frac{c\alpha_3 s\alpha_4}{s\alpha_3}, \quad k_3 \equiv \frac{c\alpha_1 s\alpha_4}{s\alpha_1}, \quad k_4 \equiv c\alpha_4 \tag{6b}$$

the inverse relations being

$$
\begin{aligned}
\cos\alpha_1 &= \frac{k_3}{\sqrt{1+k_3^2-k_4^2}}, & \sin\alpha_1 &= \sqrt{\frac{1-k_4^2}{1+k_3^2-k_4^2}} \\
\cos\alpha_2 &= \frac{k_2k_3k_4-k_1(1-k_4^2)}{\sqrt{(1+k_3^2-k_4^2)(1+k_2^2-k_4^2)}}, & \sin\alpha_2 &= \sqrt{1-\frac{[k_2k_3k_4-k_1(1-k_4^2)]^2}{(1+k_3^2-k_4^2)(1+k_2^2-k_4^2)}} \\
\cos\alpha_3 &= \frac{k_2}{\sqrt{1+k_2^2-k_4^2}}, & \sin\alpha_3 &= \sqrt{\frac{1-k_4^2}{1+k_2^2-k_4^2}} \\
\cos\alpha_4 &= k_4, & \sin\alpha_4 &= \sqrt{1-k_4^2}
\end{aligned}
$$