MECH 576
Geometry in Mechanics

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Centre of a Hyperboloid of One Sheet via Line Geometry

1 Introduction

A unique hyperboloid of one sheet may be defined by ruling three given skew lines $\mathcal{P}$,$\mathcal{Q}$,$\mathcal{R}$ with a one parameter set of lines that intersects all three. M, the centre point of this quadric, may be found on the intersection of three planes p,q,r such that

$$M = p \cap q \cap r, \;\; p = \mathcal{P} \cap \mathcal{P}_{qr}, \;\; q = \mathcal{Q} \cap \mathcal{Q}_{rp}, \;\; r = \mathcal{R} \cap \mathcal{R}_{pq}$$

where $\mathcal{P}_{qr}$,$\mathcal{Q}_{rp}$,$\mathcal{R}_{pq}$ are the respective "parallel partners" of $\mathcal{P}$,$\mathcal{Q}$,$\mathcal{R}$. $\mathcal{P}_{qr}$ intersects $\mathcal{Q}$ and $\mathcal{R}$. $\mathcal{Q}_{rp}$ intersects $\mathcal{R}$ and $\mathcal{P}$. $\mathcal{R}_{pq}$ intersects $\mathcal{P}$ and $\mathcal{Q}$.

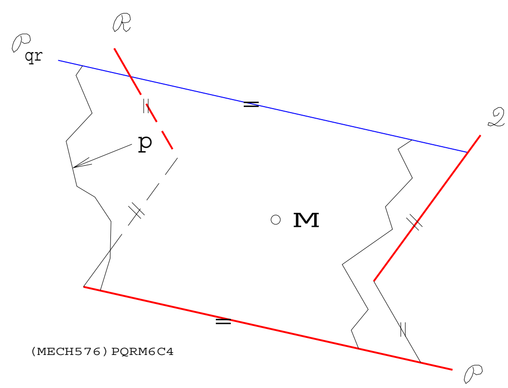

Figure 1: Relation of Parallel Partner $\mathcal{P}_{qr}$ to Lines $\mathcal{P}, \mathcal{Q}, \mathcal{R}$

The approach outlined below makes extensive use of radial Plücker line coordinates. Axial coordinates would serve just as well.

2 Parallel Partners

Direction numbers of parallel partner lines are immediately available. One needs only to determine their moment vector.

$$\mathcal{P}_{qr}\{p_{01} : p_{02} : p_{03} : p_{23}^{qr} : p_{31}^{qr} : p_{12}^{qr}\}$$
$$\mathcal{Q}_{rp}\{q_{01} : q_{02} : q_{03} : q_{23}^{rp} : q_{31}^{rp} : q_{12}^{rp}\}$$
$$\mathcal{R}_{pq}\{r_{01} : r_{02} : r_{03} : r_{23}^{pq} : r_{31}^{pq} : r_{12}^{pq}\}$$

2.1 Homogeneous Constraints and Solutions

Parallel partners intersect on an absolute point. The partner in the opposite regulus intersects the other two given skew lines on Euclidean points. Multipliers λ, μ and ν appear in Eqs. 1,2,3,4 because homogeneous, not direction vector normalized, Plücker coordinates are being used.

$$\begin{aligned}(p_{01}q_{23}+p_{02}q_{31}+p_{03}q_{12})\lambda + q_{01}p_{23}^{qr} + q_{02}p_{31}^{qr} + q_{03}p_{12}^{qr} &= 0\\ (p_{01}r_{23}+p_{02}r_{31}+p_{03}r_{12})\lambda + r_{01}p_{23}^{qr} + r_{02}p_{31}^{qr} + r_{03}p_{12}^{qr} &= 0\\ 0\lambda + p_{01}p_{23}^{qr} + p_{02}p_{31}^{qr} + p_{03}p_{12}^{qr} &= 0\end{aligned} \tag{1}$$

The first two of Eqs. 1 represent the intersection of $\mathcal{P}_{qr}$ with $\mathcal{Q}$ and $\mathcal{R}$, respectively. The third represents the Plücker condition imposed on $\mathcal{P}_{qr}$. The solution of three simultaneous equations in four homogeneous variables appears below as four determinants. λ is not required but is included to complete the solution.

$$\lambda = \begin{vmatrix} q_{01} & q_{02} & q_{03}\\ r_{01} & r_{02} & r_{03}\\ p_{01} & p_{02} & p_{03}\end{vmatrix},\quad p_{23}^{qr} = -\begin{vmatrix} p_{01}q_{23}+p_{02}q_{31}+p_{03}q_{12} & q_{02} & q_{03}\\ p_{01}r_{23}+p_{02}r_{31}+p_{03}r_{12} & r_{02} & r_{03}\\ 0 & p_{02} & p_{03}\end{vmatrix}$$

$$p_{31}^{qr} = \begin{vmatrix} p_{01}q_{23}+p_{02}q_{31}+p_{03}q_{12} & q_{01} & q_{03}\\ p_{01}r_{23}+p_{02}r_{31}+p_{03}r_{12} & r_{01} & r_{03}\\ 0 & p_{01} & p_{03}\end{vmatrix},\quad p_{12}^{qr} = -\begin{vmatrix} p_{01}q_{23}+p_{02}q_{31}+p_{03}q_{12} & q_{01} & q_{02}\\ p_{01}r_{23}+p_{02}r_{31}+p_{03}r_{12} & r_{01} & r_{02}\\ 0 & p_{01} & p_{02}\end{vmatrix} \tag{2}$$

Similarly one may write the multipliers and moment vector elements for $\mathcal{Q}_{rp}$ and $\mathcal{R}_{pq}$. Notice that $\lambda = \mu = \nu$.

$$\mu = \begin{vmatrix} r_{01} & r_{02} & r_{03}\\ p_{01} & p_{02} & p_{03}\\ q_{01} & q_{02} & q_{03}\end{vmatrix},\quad q_{23}^{rp} = -\begin{vmatrix} q_{01}r_{23}+q_{02}r_{31}+q_{03}r_{12} & r_{02} & r_{03}\\ q_{01}p_{23}+q_{02}p_{31}+q_{03}p_{12} & p_{02} & p_{03}\\ 0 & q_{02} & q_{03}\end{vmatrix}$$

$$q_{31}^{rp} = \begin{vmatrix} q_{01}r_{23}+q_{02}r_{31}+q_{03}r_{12} & r_{01} & r_{03}\\ q_{01}p_{23}+q_{02}p_{31}+q_{03}p_{12} & p_{01} & p_{03}\\ 0 & q_{01} & q_{03}\end{vmatrix},\quad q_{12}^{rp} = -\begin{vmatrix} q_{01}r_{23}+q_{02}r_{31}+q_{03}r_{12} & r_{01} & r_{02}\\ q_{01}p_{23}+q_{02}p_{31}+q_{03}p_{12} & p_{01} & p_{02}\\ 0 & q_{01} & q_{02}\end{vmatrix} \tag{3}$$

$$\nu = \begin{vmatrix} p_{01} & p_{02} & p_{03}\\ q_{01} & q_{02} & q_{03}\\ r_{01} & r_{02} & r_{03}\end{vmatrix},\quad r_{23}^{pq} = -\begin{vmatrix} r_{01}p_{23}+r_{02}p_{31}+r_{03}p_{12} & p_{02} & p_{03}\\ r_{01}q_{23}+r_{02}q_{31}+r_{03}q_{12} & q_{02} & q_{03}\\ 0 & r_{02} & r_{03}\end{vmatrix}$$

$$r_{31}^{pq} = \begin{vmatrix} r_{01}p_{23}+r_{02}p_{31}+r_{03}p_{12} & p_{01} & p_{03}\\ r_{01}q_{23}+r_{02}q_{31}+r_{03}q_{12} & q_{01} & q_{03}\\ 0 & r_{01} & r_{03}\end{vmatrix},\quad r_{12}^{pq} = -\begin{vmatrix} r_{01}p_{23}+r_{02}p_{31}+r_{03}p_{12} & p_{01} & p_{02}\\ r_{01}q_{23}+r_{02}q_{31}+r_{03}q_{12} & q_{01} & q_{02}\\ 0 & r_{01} & r_{02}\end{vmatrix} \tag{4}$$

3 Three Planes Intersect on the Centre

Planes p, q, r span points A, B, C on lines $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ paired with lines $\mathcal{P}_{qr}$,$\mathcal{Q}_{rp}$,$\mathcal{R}_{pq}$, respectively. A, B, C are on a, b, c, planes on origin O and normal to $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$. If any of these lines are on O then its point contribution is just $O\{1:0:0:0\}$. Finding the first of three planes, $p\{P_0:P_1:P_2:P_3\}$, will be done in detail. It is on point $A\{a_0:a_1:a_2:a_3\}$, on plane $a\{A_0:A_1:A_2:A_3\}=\{0:p_{01}:p_{02}:p_{03}\}$, normal to $\mathcal{P}$. Planes q,r are obtained by similarity. Point A is obtained with the piercing point relation.

$$A=\mathcal{P}\cap a,\quad a_i=\sum_{j=0}^{3}p_{ij}A_j,\quad p_{ij}=0 \text{ if } i=j,\quad p_{ji}=-p_{ij}$$

$$\begin{array}{rrrrrl}
a_0= & p_{00}A_0 & +p_{01}A_1 & +p_{02}A_2 & +p_{03}A_3= & p_{01}^2+p_{02}^2+p_{03}^2\\
a_1= & -p_{01}A_0 & +p_{11}A_1 & +p_{12}A_2 & -p_{31}A_3= & p_{02}p_{12}-p_{03}p_{31}\\
a_2= & -p_{02}A_0 & -p_{12}A_1 & +p_{22}A_2 & +p_{23}A_3= & p_{03}p_{23}-p_{01}p_{12}\\
a_3= & -p_{03}A_0 & +p_{31}A_1 & -p_{23}A_2 & +p_{33}A_3= & p_{01}p_{31}-p_{02}p_{23}
\end{array}\tag{5}$$

Similarly

$$b_0=q_{01}^2+q_{02}^2+q_{03}^2,\quad b_1=q_{02}q_{12}-q_{03}q_{31},\quad b_2=q_{03}q_{23}-q_{01}q_{12},\quad b_3=p_{01}p_{31}-p_{02}p_{23}$$
$$c_0=r_{01}^2+r_{02}^2+r_{03}^2,\quad c_1=r_{02}r_{12}-r_{03}r_{31},\quad c_2=r_{03}r_{23}-r_{01}r_{12},\quad c_3=r_{01}r_{31}-r_{02}r_{23}$$

Planes p,q,r are obtained with the spanning plan relation. Note that P_{ij} are *axial* Plücker coordinates.

$$p=\mathcal{P}_{qr}\cap A,\quad P_i=\sum_{j=0}^{3}P_{ij}^{qr}a_j,\quad P_{ij}=0 \text{ if } i=j,\quad P_{ji}=-P_{ij}$$

$$\begin{aligned}
P_0&=P_{00}a_0+P_{01}^{qr}a_1+P_{02}^{qr}a_2+P_{03}^{qr}a_3\\
P_1&=-P_{01}^{qr}a_0+P_{11}a_1+P_{12}a_2-P_{31}a_3\\
P_2&=-P_{02}^{qr}a_0-P_{12}a_1+P_{22}a_2+P_{23}a_3\\
P_3&=-P_{03}^{qr}a_0+P_{31}a_1-P_{23}a_2+P_{33}a_3\\
P_0&=p_{23}^{qr}(p_{02}p_{12}-p_{03}p_{31})+p_{31}^{qr}(p_{03}p_{23}-p_{01}p_{12})+p_{12}^{qr}(p_{01}p_{31}-p_{02}p_{23})\\
P_1&=p_{03}(p_{03}p_{23}-p_{01}p_{12})-p_{02}(p_{01}p_{31}-p_{02}p_{23})-p_{23}^{qr}(p_{01}^2+p_{02}^2+p_{03}^2)\\
P_2&=p_{01}(p_{01}p_{31}-p_{02}p_{23})-p_{03}(p_{01}p_{12}-p_{03}p_{31})-p_{31}^{qr}(p_{01}^2+p_{02}^2+p_{03}^2)\\
P_3&=p_{02}(p_{01}p_{12}-p_{03}p_{31})-p_{01}(p_{01}p_{31}-p_{02}p_{23})-p_{12}^{qr}(p_{01}^2+p_{02}^2+p_{03}^2)
\end{aligned}\tag{6}$$

By similarity one writes the coordinates of planes q and r.

$$\begin{aligned}
Q_0&=q_{23}^{rp}(q_{02}q_{12}-q_{03}q_{31})+q_{31}^{rp}(q_{03}q_{23}-q_{01}q_{12})+q_{12}^{rp}(q_{01}q_{31}-q_{02}q_{23})\\
Q_1&=q_{03}(q_{03}q_{23}-q_{01}q_{12})-q_{02}(q_{01}q_{31}-q_{02}q_{23})-q_{23}^{rp}(q_{01}^2+q_{02}^2+q_{03}^2)\\
Q_2&=q_{01}(q_{01}q_{31}-q_{02}q_{23})-q_{03}(q_{01}q_{12}-q_{03}q_{31})-q_{31}^{rp}(q_{01}^2+q_{02}^2+q_{03}^2)\\
Q_3&=q_{02}(q_{01}q_{12}-q_{03}q_{31})-q_{01}(q_{01}q_{31}-q_{02}q_{23})-q_{12}^{rp}(q_{01}^2+q_{02}^2+q_{03}^2)
\end{aligned}\tag{7}$$

$$\begin{aligned}
R_0&=r_{23}^{pq}(r_{02}r_{12}-r_{03}r_{31})+r_{31}^{pq}(r_{03}r_{23}-r_{01}r_{12})+r_{12}^{pq}(r_{01}r_{31}-r_{02}r_{23})\\
R_1&=r_{03}(r_{03}r_{23}-r_{01}r_{12})-r_{02}(r_{01}r_{31}-r_{02}r_{23})-r_{23}^{pq}(r_{01}^2+r_{02}^2+r_{03}^2)\\
R_2&=r_{01}(r_{01}r_{31}-r_{02}r_{23})-r_{03}(r_{01}r_{12}-r_{03}r_{31})-r_{31}^{pq}(r_{01}^2+r_{02}^2+r_{03}^2)\\
R_3&=r_{02}(r_{01}r_{12}-r_{03}r_{31})-r_{01}(r_{01}r_{31}-r_{02}r_{23})-r_{12}^{pq}(r_{01}^2+r_{02}^2+r_{03}^2)
\end{aligned}\tag{8}$$

Finally the centre point $M\{m_0 : m_1 : m_2 : m_3\}$ is defined by the point equation on planes p, q, r.

$$m_0 = \begin{vmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{vmatrix},\ m_1 = \begin{vmatrix} P_2 & P_0 & P_3 \\ Q_2 & Q_0 & Q_3 \\ R_2 & R_0 & R_3 \end{vmatrix},\ m_2 = \begin{vmatrix} P_0 & P_1 & P_3 \\ Q_0 & Q_1 & Q_3 \\ R_0 & R_1 & R_3 \end{vmatrix},\ m_3 = \begin{vmatrix} P_1 & P_0 & P_2 \\ Q_1 & Q_0 & Q_2 \\ R_1 & R_0 & R_2 \end{vmatrix} \qquad (9)$$