MECH541

A SUMMARY OF DUAL ALGEBRA

Appendix

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Contents

Chapter 1

A Summary of Dual Algebra

The algebra of dual numbers is recalled here, with extensions to vector and matrix operations. This material is reproduced from a chapter in a NATO Advanced Study Institute book[1]

1.1 Introduction

The aim of this Appendix is to outine the applications of dual algebra to kinematic analysis. To this end, the algebra of dual scalars, vectors, and matrices is first recalled. The applications included here refer to the computation of the parameters of the screw of a rigid body between two finitely-separated positions and of the instant screw. However, the applications of dual numbers go beyond that in kinematics. Indeed, the well-known *Principle of Transference* (Dimentberg, 1965; Bottema and Roth, 1978; Martínez and Duffy, 1994) has been found extremely useful in spatial kinematics, since it allows the derivation of spatial kinematic relations by simply *dualizing* the corresponding relations of spherical kinematics.

Dual numbers were first proposed by Clifford (1873), their first applications to kinematics being attributed to both Kotel'nikov (1895) and Study (1903). A comprehensive analysis of dual numbers and their applications to the kinematic analysis of spatial linkages was conducted by Yang (1963) and Yang and Freudenstein (1964). Bottema and Roth (1978) include a treatment of theoretical kinematics using dual numbers. More recently, Agrawal (1987) reported on the application of dual quaternions to spatial kinematics, while Pradeep et al. (1989) used the dual-matrix exponential in the analysis of robotic manipulators. Shoham and Brodsky (1993, 1994) have proposed a dual inertia operator for the dynamical analysis of mechanical systems. A comprehensive introduction

[1]Angeles, J., 1998, "The Application of Dual Algebra to Kinematic Analysis", in Angeles, J. and Zakhariev, E. (editors), Computational Methods in Mechanical Systems, Springer-Verlag, Heidelberg, Vol. 161, pp. 3-31.

to dual quaternions is to be found in (McCarthy, 1990), while an abstract treatment is found in (Chevallier, 1991)

1.2 Definitions

A *dual number* $\hat{a}$ is defined as the sum of a *primal* part a, and a *dual* part a_0, namely,

$$\hat{a} = a + \epsilon a_0 \ , \tag{1.1}$$

where ϵ is the dual unity, which verifies $\epsilon \neq 0$, $\epsilon^2 = 0$, and a and a_0 are real numbers, the former being the *primal part* of $\hat{a}$, the latter its *dual part.* Actually, dual numbers with complex parts can be equally defined (Cheng and Thompson, 1996). For the purposes of this chapter, real numbers will suffice.

If $a_0 = 0$, $\hat{a}$ is called a *real number*, or, correspondingly, a *complex number*; if $a = 0$, $\hat{a}$ is called a *pure dual number*; and if neither is zero $\hat{a}$ is called a *proper dual number.*

Let $\hat{b} = b + \epsilon b_0$ be another dual number. Equality, addition, multiplication, and division are defined, respectively, as

$$\hat{a} = \hat{b} \Leftrightarrow a = b, \quad a_0 = b_0 \tag{1.2a}$$

$$\hat{a} + \hat{b} = (a + b) + \epsilon(a_0 + b_0) \tag{1.2b}$$

$$\hat{a}\hat{b} = ab + \epsilon(ab_0 + a_0 b) \tag{1.2c}$$

$$\frac{\hat{a}}{\hat{b}} = \frac{a}{b} - \epsilon \left(\frac{ab_0 - a_0 b}{b^2} \right), \quad b \neq 0 \ . \tag{1.2d}$$

From eq.(1.2d) it is apparent that the division by a pure dual number is not defined. Hence, dual numbers do not form a *field* in the algebraic sense; they do form a *ring* (Simmons, 1963).

All formal operations involving dual numbers are identical to those of ordinary algebra, while taking into account that $\epsilon^2 = \epsilon^3 = \cdots = 0$. Therefore, the series expansion of the *analytic function* $f(\hat{x})$ of a dual argument $\hat{x}$ is given by

$$f(\hat{x}) = f(x + \epsilon x_0) = f(x) + \epsilon x_0 \frac{\mathrm{df(x)}}{\mathrm{dx}} \ . \tag{1.3}$$

As a direct consequence of eq.(1.3), we have the expression below for the exponential of a dual number $\hat{x}$:

$$\mathrm{e}^{\hat{x}} = \mathrm{e}^{x} + \epsilon\, x_0 \mathrm{e}^{x} = \mathrm{e}^{x}(1 + \epsilon\, x_0) \ , \tag{1.4}$$

and hence, *the dual exponential cannot be a pure dual number.*

The *dual angle* $\hat{\theta}$ between two skew lines $\mathcal{L}_1$ and $\mathcal{L}_2$, introduced by Study (1903), is defined as

$$\hat{\theta} = \theta + \epsilon s \ , \tag{1.5}$$

where θ and s are, respectively, the twist angle and the distance between the two lines. The *dual trigonometric functions* of the dual angle $\hat{\theta}$ are derived directly from eq.(1.3), namely,

$$\cos\hat{\theta} = \cos\theta - \epsilon s \sin\theta, \quad \sin\hat{\theta} = \sin\theta + \epsilon s \cos\theta, \quad \tan\hat{\theta} = \tan\theta + \epsilon s \sec^2\theta \; . \tag{1.6}$$

Moreover, all identities for ordinary trigonometry hold for dual angles. Likewise, the square root of a dual number can be readily found by a straightforward application of eq.(1.3), namely,

$$\sqrt{\hat{x}} = \sqrt{x} + \epsilon \frac{x_0}{2\sqrt{x}} \; , \tag{1.7}$$

A *dual vector* $\hat{\mathbf{a}}$ is defined as the sum of a primal vector part $\mathbf{a}$, and a dual vector part $\mathbf{a}_0$, namely,

$$\hat{\mathbf{a}} = \mathbf{a} + \epsilon \mathbf{a}_0, \tag{1.8}$$

where both $\mathbf{a}$ and $\mathbf{a}_0$ are Cartesian, 3-dimensional vectors. Henceforth, all vectors are assumed to be of this kind. Further, let $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ be two dual vectors and $\hat{c}$ be a dual scalar. The concepts of dual-vector equality, multiplication of a dual vector by a dual scalar, inner product and vector product of two dual vectors are defined below:

$$\hat{\mathbf{a}} = \hat{\mathbf{b}} \quad \Leftrightarrow \quad \mathbf{a} = \mathbf{b} \quad \text{and} \quad \mathbf{a}_0 = \mathbf{b}_0 \; ; \tag{1.9a}$$

$$\hat{c}\,\hat{\mathbf{a}} = c\,\mathbf{a} + \epsilon\,(c_0\mathbf{a} + c\,\mathbf{a}_0) \; ; \tag{1.9b}$$

$$\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = \mathbf{a} \cdot \mathbf{b} + \epsilon\,(\mathbf{a} \cdot \mathbf{b}_0 + \mathbf{a}_0 \cdot \mathbf{b}) \; ; \tag{1.9c}$$

$$\hat{\mathbf{a}} \times \hat{\mathbf{b}} = \mathbf{a} \times \mathbf{b} + \epsilon\,(\mathbf{a} \times \mathbf{b}_0 + \mathbf{a}_0 \times \mathbf{b}) \; . \tag{1.9d}$$

In particular, when $\hat{\mathbf{b}} = \hat{\mathbf{a}}$, eq.(1.9c) leads to the *Euclidean norm* of the dual vector $\hat{\mathbf{a}}$, i.e.,

$$\|\hat{\mathbf{a}}\|^2 = \|\mathbf{a}\|^2 + \epsilon\, 2\mathbf{a} \cdot \mathbf{a}_0 \; . \tag{1.9e}$$

Furthermore, the six *normalized Plücker coordinates* of a line $\mathcal{L}$ passing through a point P of position vector $\mathbf{p}$ and parallel to the unit vector $\mathbf{e}$ are given by the pair $(\,\mathbf{e},\, \mathbf{p} \times \mathbf{e}\,)$, where the product $\mathbf{e}_0 \equiv \mathbf{p} \times \mathbf{e}$ denotes the *moment* of the line. The foregoing coordinates can be represented by a *dual unit vector* $\hat{\mathrm{e}}^*$, whose six real components in $\mathbf{e}$ and $\mathbf{e}_0$ are the Plücker coordinates of $\mathcal{L}$, namely,

$$\hat{\mathrm{e}}^* = \mathbf{e} + \epsilon\,\mathbf{e}_0, \quad \text{with} \quad \|\mathbf{e}\| = 1 \quad \text{and} \quad \mathbf{e} \cdot \mathbf{e}_0 = 0 \; . \tag{1.10}$$

The reader is invited to verify the results summarized below:

Lemma 1.2.1 *For* $\hat{\mathbf{e}}^* \equiv \mathbf{e} + \epsilon\,\mathbf{e}_0$ *and* $\hat{\mathbf{f}}^* \equiv \mathbf{f} + \epsilon\,\mathbf{f}_0$ *defined as two dual unit vectors representing lines* $\mathcal{L}$ *and* $\mathcal{M}$*, respectively, we have:*

(*i*) *If* $\hat{\mathbf{e}}^* \times \hat{\mathbf{f}}^*$ *is a pure dual vector, then* $\mathcal{L}$ *and* $\mathcal{M}$ *are parallel;*

(ii) *if* $\hat{\mathbf{e}}^* \cdot \hat{\mathbf{f}}^*$ *is a pure dual number, then* $\mathcal{L}$ *and* $\mathcal{M}$ *are perpendicular;*

(iii) $\mathcal{L}$ *and* $\mathcal{M}$ *are coincident if and only if* $\hat{\mathbf{e}}^* \times \hat{\mathbf{f}}^* = \mathbf{0}$*; and*

(iv) $\mathcal{L}$ *and* $\mathcal{M}$ *intersect at right angles if and only if* $\hat{\mathbf{e}}^* \cdot \hat{\mathbf{f}}^* = 0$.

Dual matrices can be defined likewise, i.e., if $\mathbf{A}$ and $\mathbf{A}_0$ are two real $n \times n$ matrices, then the dual $n \times n$ matrix $\hat{\mathbf{A}}$ is defined as

$$\hat{\mathbf{A}} \equiv \mathbf{A} + \epsilon\, \mathbf{A}_0 \,. \tag{1.11}$$

We will work with 3×3 matrices in connection with dual vectors, but the above definition can be applied to any square matrices, which is the reason why n has been left arbitrary. Equality, multiplication by a dual scalar, and multiplication by a dual vector are defined as in the foregoing cases. Moreover, matrix multiplication is defined correspondingly, but then the order of multiplication must be respected. We thus have that, if $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ are two $n \times n$ dual matrices, with their primal and dual parts self-understood, then

$$\hat{\mathbf{A}}\hat{\mathbf{B}} = \mathbf{A}\mathbf{B} + \epsilon\,(\mathbf{A}\mathbf{B}_0 + \mathbf{A}_0\mathbf{B}) \,. \tag{1.12}$$

Therefore, matrix $\hat{\mathbf{A}}$ is real if $\mathbf{A}_0 = \mathbf{O}$, where $\mathbf{O}$ denotes the $n \times n$ zero matrix; if $\mathbf{A} = \mathbf{O}$, then $\hat{\mathbf{A}}$ is called a *pure dual matrix*. Moreover, as we shall see below, a square dual matrix admits an inverse if and only if its primal part is nonsingular.

Now we can define the inverse of a dual matrix, if this is nonsingular. Indeed, it suffices to make $\hat{\mathbf{B}} = \hat{\mathbf{A}}^{-1}$ in eq.(1.12) and the right-hand side of this equation equal to the $n \times n$ identity matrix, $\mathbf{1}$, thereby obtaining two matrix equations that allow us to find the primal and the dual parts of $\hat{\mathbf{A}}^{-1}$, namely,

$$\mathbf{A}\mathbf{B} = \mathbf{1}, \quad \mathbf{A}\mathbf{B}_0 + \mathbf{A}_0\mathbf{B} = \mathbf{O} \,,$$

whence

$$\mathbf{B} = \mathbf{A}^{-1}, \quad \mathbf{B}_0 = -\mathbf{A}^{-1}\mathbf{A}_0\mathbf{A}^{-1} \,,$$

which are defined because $\mathbf{A}$ is invertible by hypothesis, and hence, for any nonsingular dual matrix $\hat{\mathbf{A}}$,

$$\hat{\mathbf{A}}^{-1} = \mathbf{A}^{-1} - \epsilon\, \mathbf{A}^{-1}\mathbf{A}_0\mathbf{A}^{-1} \,. \tag{1.13}$$

Note the striking similarity of the dual part of the foregoing expression with the time-derivative of the inverse of $\mathbf{A}(t)$, namely,

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{A}^{-1}(t)] = -\mathbf{A}^{-1}(t)\dot{\mathbf{A}}(t)\mathbf{A}^{-1}(t) \,.$$

In order to find an expression for the determinant of an $n \times n$ dual matrix, we need to recall the general expression for the dual function defined in eq.(1.3). However, that expression has to be adapted to a dual-matrix argument, which leads to

$$f(\hat{\mathbf{A}}) = f(\mathbf{A}) + \epsilon \operatorname{tr} \left[\mathbf{A}_0 \left(\frac{\mathrm{d}f}{\mathrm{d}\hat{\mathbf{A}}} \right)^T \right] \Bigg|_{\hat{\mathbf{A}}=\mathbf{A}} . \tag{1.14}$$

In particular, when $f(\hat{\mathbf{A}}) = \det(\hat{\mathbf{A}})$, we have, recalling the formula for the derivative of the determinant with respect to its matrix argument (Angeles, 1982), for any $n \times n$ matrix $\mathbf{X}$,

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{X}}[\det(\mathbf{X})] = \det(\mathbf{X})\mathbf{X}^{-T} ,$$

where $\mathbf{X}^{-T}$ denotes the transpose of the inverse of $\mathbf{X}$ or, equivalently, the transpose of $\mathbf{X}^{-1}$. Therefore,

$$\operatorname{tr} \left[\mathbf{A}_0 \left(\frac{\mathrm{d}f}{\mathrm{d}\hat{\mathbf{A}}} \right)^T \right] \Bigg|_{\hat{\mathbf{A}}=\mathbf{A}} = \det(\mathbf{A}) \operatorname{tr}(\mathbf{A}_0 \mathbf{A}^{-1}) ,$$

and hence,

$$\det(\hat{\mathbf{A}}) = \det(\mathbf{A})[1 + \epsilon \operatorname{tr}(\mathbf{A}_0 \mathbf{A}^{-1})] . \tag{1.15}$$

Now we can define the eigenvalue problem for the dual matrix $\hat{\mathbf{A}}$ defined above. Let $\hat{\lambda}$ and $\hat{\mathbf{e}}$ be a dual eigenvalue and a dual (unit) eigenvector of $\hat{\mathbf{A}}$, respectively. Then,

$$\hat{\mathbf{A}}\hat{\mathbf{e}} = \hat{\lambda}\hat{\mathbf{e}}, \quad \|\hat{\mathbf{e}}\| = 1 . \tag{1.16a}$$

For the foregoing linear homogeneous equation to admit a nontrivial solution, we must have

$$\det(\hat{\lambda}\mathbf{1} - \hat{\mathbf{A}}) = 0 , \tag{1.16b}$$

which yields an nth-order dual polynomial in the dual number $\hat{\lambda}$. Its n dual roots, real and complex, constitute the n *dual eigenvalues* of $\hat{\mathbf{A}}$. Note that, associated with each dual eigenvalue $\hat{\lambda}_i$, a corresponding *dual (unit) eigenvector* $\hat{\mathbf{e}}_i^*$ is defined, for $i = 1, 2, \ldots, n$. Moreover, if we recall eq.(1.4), we can write

$$\mathrm{e}^{\hat{\mathbf{A}}} = \mathrm{e}^{\mathbf{A}} + \epsilon \, \mathbf{A}_0 \mathrm{e}^{\mathbf{A}} . \tag{1.17}$$

Upon expansion, the foregoing expression can be cast in the form

$$\mathrm{e}^{\hat{\mathbf{A}}} = (\mathbf{1} + \epsilon \, \mathbf{A}_0)\mathrm{e}^{\mathbf{A}} \neq \mathrm{e}^{\mathbf{A}}(\mathbf{1} + \epsilon \, \mathbf{A}_0) , \tag{1.18}$$

the inequality arising because, in general, $\mathbf{A}$ and $\mathbf{A}_0$ do not commute. They do so only in the case in which they share the same set of eigenvectors. A special case in which the two matrices share the same set of eigenvectors is when one matrix is an *analytic function* of the other. More formally, we have

Lemma 1.2.2 *If* $\mathbf{F}$ *is an analytic matrix function of matrix* $\mathbf{A}$*, then the two matrices*

(i) share the same set of eigenvectors, and

(ii) commute under multiplication.

Typical examples of analytic matrix functions are $\mathbf{F} = \mathbf{A}^N$ and $\mathbf{F} = \mathrm{e}^{\mathbf{A}}$, for an integer N.

1.3 Fundamentals of Rigid-Body Kinematics

We review in this section some basic facts from rigid-body kinematics. For the sake of conciseness, some proofs are not given, but the pertinent references are cited whenever necessary.

1.3.1 Finite Displacements

A rigid body is understood as a particular case of the continuum with the special feature that, under any given motion, *any* two points of the rigid body remain equidistant. A rigid body is available through a *configuration* or *pose* that will be denoted by $\mathcal{B}$. Whenever a *reference configuration* is needed, this will be labelled $\mathcal{B}^0$. Moreover, the position vector of a point P of the body in configuration $\mathcal{B}$ will be denoted by $\mathbf{p}$, that in $\mathcal{B}^0$ being denoted correspondingly by $\mathbf{p}^0$.

A rigid-body motion leaving a point O of the body fixed is called a *pure rotation*, and is represented by a *proper orthogonal matrix* $\mathbf{Q}$, i.e., $\mathbf{Q}$ verifies the two properties below:

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{1}, \quad \det(\mathbf{Q}) = +1 \ . \tag{1.19}$$

According to Euler's Theorem (Euler, 1775), a pure rotation leaves a set of points of the body immutable, this set lying on a line $\mathcal{L}$, which is termed the *axis of rotation*. If we draw the perpendicular from an arbitrary point P of the body to $\mathcal{L}$ and denote its intersection with $\mathcal{L}$ by P', the angle ϕ between $P'P^0$ and $P'P$, where, according to our convention, P^0 denotes the point P in the reference configuration $\mathcal{B}^0$ of the body, is called the *angle of rotation*. Note that a direction must be specified along this line to define the sign of the angle. Furthermore, the direction of the line is specified by the unit vector $\mathbf{e}$. We term $\mathbf{e}$ and ϕ the *natural invariants* of $\mathbf{Q}$.

As a result of Euler's Theorem, the rotation $\mathbf{Q}$ can be represented in terms of its natural invariants. This representation takes the form

$$\mathbf{Q} = \mathbf{e}\mathbf{e}^T + \cos\phi(\mathbf{1} - \mathbf{e}\mathbf{e}^T) + \sin\phi\mathbf{E} \ , \tag{1.20}$$

where $\mathbf{E}$ denotes the *cross-product matrix* of $\mathbf{e}$, i.e., for any 3-dimensional vector $\mathbf{v}$,

$$\mathbf{e} \times \mathbf{v} = \mathbf{E}\mathbf{v} \ .$$

As a result of the foregoing definition, $\mathbf{E}$ is skew-symmetric, i.e., $\mathbf{E} = -\mathbf{E}^T$ and, moreover, it has the properties below:

$$\mathbf{E}^{2k+1} = (-1)^k \mathbf{E}, \quad \mathbf{E}^{2k} = (-1)^k (\mathbf{1} - \mathbf{e}\mathbf{e}^T), \quad \text{for} \quad k = 1, 2, \ldots$$

By virtue of the foregoing properties of the cross-product matrix $\mathbf{E}$ of $\mathbf{e}$, the rotation matrix $\mathbf{Q}$ can be written in the alternative form

$$\mathbf{Q} = \mathbf{1} + \sin\phi \mathbf{E} + (1 - \cos\phi)\mathbf{E}^2 \ . \tag{1.21}$$

Now, if we recall the *Cayley-Hamilton Theorem* (Halmos, 1974), we can realize that the right-hand side of the foregoing equation is nothing but the exponential of $\phi\mathbf{E}$, i.e.,

$$\mathbf{Q} = \mathrm{e}^{\phi \mathbf{E}} \ , \tag{1.22}$$

which is the *exponential form of the rotation matrix.* Now it is a simple matter to obtain the eigenvalues of the rotation matrix if we first notice that one eigenvalue of $\mathbf{E}$ is 0, the other eigenvalues being readily derived as $\pm\sqrt{-1}$, where $\sqrt{-1}$ is the imaginary unit, i.e., $\sqrt{-1} \equiv \sqrt{-1}$. Therefore, if $\mathbf{Q}$ is the exponential of $\phi\mathbf{E}$, then the eigenvalues of $\mathbf{Q}$ are the exponentials of the eigenvalues of $\phi\mathbf{E}$:

$$\lambda_1 = \mathrm{e}^0 = 1, \quad \lambda_{2,3} = \mathrm{e}^{\pm\sqrt{-1}\phi} = \cos\phi \pm \sqrt{-1}\sin\phi \ . \tag{1.23}$$

Moreover, we recall below the *Cartesian decomposition* of an $n \times n$ matrix $\mathbf{A}$, namely,

$$\mathbf{A} = \mathbf{A}_s + \mathbf{A}_{ss} \ , \tag{1.24a}$$

where $\mathbf{A}_s$ is symmetric and $\mathbf{A}_{ss}$ is skew-symmetric. These matrices are given by

$$\mathbf{A}_s \equiv \frac{1}{2}(\mathbf{A} + \mathbf{A}^T), \quad \mathbf{A}_{ss} \equiv \frac{1}{2}(\mathbf{A} - \mathbf{A}^T) \ . \tag{1.24b}$$

Any 3×3 skew-symmmetric matrix is fully defined by three scalars, which means that such a matrix can then be made isomorphic to a 3-dimensional vector. Indeed, let $\mathbf{S}$ be a 3×3 skew-symmetric matrix and $\mathbf{v}$ be an arbitrary 3-dimensional vector. Then, we have

$$\mathbf{S}\mathbf{v} \equiv \mathbf{s} \times \mathbf{v} \ . \tag{1.25}$$

When the above items are expressed in a given coordinate frame $\mathcal{F}$, the components of $\mathbf{S}$, indicated as $\{ s_{i,j} \}_{i,j=1}^3$, and of $\mathbf{s}$, indicated as $\{ s_i \}_1^3$, bear the relations below:

$$\mathbf{S} = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}, \quad \mathbf{s} = \frac{1}{2}\begin{bmatrix} s_{32} - s_{23} \\ s_{13} - s_{31} \\ s_{21} - s_{12} \end{bmatrix} \ . \tag{1.26}$$

In general, we define the *axial vector* of an arbitrary 3×3 matrix $\mathbf{A}$ in terms of the difference of its off-diagonal entries, as appearing in eq.(1.26) for the entries of matrix $\mathbf{S}$.

Apparently, the axial vector of any 3×3 matrix is identical to that of its skew-symmetric component; this vector, represented as $\mathbf{a} \equiv \text{vect}(\mathbf{A})$, is the *vector linear invariant* of $\mathbf{A}$. The *scalar linear invariant* of the same matrix is its trace, $\text{tr}(\mathbf{A})$. With this notation, note that

$$\frac{1}{2}(\mathbf{A} - \mathbf{A}^T)\mathbf{v} = \mathbf{a} \times \mathbf{v} .$$

Further, with reference to Fig. 1.3.1, let A and P be two points of a rigid body, which is shown in its reference and its current configurations.

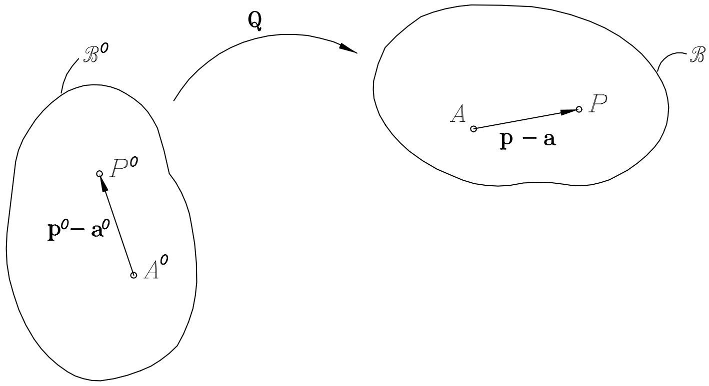

Figure 1.1: Displacements of two points of a rigid body in two finitely-separated configurations

We can regard vector $\mathbf{p} - \mathbf{a}$ as the image of $\mathbf{p}^0 - \mathbf{a}^0$ under the rotation $\mathbf{Q}$, namely,

$$\mathbf{p} - \mathbf{a} = \mathbf{Q}(\mathbf{p}^0 - \mathbf{a}^0) , \tag{1.27}$$

whence an expression for $\mathbf{p}$ can be derived as

$$\mathbf{p} = \mathbf{a} + \mathbf{Q}(\mathbf{p}^0 - \mathbf{a}^0) . \tag{1.28}$$

Furthermore, the *displacement* $\mathbf{d}_A$ of A is defined as the difference $\mathbf{a} - \mathbf{a}^0$, with a similar definition for the displacement $\mathbf{d}_P$ of P. From the above equation, it is now apparent that a linear relation between the two displacements follows:

$$\mathbf{d}_P = \mathbf{d}_A + (\mathbf{Q} - \mathbf{1})(\mathbf{p}^0 - \mathbf{a}^0) . \tag{1.29}$$

Therefore,

Theorem 1.3.1 *The displacements of all the points of a rigid body have identical projections onto the axis of the concomitant rotation.*

The proof of the foregoing result follows upon dot-multiplying both sides of eq.(1.29) by $\mathbf{e}$:

$$\mathbf{e} \cdot \mathbf{d}_P = \mathbf{e} \cdot \mathbf{d}_A .$$

From the previous result it is apparent that $\|\mathbf{d}_P\|$ can attain infinitely large values, depending on $\|\mathbf{p}^0 - \mathbf{a}^0\|$, but, in general, $\mathbf{d}_P$ does not vanish. Hence, a minimum of $\|\mathbf{d}_P\|$ can be found, a result summarized in the Mozzi-Chasles Theorem (Mozzi, 1763; Chasles, 1830). This theorem states that the points of $\mathcal{B}$ of minimum-norm displacement lie in a line $\mathcal{M}$ that is parallel to the axis of the rotation represented by matrix $\mathbf{Q}$, the minimum-norm displacement being a vector parallel to the same axis. If we recall that $\mathbf{e}$ and ϕ denote the natural invariants of $\mathbf{Q}$, then the position vector $\mathbf{p}^*$ of the point P^* of $\mathcal{M}$ lying closest to the origin O is given by Angeles (2002)

$$\mathbf{p}^* = \frac{(\mathbf{Q} - \mathbf{1})^T(\mathbf{Q}\mathbf{a}^0 - \mathbf{a})}{2(1 - \cos\phi)}, \quad \text{for} \quad \phi \neq 0 \; , \tag{1.30}$$

the special case in which $\phi = 0$ corresponding to a *pure translation*, whereby all points of $\mathcal{B}$ undergo identical displacements. In this case, then, the axis $\mathcal{M}$ is indeterminate, because all points of the body can be thought of as undergoing minimum-norm displacements. Henceforth, line $\mathcal{M}$ will be termed the *Mozzi-Chasles axis*. Note that the Plücker coordinates of the Mozzi-Chasles axis are given by $\mathbf{e}$ and $\mathbf{e}_0 \equiv \mathbf{p}^* \times \mathbf{e}$. We shall denote with $\mathbf{d}^*$ the minimum-norm displacement, which can be represented in the form

$$\mathbf{d}^* = d^*\mathbf{e}, \quad d^* = \mathbf{d}_P \cdot \mathbf{e} \; . \tag{1.31}$$

Therefore, the body under study can be regarded as undergoing, from $\mathcal{B}^0$ to $\mathcal{B}$, a screw motion, as if the body were rigidly attached to the bolt of a screw of axis $\mathcal{M}$ and *pitch* p given by

$$p = \frac{d^*}{\phi} = \frac{\mathbf{e} \cdot \mathbf{d}_P}{\phi} \; . \tag{1.32}$$

We list below further results:

Lemma 1.3.1 *Let A and P be two points of a rigid body undergoing a general motion from a reference pose $\mathcal{B}^0$ to a current pose $\mathcal{B}$. Then, under the notation adopted above, the difference $\mathbf{p} - \mathbf{Q}\mathbf{p}^0$ remains constant and is denoted by $\mathbf{d}$, i.e.,*

$$\mathbf{p} - \mathbf{Q}\mathbf{p}^0 = \mathbf{a} - \mathbf{Q}\mathbf{a}^0 = \mathbf{d} \; . \tag{1.33}$$

Proof: If we recall eq.(1.28) and substitute the expression therein for $\mathbf{p}$ in the difference $\mathbf{p} - \mathbf{Q}\mathbf{p}^0$, we obtain

$$\mathbf{p} - \mathbf{Q}\mathbf{p}^0 = \mathbf{a} + \mathbf{Q}(\mathbf{p}^0 - \mathbf{a}^0) - \mathbf{Q}\mathbf{p}^0 = \mathbf{a} - \mathbf{Q}\mathbf{a}^0 = \mathbf{d} \; ,$$

thereby completing the intended proof.

Note that the kinematic interpretation of $\mathbf{d}$ follows directly from eq.(1.33): $\mathbf{d}$ represents the displacement of the point of $\mathcal{B}$ that coincides with the origin in the reference pose $\mathcal{B}^0$.

The geometric interpretation of the foregoing lemma is given in Fig. 1.2. What this figure indicates is that the pose $\mathcal{B}$ can be attained from $\mathcal{B}^0$ in two stages: (a) first, the body is given a rotation $\mathbf{Q}$ about the origin O, that takes the body to the intermediate pose $\mathcal{B}'$; (b) then, from $\mathcal{B}'$, the body is given a pure translation of displacement $\mathbf{d}$ that takes the body into $\mathcal{B}$.

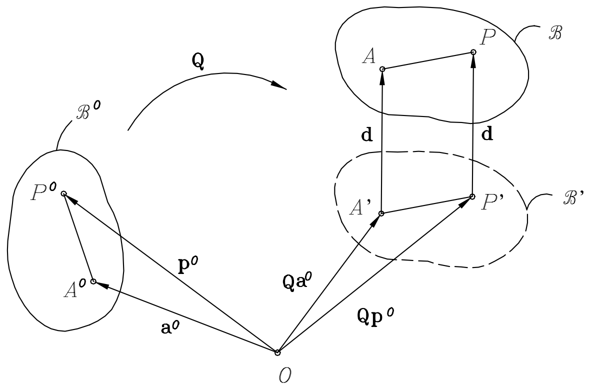

Figure 1.2: Geometric interpretation of Lemma 3.1

Therefore, eq.(1.30) for the position vector of the point of the Mozzi-Chasles axis lying closest to the origin can be expressed in terms of vector $\mathbf{d}$ as

$$\mathbf{p}^* = \frac{(\mathbf{1} - \mathbf{Q})^T\mathbf{d}}{2(1 - \cos\phi)}, \quad \text{for} \quad \phi \neq 0 \, . \tag{1.34}$$

Note that, in general, $\mathbf{d}$ is not of minimum norm. Additionally, $\mathbf{d}$ is origin-dependent, and hence, is not an invariant of the motion under study. Now, if we choose the origin on the Mozzi-Chasles axis $\mathcal{M}$, then we have the layout of Fig. 1.3, and vector $\mathbf{d}$ becomes a multiple of $\mathbf{e}$, namely, $\mathbf{d} = d^*\mathbf{e}$.

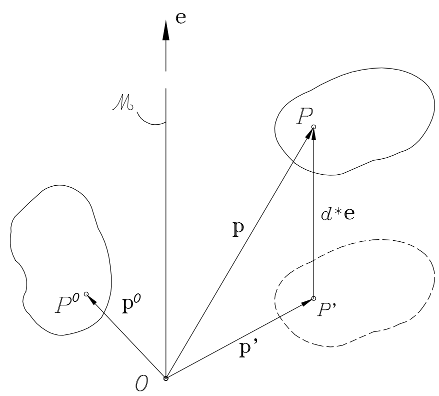

Figure 1.3: Rigid-body displacement with origin on the Mozzi-Chasles axis

We can now express the Plücker coordinates of a line $\mathcal{L}$ of a rigid body $\mathcal{B}$ in terms of those of the line in its reference configuration $\mathcal{L}^0$ (Bottema and Roth, 1978; Pradeep et al., 1989), as shown in Fig. 1.3.1. To this end, we let $\mathbf{f}$ be the unit vector parallel to $\mathcal{L}$ and P be a point of $\mathcal{L}$, and arrange the Plücker coordinates of $\mathcal{L}^0$ and $\mathcal{L}$ in the 6-dimensional arrays $\boldsymbol{\lambda}^0$ and $\boldsymbol{\lambda}$, respectively, defined as

$$\boldsymbol{\lambda}^0 \equiv \begin{bmatrix} \mathbf{f}^0 \\ \mathbf{p}^0 \times \mathbf{f}^0 \end{bmatrix}, \quad \boldsymbol{\lambda} \equiv \begin{bmatrix} \mathbf{f} \\ \mathbf{p} \times \mathbf{f} \end{bmatrix} . \tag{1.35}$$

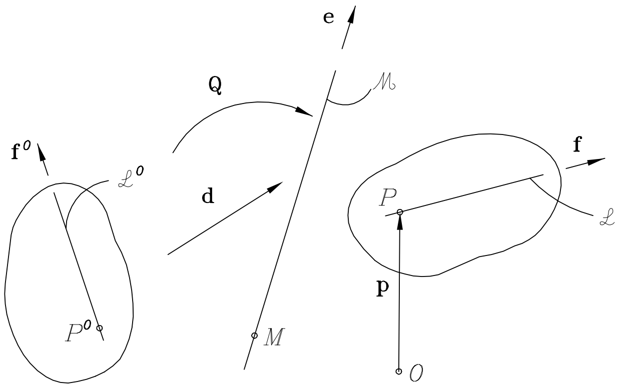

Figure 1.4: The reference and the current configurations of a body and one of its lines

We thus have

$$\mathbf{f} = \mathbf{Q}\mathbf{f}^0, \quad \mathbf{p} = \mathbf{Q}\mathbf{p}^0 + \mathbf{d} ,$$

and hence,

$$\mathbf{p} \times \mathbf{f} = (\mathbf{Q}\mathbf{p}^0 + \mathbf{d}) \times \mathbf{Q}\mathbf{f}^0 = (\mathbf{Q}\mathbf{p}^0) \times \mathbf{Q}\mathbf{f} + \mathbf{d} \times \mathbf{Q}\mathbf{f}^0 .$$

Now, the first term of the rightmost-hand side of the above equation can be simplified upon noticing that the cross product of two rotated vectors is identical to the rotated cross product. Furthermore, the second term of the same side can be expressed in terms of $\mathbf{D}$, the cross-product matrix of $\mathbf{d}$, thereby obtaining

$$\mathbf{p} \times \mathbf{f} = \mathbf{Q}(\mathbf{p}^0 \times \mathbf{f}^0) + \mathbf{D}\mathbf{Q}\mathbf{f}^0 .$$

Upon substituting the foregoing expressions for $\mathbf{f}$ and $\mathbf{p} \times \mathbf{f}$ into eq.(1.35), we obtain

$$\boldsymbol{\lambda} = \begin{bmatrix} \mathbf{Q}\mathbf{f}^0 \\ \mathbf{D}\mathbf{Q}\mathbf{f}^0 + \mathbf{Q}(\mathbf{p}^0 \times \mathbf{f}^0) \end{bmatrix} ,$$

which can be readily cast in the form of a linear transformation of $\boldsymbol{\lambda}^0$, i.e.,

$$\begin{bmatrix} \mathbf{f} \\ \mathbf{p} \times \mathbf{f} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} & \mathbf{O} \\ \mathbf{D}\mathbf{Q} & \mathbf{Q} \end{bmatrix} \begin{bmatrix} \mathbf{f}^0 \\ \mathbf{p}^0 \times \mathbf{f}^0 \end{bmatrix} , \tag{1.36a}$$

where $\mathbf{O}$ denotes the 3×3 zero matrix.

As the reader can readily verify, the inverse relation of eq.(1.36a) takes the form

$$\begin{bmatrix} \mathbf{f}^0 \\ \mathbf{p}^0 \times \mathbf{f}^0 \end{bmatrix} = \begin{bmatrix} \mathbf{Q}^T & \mathbf{O} \\ -\mathbf{Q}^T\mathbf{D} & \mathbf{Q}^T \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \mathbf{p} \times \mathbf{f} \end{bmatrix} . \tag{1.36b}$$

By inspection of eq.(1.36a), and recalling the dual-unit-vector representation of a line, as given in eq.(1.10), we can realize that the dual unit vector of $\mathcal{L}$ can be expressed as the image of the dual unit vector of $\mathcal{L}^0$ upon a linear transformation given by a dual matrix $\hat{\mathbf{Q}}$. Moreover, the dual matrix of interest can be readily derived from the real matrix of eq.(1.36a). Indeed, it can be realized from Section 2 that the difference between the primal and the dual parts of a dual quantity is that the units of the dual part are those of the primal part times units of length. Hence, the primal part of the dual matrix sought is bound to be $\mathbf{Q}$, which is dimensionless, the corresponding dual part being $\mathbf{DQ}$, which has units of length. A plausible form of the matrix sought is, then,

$$\hat{\mathbf{Q}} = \mathbf{Q} + \epsilon\,\mathbf{DQ}\,. \tag{1.37}$$

The correctness of the above expression can be readily realized. Indeed, let $\hat{\mathbf{f}}^* = \mathbf{f} + \epsilon\,\mathbf{p} \times \mathbf{f}$ and $\hat{\mathbf{f}}^{0*} = \mathbf{f}^0 + \epsilon\,\mathbf{p}^0 \times \mathbf{f}^0$ be the dual unit vectors of $\mathcal{L}$ and $\mathcal{L}^0$, respectively. Then upon performing the product $\hat{\mathbf{Q}}\hat{\mathbf{f}}^{0*}$, we note that the product is rightfully $\hat{\mathbf{f}}^*$, i.e., $\hat{\mathbf{f}}^* = \hat{\mathbf{Q}}\hat{\mathbf{f}}^{0*}$. In the derivations below, we will need expressions for the vector and scalar linear invariants of the product of two matrices, one of which is skew-symmetric. These expressions are derived in detail in (Angeles, 2002). For quick reference, we recall these relations below:

Theorem 1.3.2 *Let both* $\mathbf{A}$ *and* $\mathbf{S}$ *be* 3×3 *matrices, the former arbitrary, the latter skew-symmetric. Then,*

$$\operatorname{vect}(\,\mathbf{SA}\,) = \frac{1}{2}[\operatorname{tr}(\mathbf{A})\mathbf{1} - \mathbf{A}]\mathbf{s}\,, \tag{1.38}$$

where $\mathbf{s} \equiv \operatorname{vect}(\,\mathbf{S}\,)$.

Now, as a direct consequence of the above result, we have

Corollary 1.3.1 *If* $\mathbf{A}$ *in Theorem 1.3.2 is skew-symmetric, then the axial vector of the product* $\mathbf{SA}$ *reduces to*

$$\operatorname{vect}(\,\mathbf{SA}\,) = -\frac{1}{2}\mathbf{A}\mathbf{s} = -\frac{1}{2}\mathbf{a} \times \mathbf{s}\,, \tag{1.39}$$

where $\mathbf{a} \equiv \operatorname{vect}(\mathbf{A})$.

Moreover,

Theorem 1.3.3 *Let* $\mathbf{A}$, $\mathbf{S}$, *and* $\mathbf{s}$ *be defined as in Theorem 1.3.2. Then,*

$$\operatorname{tr}(\,\mathbf{SA}\,) = -2\mathbf{s} \cdot [\operatorname{vect}(\,\mathbf{A}\,)]\,. \tag{1.40}$$

Furthermore, we prove now that $\hat{\mathbf{Q}}$ is proper orthogonal. Indeed, orthogonality can be proven by performing the product $\hat{\mathbf{Q}}\hat{\mathbf{Q}}^T$ and noticing that this product yields the 3×3

identity matrix, i.e., $\hat{\mathbf{Q}}\hat{\mathbf{Q}}^T = \mathbf{1}$. Proper orthogonality is proven, in turn, upon application of formula (1.15) to matrix $\hat{\mathbf{Q}}$, as given by eq.(1.37), namely,

$$\det(\hat{\mathbf{Q}}) = \det(\mathbf{Q})[1 + \epsilon\,\mathrm{tr}(\mathbf{D}\mathbf{Q}\mathbf{Q}^{-1})] = \det(\mathbf{Q})[1 + \epsilon\,\mathrm{tr}(\mathbf{D})] = 1\ ,$$

thus completing the proof.

The exponential form of the dual rotation matrix can be obtained if we note that the exponential of a pure dual number $\hat{x} = \epsilon x_0$ reduces to

$$\mathrm{e}^{\epsilon x_0} = 1 + \epsilon x_0\ . \tag{1.41}$$

On the other hand, we can write

$$\hat{\mathbf{Q}} = (\mathbf{1} + \epsilon\mathbf{D})\mathbf{Q}\ . \tag{1.42}$$

In analogy with eq.(1.41), the foregoing expression takes the form

$$\hat{\mathbf{Q}} = \mathrm{e}^{\epsilon\mathbf{D}}\mathbf{Q}\ .$$

Furthermore, if we recall the exponential form of $\mathbf{Q}$, as given in eq.(1.22), the foregoing expression simplifies to

$$\hat{\mathbf{Q}} = \mathrm{e}^{\epsilon\mathbf{D}}\mathrm{e}^{\phi\mathbf{E}}\ . \tag{1.43}$$

However, since $\mathbf{D}$ and $\mathbf{E}$ are unrelated, they do not share the same set of eigenvectors, and hence, they do not commute under multiplication, the foregoing expression thus not being further reducible to one single exponential. Nevertheless, if the origin is placed on the Mozzi-Chasles axis, as depicted in Fig. 1.3, then the dual rotation matrix becomes

$$\hat{\mathbf{Q}} = \mathbf{Q} + \epsilon\, d^*\mathbf{E}\mathbf{Q}\ , \tag{1.44}$$

where $d^*\mathbf{E}$ is, apparently, the cross-product matrix of vector $d^*\mathbf{e}$. Furthermore, the exponential form of the dual rotation matrix, eq.(1.43), then simplifies to $\hat{\mathbf{Q}} = \mathrm{e}^{(\phi+\epsilon\, d^*)\mathbf{E}}$ or, if we let $\hat{\phi} = \phi + \epsilon\, d^*$, then we can write $\hat{\mathbf{Q}} = \mathrm{e}^{\hat{\phi}\mathbf{E}}$.

1.3.2 Velocity Analysis

Upon differentiation with respect to time of both sides of eq.(1.27), we obtain

$$\dot{\mathbf{p}} - \dot{\mathbf{a}} = \dot{\mathbf{Q}}(\mathbf{p}^0 - \mathbf{a}^0)\ ,$$

and, if we solve for $(\mathbf{p}^0 - \mathbf{a}^0)$ from the equation mentioned above, we obtain

$$\dot{\mathbf{p}} - \dot{\mathbf{a}} = \dot{\mathbf{Q}}\mathbf{Q}^T(\mathbf{p} - \mathbf{a})\ , \tag{1.45}$$

where $\dot{\mathbf{Q}}\mathbf{Q}^T$ is defined as the angular-velocity matrix of the motion under study, and is represented as $\mathbf{\Omega}$, namely,

$$\mathbf{\Omega} \equiv \dot{\mathbf{Q}}\mathbf{Q}^T\ . \tag{1.46a}$$

It can be readily proven that the foregoing matrix is skew-symmetric, i.e.,

$$\mathbf{\Omega}^T = -\mathbf{\Omega} \; . \tag{1.46b}$$

Moreover, the axial vector of $\mathbf{\Omega}$ is the angular-velocity vector $\boldsymbol{\omega}$:

$$\boldsymbol{\omega} = \mathrm{vect}(\mathbf{\Omega}) \; . \tag{1.46c}$$

We can now write eq.(1.45) in the form

$$\dot{\mathbf{p}} = \dot{\mathbf{a}} + \mathbf{\Omega}(\mathbf{p} - \mathbf{a}) = \dot{\mathbf{a}} + \boldsymbol{\omega} \times (\mathbf{p} - \mathbf{a}) \; , \tag{1.47}$$

whence,

$$\dot{\mathbf{p}} - \boldsymbol{\omega} \times \mathbf{p} = \dot{\mathbf{a}} - \boldsymbol{\omega} \times \mathbf{a} \equiv \mathbf{v}^0 = \mathrm{const} \; . \tag{1.48}$$

Therefore, the difference $\dot{\mathbf{p}} - \boldsymbol{\omega} \times \mathbf{p}$ is the same for all points of a rigid body. The kinematic interpretation of this quantity is straightforward: If we rewrite $\mathbf{v}^0$ in the form $\mathbf{v}^0 = \dot{\mathbf{p}} + \boldsymbol{\omega} \times (-\mathbf{p})$, then we can readily realize that, $-\mathbf{p}$ being the vector directed from point P of the rigid body to the origin O, $\mathbf{v}^0$ represents the velocity of the point of the body that coincides instantaneously with the origin. Furthermore, we express $\mathbf{d}$, as given by eq.(1.33), in terms of the position vector of an arbitrary point P, $\mathbf{p}$, thus obtaining

$$\mathbf{d} = \mathbf{p} - \mathbf{Q}\mathbf{p}^0 \; . \tag{1.49}$$

Upon differentiation of the two sides of the above expression with respect to time, we obtain

$$\dot{\mathbf{d}} = \dot{\mathbf{p}} - \dot{\mathbf{Q}}\mathbf{p}^0 \; ,$$

which can be readily expressed in terms of the current value of the position vector of P, by solving for $\mathbf{p}^0$ from eq.(1.49), namely,

$$\dot{\mathbf{d}} = \dot{\mathbf{p}} - \mathbf{\Omega}(\mathbf{p} - \mathbf{d}) \quad \text{or} \quad \dot{\mathbf{d}} - \boldsymbol{\omega} \times \mathbf{d} = \dot{\mathbf{p}} - \boldsymbol{\omega} \times \mathbf{p} \; , \tag{1.50}$$

and hence, the difference $\dot{\mathbf{d}} - \boldsymbol{\omega} \times \mathbf{d}$ is identical to the difference $\dot{\mathbf{p}} - \boldsymbol{\omega} \times \mathbf{p}$, i.e.,

$$\dot{\mathbf{d}} - \boldsymbol{\omega} \times \mathbf{d} = \mathbf{v}^0 \; . \tag{1.51}$$

Furthermore, upon dot-multiplying the two sides of eq.(1.48) by $\boldsymbol{\omega}$, we obtain an interesting result, namely,

$$\boldsymbol{\omega} \cdot \dot{\mathbf{p}} = \boldsymbol{\omega} \cdot \dot{\mathbf{a}} \; , \tag{1.52}$$

and hence,

Theorem 1.3.4 *The velocities of all points of a rigid body have the same projection onto the angular-velocity vector of the motion under study.*

Similar to the Mozzi-Chasles Theorem, we have now

Theorem 1.3.5 *Given a rigid body $\mathcal{B}$ under general motion, a set of its points, on a line $\mathcal{L}$, undergoes the identical minimum-magnitude velocity* $\mathbf{v}^*$ *parallel to the angular velocity* $\boldsymbol{\omega}$.

The Plücker coordinates of line $\mathcal{L}$, grouped in the 6-dimensional array $\boldsymbol{\lambda}$, are given as

$$\boldsymbol{\lambda} \equiv \begin{bmatrix} \mathbf{f} \\ \boldsymbol{\pi} \times \mathbf{f} \end{bmatrix}, \quad \mathbf{f} \equiv \frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|}, \quad \boldsymbol{\pi} \equiv \frac{\boldsymbol{\omega} \times \mathbf{v}^0}{\|\boldsymbol{\omega}\|^2}, \tag{1.53}$$

where $\mathbf{v}^0$ was previously introduced as the velocity of the point of $\mathcal{B}$ that coincides instantaneously with the origin. Line $\mathcal{L}$ is termed the *instant screw axis*–ISA, for brevity.

Thus, the instantaneous motion of $\mathcal{B}$ is defined by a screw of axis $\mathcal{L}$ and pitch p', given by

$$p' = \frac{\dot{\mathbf{p}} \cdot \boldsymbol{\omega}}{\|\boldsymbol{\omega}\|^2}, \tag{1.54}$$

where $\dot{\mathbf{p}}$ is the velocity of an arbitrary point P of $\mathcal{B}$, the product $\dot{\mathbf{p}} \cdot \boldsymbol{\omega}$ being constant by virtue of Theorem 1.3.4. A proof of the foregoing results is available in (Angeles, 2002).

1.3.3 The Linear Invariants of the Dual Rotation Matrix

We start by recalling the *linear invariants* of the real rotation matrix Angeles (2002). These are defined as

$$\mathbf{q} \equiv \operatorname{vect}(\mathbf{Q}) = (\sin\phi)\mathbf{e}, \quad q_0 \equiv \frac{\operatorname{tr}(\mathbf{Q}) - 1}{2} = \cos\phi\,. \tag{1.55a}$$

Note that the linear invariants of any 3×3 matrix can be obtained from simple differences of its off-diagonal entries and sums of its diagonal entries. Once the foregoing linear invariants are calculated, the natural invariants can be obtained uniquely as indicated below: First, note that the sign of $\mathbf{e}$ can be changed without affecting $\mathbf{q}$ if the sign of ϕ is changed accordingly, which means that the sign of ϕ–or that of $\mathbf{e}$, for that matter–is undefined. In order to define this sign uniquely, we will adopt a positive sign for $\sin\phi$, which means that ϕ is assumed, henceforth, to lie in the interval $0 \le \phi \le \pi$.

We can thus obtain the inverse relations of eq.(1.55a) in the form

$$\mathbf{e} = \frac{\mathbf{q}}{\|\mathbf{q}\|}, \quad \phi = \arctan\left(\frac{\|\mathbf{q}\|}{q_0}\right), \quad \mathbf{q} \neq \mathbf{0}\,, \tag{1.55b}$$

the case $\mathbf{q} = \mathbf{0}$ being handled separately. Indeed, $\mathbf{q}$ vanishes under two cases: (a) $\phi = 0$, in which case the body undergoes a pure translation and the axis of rotation is obviously undefined; and (b) $\phi = \pi$, in which case $\mathbf{Q}$ is symmetric and takes the form

$$\text{For} \quad \phi = \pi: \quad \mathbf{Q} = -\mathbf{1} + 2\mathbf{e}\mathbf{e}^T\,, \tag{1.55c}$$

whence the natural invariants become apparent and can be readily extracted from $\mathbf{Q}$.

Similar to the linear invariants of the real rotation matrix, in the dual case we have

$$\hat{\mathbf{q}} \equiv \mathrm{vect}(\hat{\mathbf{Q}}), \quad \hat{q}_0 \equiv \frac{\mathrm{tr}(\hat{\mathbf{Q}}) - 1}{2} . \tag{1.56}$$

Expressions for the foregoing quantities in terms of the motion parameters are derived below; in the sequel, we also derive expressions for the *dual natural invariants* in terms of the same parameters. We start by expanding the vector linear invariant:

$$\mathrm{vect}(\hat{\mathbf{Q}}) = \mathrm{vect}(\mathbf{Q} + \epsilon\, \mathbf{DQ}) = \mathrm{vect}(\mathbf{Q}) + \epsilon\, \mathrm{vect}(\mathbf{DQ}) . \tag{1.57a}$$

But, by virtue of eq.(1.20),

$$\mathrm{vect}(\mathbf{Q}) = (\sin\phi)\mathbf{e} . \tag{1.57b}$$

Furthermore, the second term of the rightmost-hand side of eq.(1.57a) can be readily calculated if we recall Theorem 1.3.2, with $\mathbf{d} \equiv \mathrm{vect}(\mathbf{D})$:

$$\mathrm{vect}(\mathbf{DQ}) = \frac{1}{2}[\mathrm{tr}(\mathbf{Q})\mathbf{1} - \mathbf{Q}]\mathbf{d} . \tag{1.57c}$$

Now, if we recall expression (1.20), we obtain

$$\mathrm{tr}(\mathbf{Q})\mathbf{1} - \mathbf{Q} = (1 + \cos\phi)\mathbf{1} - \sin\phi\mathbf{E} - (1 - \cos\phi)\mathbf{e}\mathbf{e}^T .$$

Upon substitution of the foregoing expression into eq.(1.57c), the desired expression for $\mathrm{vect}(\mathbf{DQ})$ is readily derived, namely,

$$\mathrm{vect}(\mathbf{DQ}) = \frac{1}{2}[(1 + \cos\phi)\mathbf{d} - \sin\phi\mathbf{e} \times \mathbf{d} - (1 - \cos\phi)(\mathbf{e} \cdot \mathbf{d})\mathbf{e}] , \tag{1.57d}$$

and hence,

$$\hat{\mathbf{q}} = (\sin\phi)\mathbf{e} + \epsilon\frac{1}{2}[(\cos\phi)(\mathbf{e} \cdot \mathbf{d})\mathbf{e} + (1 + \cos\phi)\mathbf{d} + (\sin\phi)\mathbf{d} \times \mathbf{e} \quad - (\mathbf{e} \cdot \mathbf{d})\mathbf{e}]. \tag{1.57e}$$

On the other hand, the position vector $\mathbf{p}^*$ of the Mozzi-Chasles axis, given by eq.(1.34), can be expressed as

$$\mathbf{p}^* = \frac{1}{2}\frac{\sin\phi}{1 - \cos\phi}\mathbf{e} \times \mathbf{d} + \frac{1}{2}\mathbf{d} - \frac{1}{2}(\mathbf{e} \cdot \mathbf{d})\mathbf{e} , \tag{1.58a}$$

and hence,

$$\mathbf{p}^* \times \mathbf{e} = \frac{1}{2}\frac{\sin\phi}{1 - \cos\phi}\mathbf{d} - \frac{1}{2}\frac{\sin\phi}{1 - \cos\phi}(\mathbf{e} \cdot \mathbf{d})\mathbf{e} + \frac{1}{2}\mathbf{d} \times \mathbf{e} . \tag{1.58b}$$

Moreover, let us recall the identity

$$\frac{1 + \cos\phi}{\sin\phi} = \frac{\sin\phi}{1 - \cos\phi} , \tag{1.58c}$$

which allows us to rewrite eq.(1.58b) in the form

$$\mathbf{p}^* \times \mathbf{e} = \frac{1}{2}\frac{1+\cos\phi}{\sin\phi}\mathbf{d} - \frac{1}{2}\frac{1+\cos\phi}{\sin\phi}(\mathbf{e}\cdot\mathbf{d})\mathbf{e} + \frac{1}{2}\mathbf{d}\times\mathbf{e} \ , \tag{1.58d}$$

whence,

$$(\sin\phi)\mathbf{p}^* \times \mathbf{e} = \frac{1}{2}[(1+\cos\phi)\mathbf{d} - (1+\cos\phi)(\mathbf{e}\cdot\mathbf{d})\mathbf{e} + (\sin\phi)\mathbf{d}\times\mathbf{e}] \ ,$$

and $\hat{\mathbf{q}}$ takes the form

$$\hat{\mathbf{q}} = (\sin\phi)\mathbf{e} + \epsilon\,[(\cos\phi)(\mathbf{e}\cdot\mathbf{d})\mathbf{e} + (\sin\phi)\mathbf{p}^* \times \mathbf{e}] \ . \tag{1.59}$$

If we now recall eqs.(1.31) and (1.32), $\mathbf{d}\cdot\mathbf{e} \equiv d^* = p\phi$, while $\mathbf{p}^* \times \mathbf{e}$ is the moment of the associated Mozzi-Chasles axis, $\mathbf{e}_0$, and hence, eq.(1.59) becomes

$$\hat{\mathbf{q}} = (\sin\phi)\mathbf{e} + \epsilon\,[(\cos\phi)p\phi\mathbf{e} + (\sin\phi)\mathbf{e}_0] \ , \tag{1.60}$$

and hence, $\hat{\mathbf{q}}$ can be further simplified to

$$\hat{\mathbf{q}} = \hat{\mathbf{e}}^* \sin\hat{\phi}, \quad \hat{\phi} \equiv \phi 1 + \epsilon\, p) \ , \tag{1.61}$$

where $\hat{\mathbf{e}}^*$ is the dual unit vector representing the Mozzi-Chasles axis, i.e., $\hat{\mathbf{e}}^* = \mathbf{e} + \epsilon\,\mathbf{e}_0$.

Now, such as in the real case, we can calculate the dual natural invariants of the motion under study in terms of the foregoing dual linear invariants. We do this by mimicking eqs.(1.55b), namely,

$$\hat{\mathbf{e}}^* = \frac{\hat{\mathbf{q}}}{\|\hat{\mathbf{q}}\|}, \quad \hat{\phi} = \arctan\left(\frac{\|\hat{\mathbf{q}}\|}{\hat{q}_0}\right), \quad \|\hat{\mathbf{q}}\| \neq 0 \ , \tag{1.62}$$

where $\|\hat{\mathbf{q}}\|$ is calculated from eq.(1.9e), which gives $\|\hat{\mathbf{q}}\|^2$, the square root of the latter then following from eq.(1.7), thus obtaining

$$\|\hat{\mathbf{q}}\| = \sin\hat{\phi} = \sin\phi + \epsilon\,(\mathbf{e}\cdot\mathbf{d})\cos\phi \ , \tag{1.63}$$

and hence, upon simplification,

$$\hat{\mathbf{e}}^* = \mathbf{e} + \epsilon\,\mathbf{p}^* \times \mathbf{e} = \mathbf{e} + \epsilon\,\mathbf{e}_0 \ , \tag{1.64}$$

which is rightfully the dual unit vector of the Mozzi-Chasles axis. Furthermore,

$$\mathrm{tr}(\hat{\mathbf{Q}}) = \mathrm{tr}(\mathbf{Q}) + \epsilon\,\mathrm{tr}(\mathbf{DQ}) \ , \tag{1.65a}$$

where, from Theorem 1.3.3, $\mathrm{tr}(\mathbf{DQ})$ turns out to be

$$\mathrm{tr}(\mathbf{DQ}) = -2[\mathrm{vect}(\mathbf{Q})]\cdot\mathbf{d} = -2\sin\phi(\mathbf{e}\cdot\mathbf{d}) \ , \tag{1.65b}$$

whence,

$$\mathrm{tr}(\hat{\mathbf{Q}}) = 1 + 2\cos\phi - \epsilon\, 2(\sin\phi)\mathbf{e}\cdot\mathbf{d} \;, \tag{1.65c}$$

and so, from the second of eqs.(1.56),

$$\hat{q}_0 \equiv \cos\hat{\phi} = \cos\phi - \epsilon\,(\sin\phi)(\mathbf{e}\cdot\mathbf{d}) \;,$$

which, by virtue of eqs.(1.31), leads to

$$\hat{q}_0 = \cos\phi - \epsilon\,(\sin\phi)d^*, \quad \hat{\phi} = \phi + \epsilon\, d^* = \phi(1+\epsilon\, p) \;. \tag{1.65d}$$

In summary, the dual angle of the dual rotation under study comprises the angle of rotation of $\mathbf{Q}$ in its primal part and the axial component of the displacement of all points of the moving body onto the Mozzi-Chasles axis. Upon comparison of the dual angle between two lines, as given in eq.(1.5), with the dual angle of rotation $\hat{\phi}$, it is then apparent that the primal part of the latter plays the role of the angle between two lines, while the corresponding dual part plays the role of the distance s between those lines. It is noteworthy that a pure rotation has a dual angle of rotation that is real, while a pure translation has an angle of rotation that is a pure dual number.

Example 1: Determination of the screw parameters of a rigid-body motion.

We take here an example of Angeles (2002): The cube of Fig. 1.5 is displaced from configuration $A^0B^0\ldots H^0$ into configuration $AB\ldots H$. Find the Plücker coordinates of the Mozzi-Chasles axis of the motion undergone by the cube.

Solution: We start by constructing $\hat{\mathbf{Q}}$: $\hat{\mathbf{Q}} \equiv [\,\hat{\mathbf{i}}^* \;\; \hat{\mathbf{j}}^* \;\; \hat{\mathbf{k}}^*\,]$, where $\hat{\mathbf{i}}^*$, $\hat{\mathbf{j}}^*$, and $\hat{\mathbf{k}}^*$ are the dual unit vectors of lines AB, AD, and AE, respectively. These lines are, in turn, the images of lines A^0B^0, A^0D^0, and A^0E^0 under the rigid-body motion at hand. The dual unit vectors of the latter are denoted by $\hat{\mathbf{i}}^{0*}$, $\hat{\mathbf{j}}^{0*}$, and $\hat{\mathbf{k}}^{0*}$, respectively, and are parallel to the X, Y, and Z axes of the figure. We thus have

$$\hat{\mathbf{i}}^* = -\mathbf{j}^0 + \epsilon\,\mathbf{a}\times(-\mathbf{j}^0), \quad \hat{\mathbf{j}}^* = \mathbf{k}^0 + \epsilon\,\mathbf{a}\times\mathbf{k}^0, \quad \hat{\mathbf{k}}^* = -\mathbf{i}^0 + \epsilon\,\mathbf{a}\times(-\mathbf{i}^0) \;,$$

where $\mathbf{a}$ is the position vector of A, and is given by

$$\mathbf{a} = [\,2 \quad 1 \quad -1\,]^T a \;.$$

Hence,

$$\begin{aligned}
\hat{\mathbf{i}}^* &= -\mathbf{j}^0 + \epsilon\, a(-\mathbf{i}^0 - 2\mathbf{k}^0)\\
\hat{\mathbf{j}}^* &= \mathbf{k}^0 + \epsilon\, a(\mathbf{i}^0 - 2\mathbf{j}^0)\\
\hat{\mathbf{k}}^* &= -\mathbf{i}^0 + \epsilon\, a(\mathbf{j}^0 + \mathbf{k}^0)
\end{aligned}$$

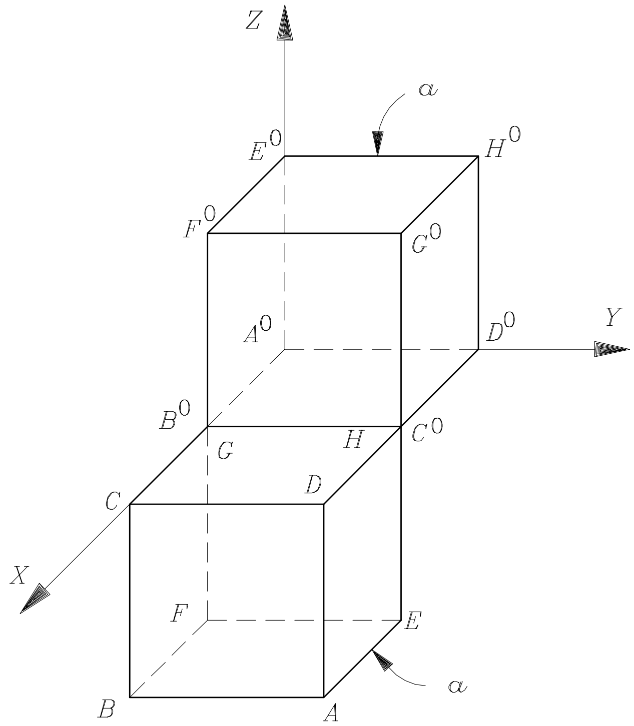

Figure 1.5: Motion of a cube

Therefore,

$$\hat{\mathbf{Q}} = \begin{bmatrix} -\epsilon\, a & +\epsilon\, a & -1 \\ -1 & -\epsilon\, 2a & +\epsilon\, a \\ -\epsilon\, 2a & 1 & +\epsilon\, a \end{bmatrix} ,$$

whence,

$$\text{vect}(\hat{\mathbf{Q}}) = \frac{1}{2} \begin{bmatrix} 1-\epsilon\, a \\ -1+\epsilon\, 2a \\ -1-\epsilon\, a \end{bmatrix} , \quad \text{tr}(\hat{\mathbf{Q}}) = -\epsilon\,(2a) ,$$

and

$$\|\text{vect}(\hat{\mathbf{Q}})\|^2 = \|\frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \|^2 + \epsilon\, 2\frac{1}{2} [\,1 \quad -1 \quad -1\,] \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \frac{a}{2} = \frac{3}{4} - \epsilon\, a \, .$$

Thus,

$$\|\text{vect}(\hat{\mathbf{Q}})\| = \frac{\sqrt{3}}{2} + \epsilon\, \frac{-a}{\sqrt{3}} = \frac{\sqrt{3}}{2} - \epsilon\, \frac{\sqrt{3}}{3} a \, .$$

Therefore, the unit dual vector representing the Mozzi-Chasles axis of the motion at hand, $\hat{\mathbf{e}}^*$, is given by $\hat{\mathbf{e}}^* = \text{vect}(\hat{\mathbf{Q}})/\|\text{vect}(\hat{\mathbf{Q}})\|$, i.e.,

$$\hat{\mathbf{e}}^* = \frac{1}{\sqrt{3}/2} \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \epsilon\, \frac{a}{3/4} \left(\frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \frac{-\sqrt{3}}{3} - \frac{1}{2} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \frac{\sqrt{3}}{2} \right) .$$

After various stages of simplification, the foregoing expression reduces to

$$\hat{\mathbf{e}}^* = \frac{\sqrt{3}}{3}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + \epsilon\,\frac{\sqrt{3}}{9}\begin{bmatrix} -1 \\ 4 \\ -5 \end{bmatrix} a\ .$$

Thus, the Mozzi-Chasles axis is parallel to the unit vector $\mathbf{e}$, which is given by the primal part of $\hat{\mathbf{e}}$, while the dual part of the same dual unit vector represents the moment of the Mozzi-Chasles axis, from which the position vector $\mathbf{p}^*$ of P^*, the point of the Mozzi-Chasles axis closest to the origin, is readily found as

$$\mathbf{p}^* = \mathbf{e} \times \mathbf{e}_0 = \frac{a}{3}\,[3 \quad 2 \quad 1]^T\ .$$

1.3.4 The Dual Euler-Rodrigues Parameters of a Rigid-Body Motion

We first recall the definition of the Euler-Rodrigues parameters of a pure rotation, which are isomorphic to the *quaternion* of the rotation (Hamilton, 1844). These are most naturally introduced as the linear invariants of the square root of the rotation at hand, and represented, paralleling the definition of the linear invariants, as

$$\mathbf{r} \equiv \mathrm{vect}(\sqrt{\mathbf{Q}}), \quad r_0 \equiv \frac{\mathrm{tr}(\sqrt{\mathbf{Q}}) - 1}{2}\ , \tag{1.66}$$

the *proper orthogonal* square root of $\mathbf{Q}$ being given as (Angeles, 2002):

$$\sqrt{\mathbf{Q}} = \mathbf{1} + \sin\left(\frac{\phi}{2}\right)\mathbf{E} + \left[1 - \cos\left(\frac{\phi}{2}\right)\right])\mathbf{E}^2\ . \tag{1.67}$$

The *dual Euler-Rodrigues parameters* of a rigid-body motion are thus defined as

$$\hat{\mathbf{r}} \equiv \mathrm{vect}(\sqrt{\hat{\mathbf{Q}}}), \quad \hat{r}_0 \equiv \frac{\mathrm{tr}(\sqrt{\hat{\mathbf{Q}}}) - 1}{2}\ . \tag{1.68}$$

Below we derive an expression for $\sqrt{\hat{\mathbf{Q}}}$. Prior to this, we introduce a relation that will prove useful:

Lemma 1.3.2 *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be arbitrary 3-dimensional vectors, and* $\mathbf{c} \equiv \mathbf{a} \times \mathbf{b}$. *The cross-product matrix* $\mathbf{C}$ *of* $\mathbf{c}$ *is given by*

$$\mathbf{C} = \mathbf{b}\mathbf{a}^T - \mathbf{a}\mathbf{b}^T\ . \tag{1.69}$$

Proof: This follows by noticing that, for any 3-dimensional vector $\mathbf{u}$,

$$\mathbf{c} \times \mathbf{u} = (\mathbf{a} \times \mathbf{b}) \times \mathbf{u} = \mathbf{b}(\mathbf{a}^T\mathbf{u}) - \mathbf{a}(\mathbf{b}^T\mathbf{u}) \ ,$$

which readily leads to

$$\mathbf{C}\mathbf{u} = (\mathbf{b}\mathbf{a}^T - \mathbf{a}\mathbf{b}^T)\mathbf{u} \ ,$$

thereby completing the proof.

Now we proceed to determine $\sqrt{\hat{\mathbf{Q}}}$. To this end, we regard the motion at hand, from a reference configuration $\mathcal{B}^0$ to a current configuration $\mathcal{B}$, as consisting of a rotation $\mathbf{Q}$ about the origin O followed by a translation $\mathbf{d}$. Then, this motion is decomposed into two parts, as shown in Fig. 1.3.4: First, the body is rotated about the origin O by a rotation $\sqrt{\mathbf{Q}}$ and a translation $\mathbf{d}_s$; then, from the configuration $\mathcal{B}'$ thus attained, the body is given a new rotation $\sqrt{\mathbf{Q}}$ about O as well, followed by the same translation $\mathbf{d}_s$.

It is apparent that, from the general expression for the dual rotation matrix, eq.(1.42), $\sqrt{\hat{\mathbf{Q}}}$ can be represented as

$$\sqrt{\hat{\mathbf{Q}}} = (\mathbf{1} + \epsilon\,\mathbf{D}_s)\sqrt{\mathbf{Q}} \ , \tag{1.70}$$

the calculation of $\sqrt{\hat{\mathbf{Q}}}$ thus reducing to that of the skew-symmetric matrix $\mathbf{D}_s$, which is the cross-product matrix of $\mathbf{d}_s$. This matrix is calculated below in terms of $\sqrt{\mathbf{Q}}$ and $\mathbf{D}$. We thus have

$$\mathbf{p}^2 = \sqrt{\mathbf{Q}}\mathbf{p}^0 + \mathbf{d}_s \ , \tag{1.71}$$

$$\mathbf{p}^4 = \sqrt{\mathbf{Q}}\mathbf{p}^2 + \mathbf{d}_s = \mathbf{Q}\mathbf{p}^0 + (\mathbf{1} + \sqrt{\mathbf{Q}})\mathbf{d}_s \ . \tag{1.72}$$

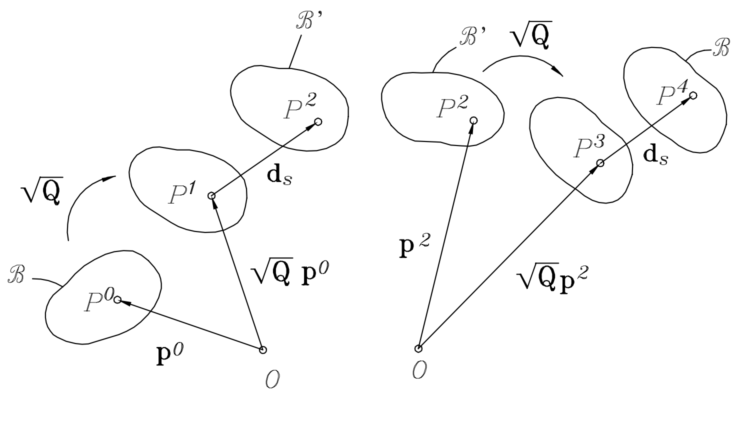

Figure 1.6: Decomposition of the motion of a rigid body

But $\mathbf{p}^4$ is the position vector of point P in $\mathcal{B}$, which can be attained by a rotation $\mathbf{Q}$ about O followed by a translation $\mathbf{d}$, i.e.,

$$\mathbf{p}^4 = \mathbf{Q}\mathbf{p}^0 + \mathbf{d} \ . \tag{1.73}$$

Upon comparing the right-hand sides of eqs.(1.72) and (1.73), we obtain

$$(\mathbf{1} + \sqrt{\mathbf{Q}})\mathbf{d}_s = \mathbf{d} \ ,$$

whence,

$$\mathbf{d}_s = (\mathbf{1} + \sqrt{\mathbf{Q}})^{-1}\mathbf{d} \ . \tag{1.74}$$

An expression for the above inverse can be derived if we realize that this inverse is an analytic function of $\sqrt{\mathbf{Q}}$, which is, in turn, an analytic function of $\mathbf{Q}$. We can thus conclude that by virtue of the Cayley-Hamilton Theorem, invoked when deriving the exponential form of the rotation matrix in eq.(1.22), the inverse sought must be a linear combination of the first three powers of $\mathbf{E}$: $\mathbf{e}^0 \equiv \mathbf{1}$, $\mathbf{E}$, and $\mathbf{E}^2$, namely,

$$(\mathbf{1} + \sqrt{\mathbf{Q}})^{-1} = \alpha\mathbf{1} + \beta\mathbf{E} + \gamma\mathbf{E}^2 \ , \tag{1.75}$$

where α, β, and γ are to be determined. To this end, we write

$$(\mathbf{1} + \sqrt{\mathbf{Q}})^{-1}(\alpha\mathbf{1} + \beta\mathbf{E} + \gamma\mathbf{E}^2) = \mathbf{1} \ .$$

If we now substitute in the above equation the expression for $\sqrt{\mathbf{Q}}$ displayed in eq.(1.67), we obtain three equations for the three unknowns α, β, and γ, from which it is a simple matter to solve for these unknowns, namely,

$$\alpha = \frac{1}{2}, \quad \beta = -\frac{\sin(\phi/2)}{2[1+\cos(\phi/2)]}, \quad \gamma = 0 \ , \tag{1.76}$$

the inverse sought thus taking the form

$$(\mathbf{1} + \sqrt{\mathbf{Q}})^{-1} = \frac{1}{2}\left[\mathbf{1} - \frac{\sin(\phi/2)}{1+\cos(\phi/2)}\mathbf{E}\right] \ . \tag{1.77}$$

Therefore, eq.(1.74) yields

$$\mathbf{d}_s = (\mathbf{1} + \sqrt{\mathbf{Q}})^{-1}\mathbf{d} = \frac{1}{2}\left[\mathbf{1} - \frac{\sin(\phi/2)}{1+\cos(\phi/2)}\mathbf{E}\right]\mathbf{d} \ ,$$

i.e.,

$$\mathbf{d}_s = \frac{1}{2}\left[\mathbf{d} - \frac{\sin(\phi/2)}{1+\cos(\phi/2)}\mathbf{e}\times\mathbf{d}\right] \ . \tag{1.78}$$

Thus, $\mathbf{D}_s$ is the cross-product matrix of the sum of two vectors, and hence, $\mathbf{D}_s$ reduces to the sum of the corresponding corss-product matrices. The cross-product matrix of the first term of the right-hand side of the foregoing equation is apparently proportional to $\mathbf{D}$, that of the second term being proportional to the cross-product matrix of $\mathbf{e}\times\mathbf{d}$. The latter can be readily obtained by application of Lemma 1.3.2, which leads to

$$\mathbf{D}_s = \frac{1}{2}\left[\mathbf{D} - \frac{\sin(\phi/2)}{1+\cos(\phi/2)}(\mathbf{d}\mathbf{e}^T - \mathbf{e}\mathbf{d}^T)\right] \ . \tag{1.79}$$

Hence,

$$\sqrt{\hat{\mathbf{Q}}} = \mathbf{1} + \epsilon \frac{1}{2}\left[\mathbf{D} - \frac{\sin(\phi/2)}{1+\cos(\phi/2)}(\mathbf{d}\mathbf{e}^T - \mathbf{e}\mathbf{d}^T)\right]\sqrt{\mathbf{Q}} \, . \tag{1.80}$$

Now, the linear invariants of $\sqrt{\hat{\mathbf{Q}}}$ are

$$\operatorname{vect}(\sqrt{\hat{\mathbf{Q}}}) = \operatorname{vect}(\sqrt{\mathbf{Q}}) + \epsilon \operatorname{vect}(\mathbf{D}_s\sqrt{\mathbf{Q}}) \tag{1.81a}$$

and

$$\operatorname{tr}(\sqrt{\hat{\mathbf{Q}}}) = \operatorname{tr}(\sqrt{\mathbf{Q}}) + \epsilon \operatorname{tr}(\mathbf{D}_s\sqrt{\mathbf{Q}}) \, . \tag{1.81b}$$

An expression for $\operatorname{vect}(\sqrt{\mathbf{Q}})$, appearing in the first term of $\operatorname{vect}(\sqrt{\hat{\mathbf{Q}}})$, can be obtained from eq.(1.67), namely,

$$\operatorname{vect}(\sqrt{\mathbf{Q}}) = \sin\left(\frac{\phi}{2}\right)\operatorname{vect}(\mathbf{E}) = \sin\left(\frac{\phi}{2}\right)\mathbf{e} \, , \tag{1.82}$$

while an expression for the second term of the right-hand side of eq.(1.81b) is obtained by application of Theorem 1.3.2:

$$\operatorname{vect}(\mathbf{D}_s\sqrt{\mathbf{Q}}) = \frac{1}{2}[\operatorname{tr}(\sqrt{\mathbf{Q}})\mathbf{1} - \sqrt{\mathbf{Q}}]\mathbf{d}_s \, ,$$

which can be further expanded without intermediate lengthy derivations if we realize that the above expression is the counterpart of that appearing in eq.(1.57c); the latter is expanded in eq.(1.57d). Thus, all we need now is mimic eq.(1.57d), if with ϕ and $\mathbf{d}$ substituted by their counterparts $\phi/2$ and $\mathbf{d}_s$, respectively, i.e.,

$$\begin{aligned}\operatorname{vect}(\mathbf{D}_s\sqrt{\mathbf{Q}}) = \frac{1}{2}\Bigg\{&\left[1+\cos\left(\frac{\phi}{2}\right)\right]\mathbf{d}_s - \sin\left(\frac{\phi}{2}\right)\mathbf{e}\times\mathbf{d}_s \\ &- \left[1-\cos\left(\frac{\phi}{2}\right)\right](\mathbf{e}\cdot\mathbf{d})\mathbf{e}\Bigg\} \, .\end{aligned} \tag{1.83}$$

If we now simplify the above expression for $\operatorname{vect}(\mathbf{D}_s\sqrt{\mathbf{Q}})$, and substitute the simplified expression into eq.(1.81a), along with eq.(1.82), we obtain the desired expression for $\hat{\mathbf{r}}$. Note that the latter is defined in eq.(1.68), and hence,

$$\hat{\mathbf{r}} = \sin\left(\frac{\phi}{2}\right)\mathbf{e} + \epsilon\left[\cos\left(\frac{\phi}{2}\right)p_s\frac{\phi}{2}\mathbf{e} + \sin\left(\frac{\phi}{2}\right)\mathbf{e}_0\right] , \tag{1.84}$$

where p_s is the pitch associated with the motion represented by $\sqrt{\hat{\mathbf{Q}}}$, namely,

$$p_s \equiv \mathbf{d}_s \cdot \mathbf{e} = \frac{1}{2}\mathbf{d} \, , \tag{1.85}$$

where we have recalled the expression for $\mathbf{d}_s$ displayed in eq.(1.78). Similar to eq.(1.61), then, the dual vector of the Euler-Rodrigues parameters is given by

$$\hat{\mathbf{r}} = \hat{\mathbf{e}}^*\sin\left(\frac{\hat{\phi}}{2}\right) , \quad \hat{\phi} \equiv \phi + \epsilon\, d_s^*, \quad d_s^* \equiv \mathbf{d}_s \cdot \mathbf{e} \, . \tag{1.86}$$

The scalar of the Euler-Rodrigues parameters under study, $\hat{r}_0$, is now found in terms of the trace of $\sqrt{\hat{\mathbf{Q}}}$, which is displayed in eq.(1.81b). In that equation,

$$\mathrm{tr}(\sqrt{\mathbf{Q}}) = 1 + 2\cos\left(\frac{\phi}{2}\right) ,$$

the dual part of the right-hand side of eq.(1.81b) being calculated by application of Theorem 1.3.3:

$$\mathrm{tr}(\mathbf{D}_s\sqrt{\mathbf{Q}}) = -2\mathbf{d}_s \cdot \mathrm{vect}(\sqrt{\mathbf{Q}}) = -2\mathbf{d}_s \cdot \mathbf{e}\sin\left(\frac{\phi}{2}\right)$$

or, in terms of the corresponding pitch p_s,

$$\mathrm{tr}(\mathbf{D}_s\sqrt{\mathbf{Q}}) = -2p_s\sin\left(\frac{\phi}{2}\right) .$$

Therefore,

$$\mathrm{tr}(\sqrt{\hat{\mathbf{Q}}}) = 1 + 2\cos\left(\frac{\phi}{2}\right) - \epsilon\, 2p_s\sin\left(\frac{\phi}{2}\right) ,$$

and hence,

$$\hat{r}_0 = \cos\left(\frac{\phi}{2}\right) - \epsilon\, p_s\sin\left(\frac{\phi}{2}\right) , \tag{1.87}$$

which is the counterpart of the second of eqs.(1.55a). The set ($\hat{\mathbf{r}}$, $\hat{r}_0$) constitutes the *dual quaternion* of the motion under study (McCarthy, 1990).

1.4 The Dual Angular Velocity

Similar to the angular-velocity matrix $\mathbf{\Omega}$ introduced in eq.(1.46a), the *dual angular velocity matrix* $\hat{\mathbf{\Omega}}$ is defined as

$$\hat{\mathbf{\Omega}} \equiv \dot{\hat{\mathbf{Q}}}\hat{\mathbf{Q}}^T . \tag{1.88}$$

Now we differentiate with respect to time the expression for $\hat{\mathbf{Q}}$ introduced in eq.(1.42), which yields

$$\dot{\hat{\mathbf{Q}}} = (\mathbf{1} + \epsilon\,\mathbf{D})\dot{\mathbf{Q}} + \epsilon\dot{\mathbf{D}}\mathbf{Q} .$$

Upon substitution of the above expression for $\dot{\hat{\mathbf{Q}}}$ and of the expression for $\hat{\mathbf{Q}}$ of eq.(1.42) into eq.(1.88), we obtain

$$\hat{\mathbf{\Omega}} = \mathbf{\Omega} + \epsilon\,(\mathbf{D}\mathbf{\Omega} - \mathbf{\Omega}\mathbf{D} + \dot{\mathbf{D}}) . \tag{1.89}$$

The *dual angular-velocity vector* $\hat{\boldsymbol{\omega}}$ of the motion under study is then obtained as the axial vector of the foregoing expression, namely,

$$\hat{\boldsymbol{\omega}} = \mathrm{vect}(\hat{\mathbf{\Omega}}) = \boldsymbol{\omega} + \epsilon\,[\mathrm{vect}(\mathbf{D}\mathbf{\Omega} - \mathbf{\Omega}\mathbf{D}) + \dot{\mathbf{d}}] , \tag{1.90}$$

with $\dot{\mathbf{d}}$ being the time-derivative of vector $\mathbf{d}$, introduced in eq.(1.33). Thus, in order to determine $\hat{\boldsymbol{\omega}}$, all we need is the axial vector of the difference $\mathbf{D}\mathbf{\Omega} - \mathbf{\Omega}\mathbf{D}$. An expression for this difference can be obtained in various manners, one of which is outlined below: First, note that this difference is skew-symmetric, and hence,

$$\text{vect}(\mathbf{D}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{D}) = 2\,\text{vect}(\mathbf{D}\boldsymbol{\Omega}) \ .$$

Further, the vector of $\mathbf{D}\boldsymbol{\Omega}$ is computed by means of Corollary 1.3.1, eq.(1.39), upon substituting $\mathbf{A}$ by $\boldsymbol{\Omega}$ in that expression. Thus,

$$\text{vect}(\mathbf{D}\boldsymbol{\Omega}) = -\frac{1}{2}\boldsymbol{\omega} \times \mathbf{d} \ . \tag{1.91}$$

Therefore,

$$\hat{\boldsymbol{\omega}} = \boldsymbol{\omega} + \epsilon\,(\dot{\mathbf{d}} - \boldsymbol{\omega} \times \mathbf{d}) \ , \tag{1.92}$$

and, if we recall eq.(1.51), the foregoing expression takes the alternative form

$$\hat{\boldsymbol{\omega}} = \boldsymbol{\omega} + \epsilon\,\mathbf{v}^0 \ . \tag{1.93}$$

In consequence, the dual angular velocity is the dual representation of the *twist* $\mathbf{t}$ of $\mathcal{B}$, defined as the 6-dimensional array

$$\mathbf{t} \equiv \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v}^0 \end{bmatrix} \ . \tag{1.94}$$

We can therefore find the angular velocity vector and the moment of the ISA about the given origin–i.e., the *instant screw parameters* of the motion at hand–if we are given enough information as to allow us to compute $\hat{\boldsymbol{\omega}}$. The information required to determine the screw parameters of the motion under study can be given as the position and velocity vectors of three noncollinear points of a rigid body (Angeles, 2002). However, note that the dual rotation matrix was obtained in Example 1 in terms of the dual unit vectors representing three mutually orthogonal lines. Notice that, by virtue of Lemma 1.2.1, the three lines of Example 1 were chosen concurrent and mutually orthogonal.

Now, in order to find the instant-screw parameters of interest, we need the time-derivatives of the dual unit vectors representing three concurrent, mutually orthogonal lines, but all we have at our disposal is the position and velocity vectors of three non-collinear points. Nevertheless, once we know three noncollinear points of a rigid body, say A, B, and C, along with their velocities, it is possible to find the position and velocity vectors of three pairs of points defining a triad of concurrent, mutually orthogonal lines, an issue that falls beyond the scope of this chapter. Rather than discussing the problem at hand in its fullest generality, we limit ourselves to the special case in which the position vector $\mathbf{p}$ of a point P of the rigid body under study can be determined so that the three lines PA, PB, and PC are mutually orthogonal. Further, we let the position vectors of the three given points be $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. Thus, point P of the body in this case forms a rectangular trihedron with vertex at P and edges PA, PB, and PC. We can thus express $\mathbf{p}$ as a nonlinear function of the three position vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$:

$$\mathbf{p} = \mathbf{p}(\mathbf{a},\, \mathbf{b},\, \mathbf{c}) \,. \tag{1.95}$$

Moreover, the velocity of point P, $\dot{\mathbf{p}}$, can be calculated now as a linear combination of the velocities of the three given points, by straightforward differentiation of the foregoing expression, namely,

$$\dot{\mathbf{p}} = \mathbf{P}_a\dot{\mathbf{a}} + \mathbf{P}_b\dot{\mathbf{b}} + \mathbf{P}_c\dot{\mathbf{c}} \,, \tag{1.96}$$

where $\mathbf{P}_a$, $\mathbf{P}_b$, and $\mathbf{P}_c$ denote the partial derivatives of $\mathbf{p}$ with respect to $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, respectively. Once the position and the velocity vectors of point P are known, it is possible to determine the time-rates of change of the dual unit vectors representing the three lines PA, PB and PC, as described below.

Let $\hat{\mathbf{e}}^*$ denote the dual unit vector representing the line determined by points A and P, its primary and dual parts, $\mathbf{e}$ and $\mathbf{e}_0$, being given by

$$\mathbf{e} = \frac{\mathbf{a}-\mathbf{p}}{\|\mathbf{a}-\mathbf{p}\|}, \quad \mathbf{e}_0 = \mathbf{p} \times \frac{\mathbf{a}-\mathbf{p}}{\|\mathbf{a}-\mathbf{p}\|} \,. \tag{1.97}$$

Straightforward differentiation of the foregoing expressions with respect to time leads to

$$\dot{\mathbf{e}} = \frac{1}{\|\mathbf{a}-\mathbf{p}\|}\left(\dot{\mathbf{a}} - \dot{\mathbf{p}} - \mathbf{e}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{a}-\mathbf{p}\|\right) \,,$$
$$\dot{\mathbf{e}}_0 = \dot{\mathbf{p}} \times \frac{\mathbf{a}-\mathbf{p}}{\|\mathbf{a}-\mathbf{p}\|} + \mathbf{p} \times \frac{1}{\|\mathbf{a}-\mathbf{p}\|}\left(\dot{\mathbf{a}} - \dot{\mathbf{p}} - \mathbf{e}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{a}-\mathbf{p}\|\right) \,.$$

Upon simplification, we obtain the desired expression for $\dot{\hat{\mathbf{e}}}^*$, namely,

$$\dot{\hat{\mathbf{e}}}^* = \frac{1}{\|\mathbf{a}-\mathbf{p}\|}\left[\dot{\mathbf{a}} - \dot{\mathbf{p}} + \epsilon\left(\mathbf{p}\dot{\mathbf{a}} + \dot{\mathbf{p}} \times \mathbf{a}\right)\right] \,. \tag{1.98}$$

Therefore, knowing the velocity of two points of a line, we can determine the time-rate of change of the dual unit vector representing the line. The foregoing idea is best illustrated with the aid of the example included below.

Example 2: Determination of the ISA of a rigid-body motion.

For comparison purposes, we take an example from (Angeles, 2002): The three vertices of the equilateral triangular plate of Fig. 1.4, which lie in the X-Y plane, $\{\, P_i \,\}_1^3$, have the position vectors $\{\, \mathbf{p}_i \,\}_1^3$. Moreover, the origin of the coordinate frame X, Y, Z lies at the centroid C of the triangle, and the velocities of the foregoing points, $\{\, \dot{\mathbf{p}}_i \,\}_1^3$, are given in this coordinate frame as

$$\dot{\mathbf{p}}_1 = \frac{4-\sqrt{2}}{4}\begin{bmatrix}0\\0\\1\end{bmatrix}, \quad \dot{\mathbf{p}}_2 = \frac{4-\sqrt{3}}{4}\begin{bmatrix}0\\0\\1\end{bmatrix}, \quad \dot{\mathbf{p}}_3 = \frac{4+\sqrt{2}}{4}\begin{bmatrix}0\\0\\1\end{bmatrix} \,.$$

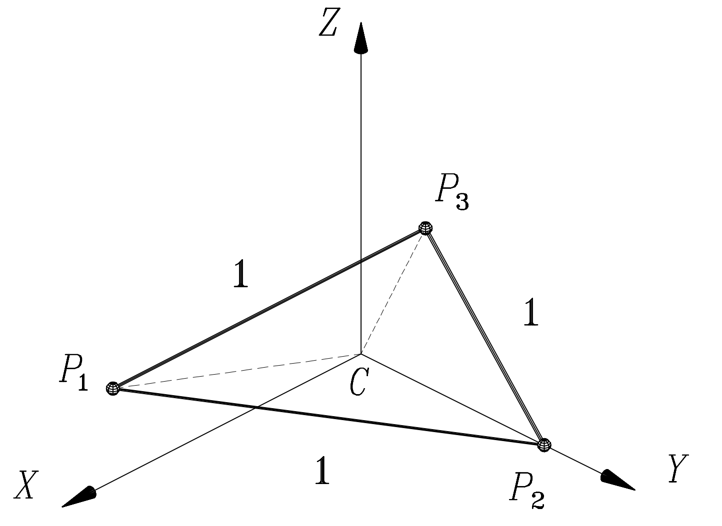

Figure 1.7: A rigid triangular plate undergoing a motion given by the velocity of its vertices

With the above information, compute the instant-screw parameters of the motion under study.

Solution: Since the centroid C of the triangle coincides with that of the three given points, we have $\mathbf{c} = \mathbf{0}$, where $\mathbf{c}$ is the position vector of C. Moreover,

$$\mathbf{p}_1 = \begin{bmatrix} 1/2 \\ -\sqrt{3}/6 \\ 0 \end{bmatrix}, \ \mathbf{p}_2 = \begin{bmatrix} 0 \\ \sqrt{3}/3 \\ 0 \end{bmatrix}, \ \mathbf{p}_3 = \begin{bmatrix} -1/2 \\ -\sqrt{3}/6 \\ 0 \end{bmatrix} .$$

First and foremost, we have to verify the compatibility of the data. To do this, we calculate the component of the relative velocities of two given points onto the line that they define. It can be readily shown that the data are compatible, and hence, the motion is possible. Next, we obtain the position vector of the point P that, along with $\{P_i\}_1^3$, forms an orthogonal trihedron. It is not difficult to realize that the position vector of point P can be expressed as[2]

$$\mathbf{p} = \mathbf{c} + \frac{\sqrt{2}}{3}(\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1) ,$$

and hence,

$$\dot{\mathbf{p}} = \dot{\mathbf{c}} + \frac{\sqrt{2}}{3}[(\mathbf{p}_3 - \mathbf{p}_2) \times \dot{\mathbf{p}}_1 + (\mathbf{p}_1 - \mathbf{p}_3) \times \dot{\mathbf{p}}_2 + (\mathbf{p}_2 - \mathbf{p}_1) \times \dot{\mathbf{p}}_3] ,$$

[2]Although $\mathbf{c} = \mathbf{0}$ in this case, $\dot{\mathbf{c}} \neq \mathbf{0}$, and hence, $\mathbf{c}$ must be written explicitly in the expression for $\mathbf{p}$.

with the numerical values of $\mathbf{p}$ and $\dot{\mathbf{p}}$ given below:

$$\mathbf{p} = \frac{\sqrt{6}}{6}\begin{bmatrix}0\\0\\1\end{bmatrix}, \quad \dot{\mathbf{p}} = \frac{1}{12}\begin{bmatrix}2\sqrt{3}\\ \sqrt{6}\\ 12-\sqrt{3}\end{bmatrix} .$$

Now, let $\hat{\mathbf{e}}_i^*$ denote the dual unit vector representing the line that passes through P and P_i, i.e.,

$$\hat{\mathbf{e}}_i^* = \frac{1}{\|\mathbf{p}_i - \mathbf{p}\|}[\mathbf{p}_i - \mathbf{p} + \epsilon\,\mathbf{p} \times \mathbf{p}_i] ,$$

where

$$\|\mathbf{p}_i - \mathbf{p}\| = \frac{\sqrt{2}}{2}, \quad i = 1, 2, 3 .$$

Next, the three foregoing dual unit vectors are stored columnwise in the dual rotation matrix $\hat{\mathbf{Q}}$, i.e.,

$$\hat{\mathbf{Q}} = [\,\hat{\mathbf{e}}_1^* \quad \hat{\mathbf{e}}_2^* \quad \hat{\mathbf{e}}_3^*\,] .$$

Upon substitution of the numerical values of these vectors into the above expression, we obtain

$$\hat{\mathbf{Q}} = \frac{\sqrt{12}}{12}\begin{bmatrix} 6+\epsilon\,2 & -\,\epsilon\,2\sqrt{2} & -6+\epsilon\,\sqrt{2} \\ -2\sqrt{3}+\epsilon\,\sqrt{6} & 4\sqrt{3} & -2\sqrt{3}-\epsilon\,\sqrt{6} \\ -2\sqrt{6} & -2\sqrt{6} & -2\sqrt{6} \end{bmatrix} .$$

Likewise, the time derivative of $\hat{\mathbf{Q}}$ is computed as

$$\dot{\hat{\mathbf{Q}}} = \frac{\sqrt{2}}{24}\left(\begin{bmatrix} -4\sqrt{3} & -4\sqrt{3} & -4\sqrt{3} \\ -2\sqrt{6} & -2\sqrt{6} & -2\sqrt{6} \\ -6\sqrt{2}+2\sqrt{3} & -4\sqrt{3} & 6\sqrt{2}+2\sqrt{3} \end{bmatrix} + \epsilon\begin{bmatrix} -1+4\sqrt{3} & 2-8\sqrt{3} & -1+4\sqrt{3} \\ 12-\sqrt{3} & 0 & -12+\sqrt{3} \\ -(2+\sqrt{6}) & 4 & -2+\sqrt{6} \end{bmatrix}\right) .$$

Therefore,

$$\hat{\boldsymbol{\Omega}} = \dot{\hat{\mathbf{Q}}}\hat{\mathbf{Q}}^T = \frac{1}{12}\begin{bmatrix} 0 & -\,\epsilon\,(12-\sqrt{3}) & 6\sqrt{2} \\ +\,\epsilon\,(12-\sqrt{3}) & 0 & 6 \\ -6\sqrt{2} & -6 & 0 \end{bmatrix} ,$$

which, as expected, is a dual skew-symmetric matrix. Hence,

$$\hat{\boldsymbol{\omega}} = \mathrm{vect}(\hat{\boldsymbol{\Omega}}) = \frac{1}{2}\begin{bmatrix}-1\\ \sqrt{2}\\ 0\end{bmatrix} + \epsilon\,\frac{12-\sqrt{3}}{12}\begin{bmatrix}0\\0\\1\end{bmatrix} ,$$

from which we can readily identify

$$\boldsymbol{\omega} = \frac{1}{2}\begin{bmatrix}-1\\ \sqrt{2}\\ 0\end{bmatrix}, \quad \mathbf{v}^0 = \frac{12-\sqrt{3}}{12}\begin{bmatrix}0\\0\\1\end{bmatrix} .$$

Furthermore, the position vector $\boldsymbol{\pi}^*$ of the point P^* of the ISA lying closest to the origin can be obtained from $\mathbf{v}^0$. Indeed, let $\mathbf{v}^*$ be the velocity of P^*, which thus allows us to write

$$\mathbf{v}^0 = \mathbf{v}^* + \boldsymbol{\omega} \times (-\mathbf{p}^*) = \mathbf{v}^* + \mathbf{p}^* \times \boldsymbol{\omega} \; .$$

Upon cross-multiplying the two sides of the foregoing expression by $\boldsymbol{\omega}$, we obtain

$$\mathbf{v}^0 \times \boldsymbol{\omega} = \mathbf{v}^* \times \boldsymbol{\omega} + (\mathbf{p}^* \times \boldsymbol{\omega}) \times \boldsymbol{\omega} \; ,$$

whose first term of the right-hand side vanishes because $\mathbf{v}^*$ and $\boldsymbol{\omega}$ are parallel. Therefore,

$$\mathbf{v}^0 \times \boldsymbol{\omega} = (\mathbf{p}^* \times \boldsymbol{\omega}) \times \boldsymbol{\omega} = (\mathbf{p}^* \cdot \boldsymbol{\omega})\boldsymbol{\omega} - \|\boldsymbol{\omega}\|^2 \mathbf{p}^* \; .$$

The first term of the rightmost-hand side of the foregoing equation vanishes because $\mathbf{p}^*$ being the position vector of the point of the ISA that lies closest to the origin, and the ISA being parallel to $\boldsymbol{\omega}$, these two vectors are orthogonal. We can thus solve for $\mathbf{p}^*$ from the above expression, which yields

$$\mathbf{p}^* = -\frac{\mathbf{v}^0 \times \boldsymbol{\omega}}{\|\boldsymbol{\omega}\|^2} \; .$$

The quantities involved in the foregoing expression are now evaluated:

$$-\mathbf{v}^0 \times \boldsymbol{\omega} = \boldsymbol{\omega} \times \mathbf{v}^0 = \frac{12 - \sqrt{3}}{24} \begin{bmatrix} \sqrt{2} \\ 1 \\ 0 \end{bmatrix} , \quad \|\boldsymbol{\omega}\|^2 = \frac{3}{4} \; .$$

Finally, $\mathbf{p}^* = \{[(12 - \sqrt{3})]/18\}[\sqrt{2} \;\; 1 \;\; 0]^T$, which coincides with the results reported in (Angeles, 2002), obtained by another method.

1.5 Conclusions

We revisited dual algebra in the context of kinematic analysis, which led us to a straightforward introduction of dual quaternions. In the process, we showed that the parameters of both the finite screw and the instant screw of a rigid-body motion can be computed from the sum of the diagonal and the difference of the off-diagonal entries of the dual rotation and, correspondingly, the dual angular-velocity matrices.

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