MECH 577 Optimum Design
November 30th, 2004

Class Test
Solution

1. a)

$$\underset{\sim}{M} = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 2 & 0 & y \\ 0 & 0 & 3 & z \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Expand $\det(\underset{\sim}{M})$ by cofactors of 4th column:

$$\det(\underset{\sim}{M}) = -x \det\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix} + y \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$- z \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} + 1 \times \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= -x\left[-2 \times (-3)\right] + y(-3) - z(2) + 1 \times 6$$

$$= -6x - 3y - 2z + 6 = 0$$

$$\Rightarrow \Pi: \; 6x + 3y + 2z = 6 \qquad \underline{\text{Ans.}}$$

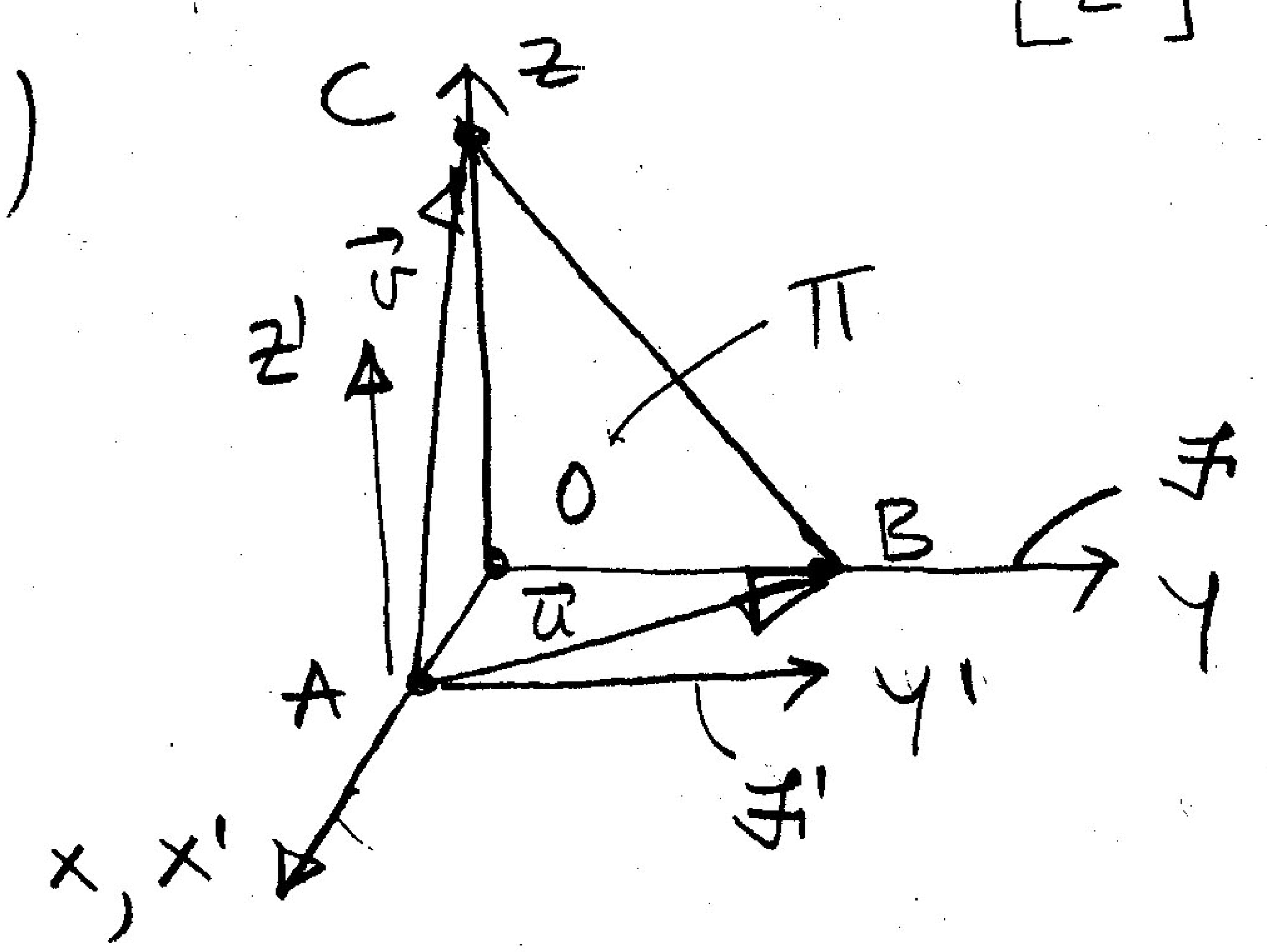
$$\Rightarrow \underset{\sim}{C} = [6 \;\; 3 \;\; 2], \quad d = [6]$$

b)

$$[\vec{p}_0]_{\mathcal{F}} = \underset{\sim}{C}^T (\underbrace{\underset{\sim}{C}\,\underset{\sim}{C}^T}_{})^{-1} d = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} \frac{1}{49} 6 = \begin{bmatrix} 36/49 \\ 18/49 \\ 12/49 \end{bmatrix} \quad \underline{\text{Ans.}}$$

$$[6 \;\; 3 \;\; 2]\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} = 49$$

c)



$$\Rightarrow [\vec{p}]_{\mathcal{F}'} = [\vec{p}]_{\mathcal{F}} - [\vec{a}]_{\mathcal{F}}$$

Express an arbitrary vector $\vec{p} \in \Pi$ as a linear combination of linearly independent vectors in Π, e.g. $\vec{u} \equiv \vec{b}-\vec{a}$ & $\vec{v} = \vec{c}-\vec{a}$: ②

$$\vec{u} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

$$\Rightarrow \vec{p} = \vec{u}x_1 + \vec{v}x_2 = \underbrace{\begin{bmatrix} -1 & -1 \\ 2 & 0 \\ 0 & 3 \end{bmatrix}}_{\underset{\sim}{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x} \in \mathbb{R}^2} \tag{1}$$

We want to reach $[\vec{o}]_{\mathcal{F}} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ (2)

$$(1) \;\&\; (2) \Rightarrow \underbrace{\begin{bmatrix} -1 & -1 \\ 2 & 0 \\ 0 & 3 \end{bmatrix}}_{\underset{\sim}{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}}_{-\vec{a}} \tag{3}$$

Let $\vec{x}_L$ be least-square approximation of eq. (3).
Then,

$$\vec{x}_L = (\underset{\sim}{A}^T \underset{\sim}{A})^{-1} \underset{\sim}{A}^T (-\vec{a})$$

$$\underset{\sim}{A}^T \underset{\sim}{A} = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 10 \end{bmatrix} \Rightarrow \det(\underset{\sim}{A}^T \underset{\sim}{A}) = 49$$

$$\Rightarrow \vec{x}_L = \frac{1}{49} \begin{bmatrix} 10 & -1 \\ -1 & 5 \end{bmatrix} \underbrace{\begin{bmatrix} -1 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}}_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \frac{1}{49} \begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

$$[\vec{p}_0]_{\mathcal{F}'} = \underset{\sim}{A}\,\vec{x}_L = \begin{bmatrix} -1 & -1 \\ 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ 4 \end{bmatrix} \frac{1}{49} = \underbrace{\frac{1}{49} \begin{bmatrix} -13 \\ 18 \\ 12 \end{bmatrix}}_{= [\vec{p}_0]_{\mathcal{F}_1} - [\vec{a}]_{\mathcal{F}_1}} \quad \underline{\text{Ans.}}$$

<u>Projection Thm:</u>

$$\vec{e}_0 = -\vec{a} - \vec{p}_0 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -13/49 \\ 18/49 \\ 12/49 \end{bmatrix} = \begin{bmatrix} -36/49 \\ -18/49 \\ -12/49 \end{bmatrix}$$

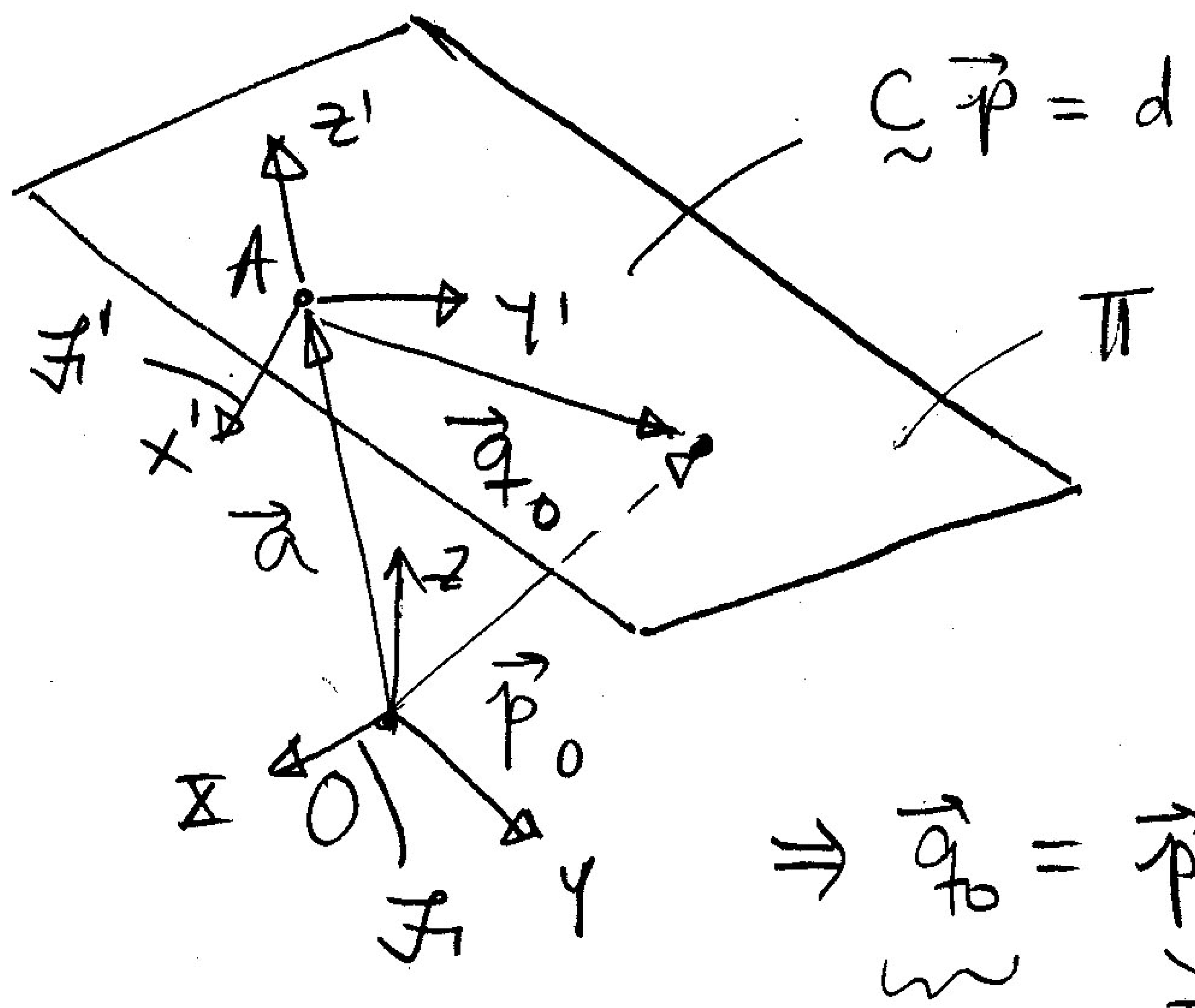
$$= -\frac{6}{49} \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$$

$$\Rightarrow \vec{e}_0^{\,T} \vec{p}_0 = -\frac{6}{49^2} \begin{bmatrix} 6 & 3 & 2 \end{bmatrix} \begin{bmatrix} -13 \\ 18 \\ 12 \end{bmatrix} = -\frac{6}{49^2}\left(\cancel{-78} + \underbrace{54+24}_{\cancel{78}}\right)$$

$$= 0 \qquad \underline{\text{Check}}$$

d) $$\underset{\sim}{C}\,\underset{\sim}{A} = \begin{bmatrix} 6 & 3 & 2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -6+6 & -6+6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \qquad \underline{\text{Check}}$$

e) In general, we have



$$\Rightarrow \underbrace{[\vec{p}_0]_{\mathcal{F}'}}_{\equiv \vec{q}_0} = \underbrace{[\vec{p}_0]_{\mathcal{F}_1}}_{\equiv \vec{p}_0} - \underbrace{[\vec{a}]_{\mathcal{F}_1}}_{\equiv \vec{a}}$$

$$\Rightarrow \underbrace{\vec{q}_0}_{\underset{\sim}{A}(\underset{\sim}{A}^T\underset{\sim}{A})^{-1}\underset{\sim}{A}^T(-\vec{a})} = \underbrace{\vec{p}_0}_{\underset{\sim}{C}^T(\underset{\sim}{C}\,\underset{\sim}{C}^T)^{-1}d} - \vec{a}$$

$$\Rightarrow \; -\underset{\sim}{A}(\underset{\sim}{A}^T\underset{\sim}{A})^{-1}\underset{\sim}{A}^T\vec{a} = \underset{\sim}{C}^T(\underset{\sim}{C}\underset{\sim}{C}^T)^{-1}d - \vec{a} \qquad (4)$$

$$\underset{\sim}{A}^T(4) \Rightarrow \; \underbrace{-\underbrace{\underset{\sim}{A}^T\underset{\sim}{A}(\underset{\sim}{A}^T\underset{\sim}{A})^{-1}}_{\underset{\sim}{1}}\underset{\sim}{A}^T\vec{a}}_{-\underset{\sim}{A}^T\vec{a}} = \underset{\sim}{A}^T\underset{\sim}{C}^T(\underset{\sim}{C}\underset{\sim}{C}^T)^{-1}d - \underset{\sim}{A}^T\vec{a}$$

$$\Rightarrow \; \underset{\sim}{A}^T\underset{\sim}{C}^T(\underset{\sim}{C}\underset{\sim}{C}^T)^{-1}d = 0$$

$$d \neq 0 \Rightarrow \; \underbrace{\underset{\sim}{A}^T}_{2\times 3}\underbrace{\underset{\sim}{C}^T}_{3\times 1}\underbrace{(\underset{\sim}{C}\underset{\sim}{C}^T)^{-1}}_{\substack{1\times 1 \\ \text{nonsingular}}} = \underset{\sim}{0}_{2\times 1} \Rightarrow \underset{\sim}{A}^T\underset{\sim}{C}^T = \underset{\sim}{0}_{2\times 1}$$

$$\Rightarrow \underset{\sim}{C}\underset{\sim}{A} = \underset{\sim}{0}_{1\times 2} \qquad \text{q.e.d.}$$

Ans.

2. a) $\underset{\sim}{1}$ is symmetric; $\underset{\sim}{A}\,\underset{\sim}{A}^T$ is symmetric $\Rightarrow (\underset{\sim}{A}\,\underset{\sim}{A}^T)^{-1}$ is symmetric $\Rightarrow \underset{\sim}{A}^T(\underset{\sim}{A}\,\underset{\sim}{A}^T)^{-1}\underset{\sim}{A}$ is symmetric $\Rightarrow \underset{\sim}{P}$ is symmetric (i) Ans.

To prove (ii), we show first that there exists a vector $\vec{v} \in \mathbb{R}^n$ that is mapped by $\underset{\sim}{P}$ onto $\vec{0}_n$. Indeed, from the form of $\underset{\sim}{P}$, it is apparent that $\vec{v} = \underset{\sim}{A}^T\vec{u}$, $\vec{u} \in \mathbb{R}^m$ is such a vector:

$$\underset{\sim}{P}\vec{v} = \underset{\sim}{P}\,\underset{\sim}{A}^T\vec{u} = \left[\underset{\sim}{1} - \underset{\sim}{A}^T(\underset{\sim}{A}\,\underset{\sim}{A}^T)^{-1}\underset{\sim}{A}\right]\underset{\sim}{A}^T\vec{u}$$

$$= \underset{\sim}{A}^T\vec{u} - \underset{\sim}{A}^T\underbrace{(\underset{\sim}{A}\,\underset{\sim}{A}^T)^{-1}\underset{\sim}{A}\,\underset{\sim}{A}^T}_{\underset{\sim}{1}}\vec{u} = \underset{\sim}{A}^T\vec{u} - \underset{\sim}{A}^T\vec{u} = \vec{0}_n$$

q.e.d. Ans.

(iii) $$\underset{\sim}{P}^2 = \left[\underset{\sim}{1} - \underset{\sim}{A}^T(\underset{\sim}{A}\,\underset{\sim}{A}^T)^{-1}\underset{\sim}{A}\right]\left[\underset{\sim}{1} - \underset{\sim}{A}^T(\underset{\sim}{A}\,\underset{\sim}{A}^T)^{-1}\underset{\sim}{A}\right]$$

$$= \underset{\sim}{1} - \underset{\sim}{A}^T(\underset{\sim}{A}\,\underset{\sim}{A}^T)^{-1}\underset{\sim}{A} - \underset{\sim}{A}^T(\underset{\sim}{A}\,\underset{\sim}{A}^T)^{-1}\underset{\sim}{A} + \underset{\sim}{A}^T(\underset{\sim}{A}\,\underset{\sim}{A}^T)^{-1}\underbrace{\underset{\sim}{A}\,\underset{\sim}{A}^T(\underset{\sim}{A}\,\underset{\sim}{A}^T)^{-1}}_{\underset{\sim}{1}}\underset{\sim}{A}$$

$$= \underset{\sim}{1} - 2\underset{\sim}{A}^T(\underset{\sim}{A}\,\underset{\sim}{A}^T)^{-1}\underset{\sim}{A} + \underset{\sim}{A}^T(\underset{\sim}{A}\,\underset{\sim}{A}^T)^{-1}\underset{\sim}{A}$$

$$= \underset{\sim}{1} - \underset{\sim}{A}^T(\underset{\sim}{A}\,\underset{\sim}{A}^T)^{-1}\underset{\sim}{A} = \underset{\sim}{P},$$ q.e.d. Ans

3. Introduce a Householder reflection $\underset{\sim}{H}$ that renders $\underset{\sim}{J}^T$ into upper-triangular form. Moreover, we want an isotropic orthogonal complement $\underset{\sim}{L}$, of $\underset{\sim}{J}$ such that

$$\underbrace{\underset{\sim}{J}}_{1\times 3}\ \underbrace{\underset{\sim}{L}_0}_{3\times 2} = \underset{\sim}{0}_{1\times 2} \Rightarrow \underset{\sim}{J}\,\underset{\sim}{H}^T \underbrace{\underset{\sim}{H}\,\underset{\sim}{L}_0}_{\underset{\sim}{1}} = \underset{\sim}{0}_{1\times 2}$$

Computation of $\underset{\sim}{H}$:

$$\alpha = \operatorname{sgn}(3)\sqrt{9+0+16} = +5$$

$$\vec{u} = [3+5 \quad 0 \quad 4]^T = [8 \quad 0 \quad 4]^T$$

$$\Rightarrow \|\vec{u}\|^2 = 64+16 = 80$$

$$\vec{u}\,\vec{u}^T = \begin{bmatrix} 64 & 0 & 32 \\ 0 & 0 & 0 \\ 32 & 0 & 16 \end{bmatrix} = 16\begin{bmatrix} 4 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$2\frac{\vec{u}\,\vec{u}^T}{\|\vec{u}\|^2} = \frac{32}{80}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \frac{2}{5}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \underset{\sim}{H} = \underset{\sim}{1} - 2\frac{\vec{u}\,\vec{u}^T}{\|\vec{u}\|^2} = \begin{bmatrix} 1-8/5 & 0 & -4/5 \\ 0 & 1-0 & 0 \\ -4/5 & 0 & 1-2/5 \end{bmatrix} = \begin{bmatrix} -3/5 & 0 & -4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \end{bmatrix}$$

$$\Rightarrow \underset{\sim}{H}\,\underset{\sim}{J}^T = \frac{1}{5}\begin{bmatrix} -3 & 0 & -4 \\ 0 & 5 & 0 \\ -4 & 0 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -25 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \underset{\sim}{H}\,\underset{\sim}{L}_0 = \begin{bmatrix} \underset{\sim}{0}_{2\times 1} \\ \underset{\sim}{1}_{2\times 2} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\underset{\sim}{E}} \Rightarrow \underset{\sim}{L}_0 = \underset{\sim}{H}^T\underset{\sim}{E} \equiv \underset{\sim}{H}\,\underset{\sim}{E} = \frac{1}{5}\begin{bmatrix} -3 & 0 & -4 \\ 0 & 5 & 0 \\ -4 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \underset{\sim}{L}_0 = \frac{1}{5}\begin{bmatrix} 0 & -4 \\ 5 & 0 \\ 0 & 3 \end{bmatrix} \quad \underline{\underline{\text{Ans.}}}$$