On the Minimum Principle and Dynamic Programming for Hybrid Systems with Low Dimensional Switching Manifolds

Ali Pakniyat, *Member, IEEE*, and Peter E. Caines, *Life Fellow, IEEE*

Abstract— A class of hybrid systems with both autonomous and controlled switchings and jumps is considered where switching manifolds corresponding to autonomous switchings and jumps are allowed to be codimension k submanifolds in $\mathbb{R}^n$ with $1 \leq k \leq n$. Optimal control problems associated to this class of hybrid systems are studied where in addition to running and terminal costs, costs associated to switching between discrete states are allowed. Statements of the Hybrid Minimum Principle and Hybrid Dynamic Programming as well as their relationship are presented in this general setting and an illustrative example is provided.

I. INTRODUCTION

There is now an extensive literature on the optimal control of hybrid systems (see e.g. [1]–[15]). On one hand, the generalization of the fundamental Pontryagin Maximum Principle (PMP) [16] results in the Hybrid Minimum Principle (HMP) [1]–[8] that gives necessary conditions for the optimality of the trajectory and the control inputs of a given hybrid system with fixed initial conditions and a sequence of autonomous and controlled switchings. These conditions are expressed in terms of the minimization of the distinct Hamiltonians defined along the hybrid trajectory of the system corresponding to a sequence of discrete states and continuous valued control inputs on the associated time intervals. A feature of special interest in the Hybrid Minimum Principle is the boundary conditions on the adjoint processes and the Hamiltonian functions at autonomous and controlled switching times and states; these boundary conditions may be viewed as a generalization of the optimal control case of the Weierstrass–Erdmann conditions of the calculus of variations [17].

The generalization of Dynamic Programming [18] for hybrid systems, on the other hand, results in the theory of Hybrid Dynamic Programming (HDP) which employs the optimal cost to go for the hybrid optimal control problem as its fundamental notion. Under the assumption of smoothness of the value function, the Principle of Optimality results in the celebrated Hamilton-Jacobi-Bellman (HJB) equation of HDP [9]–[12], [19]–[26]. In the case of non-smooth value functions, the so-called viscosity solutions give a general class of solutions to the HJB equation [9]–[11].

The usual assumption in design, analysis and control of hybrid systems is that switching manifolds corresponding to

autonomous switchings and jumps are smooth codimension 1 sub-manifolds of $\mathbb{R}^n$. In some studies like hybrid stability, this assumption reduces the analysis by decoupling the sequence of switching and the uniform convergence of hybrid executions within those with the same switching sequence. However, in the hybrid optimal control context, the assumption of codimension 1 switching manifolds is not a necessity since the optimality conditions are expressed in terms of the admissible controls and their corresponding trajectories that satisfy the desired switching conditions. While numerous hybrid optimal control problems can be considered where the system has switching manifolds with dimensions smaller than $n-1$, i.e. where switching manifolds are codimension k sub-manifold of $\mathbb{R}^n$ with $k > 1$, this class of hybrid systems has been the subject of a limited number of studies in the hybrid optimal control context.

In past work of the authors (see [24]–[28]) the results of the Hybrid Minimum Principle are given for the general class of hybrid optimal control problems with autonomous and controlled state jumps and in the presence of a large range of running, terminal and switching costs. In this paper, the class of hybrid systems under study is further generalized by letting the switching manifolds be codimension k sub-manifold of $\mathbb{R}^n$, with $k \in \{1, \cdots, n\}$. In addition, as shown in [24]–[26], the adjoint process in the HMP and the gradient of the value function in HDP are governed by the same dynamic equation and have the same boundary conditions and hence are identical to each other. The same result is shown to hold in this paper in the presence of low dimensional switching manifolds. Furthermore, an illustrative example is provided in which the continuous state lies in $\mathbb{R}^4$ and the switching manifold is in $\mathbb{R}^2$, i.e. it is a codimension 2 sub-manifold of $\mathbb{R}^4$.

II. HYBRID SYSTEMS

A *hybrid system (structure)* $\mathbb{H}$ is a septuple

$$\mathbb{H} = \{H := Q \times \mathbb{R}^n, I := \Sigma \times U, \Gamma, A, F, \Xi, \mathscr{M}\} \quad (1)$$

where the symbols in the expression are defined as below.

A0: $Q = \{1, 2, ..., |Q|\} \equiv \{q_1, q_2, ..., q_{|Q|}\}, |Q| < \infty$, is a finite set of *discrete states (components)*.

$H := Q \times \mathbb{R}^n$ is called the *(hybrid) state space* of the hybrid system $\mathbb{H}$.

$I := \Sigma \times U$ is the set of system input values, where $|\Sigma| < \infty$ and $U \subset \mathbb{R}^m$ is the set of *admissible input control values*, where U is an open bounded set in $\mathbb{R}^m$ which necessarily has compact closure $\bar{U}$.

The set of admissible (continuous) control inputs $\mathscr{U}(U) := L_\infty([t_0, T_*), U)$, is defined to be the set of all measurable functions that are bounded up to a set of measure zero on $[t_0, T_*)$, $T_* < \infty$, where the boundedness condition necessarily holds since admissible input functions take values in the open bounded set U.

$\Gamma : H \times \Sigma \to H$ is a time independent (partially defined) *discrete state transition map* which is the identity on the second ($\mathbb{R}^n$) component.

$\Xi : H \times \Sigma \to H$ is a time independent (partially defined) *continuous state jump transition map* which is the identity on the first (Q) component. All $\xi_\sigma \in \Xi$ are assumed to be injective and continuously differentiable in the continuous state x.

$A : Q \times \Sigma \to Q$ denotes both a finite automaton and the automaton's associated transition function on the state space Q and event set Σ, such that for a discrete state $q \in Q$ only the discrete controlled and uncontrolled transitions into the q-dependant subset $\{A(q, \sigma), \sigma \in \Sigma\} \subset Q$ occur under the projection of Γ on its Q components: $\Gamma : Q \times \mathbb{R}^n \times \Sigma \to H|_Q$. In other words, Γ can only make a discrete state transition in a hybrid state (q, x) if the automaton A can make the corresponding transition in q.

F is an indexed collection of *vector fields* $\{f_q\}_{q \in Q}$ such that $f_q \in C^{k_f}(\mathbb{R}^n \times U \to \mathbb{R}^n)$, $k_f \geq 1$, satisfies a uniform^x Lipschitz condition, i.e. there exists $L_f < \infty$ such that $\|f_q(x_1, u) - f_q(x_2, u)\| \leq L_f \|x_1 - x_2\|$, $x_1, x_2 \in \mathbb{R}^n$, $u \in U$, $q \in Q$. We also assume that there exists $K_f < \infty$ such that $\max_{q \in Q} \left(\sup_{u \in U} (\|f_q(0, u)\|) \right) \leq K_f$.

$\mathscr{M} = \{m_\alpha : \alpha \in Q \times Q,\}$ denotes a collection of *switching manifolds* such that, for any ordered pair $\alpha = (p, q)$, m_α is a smooth, i.e. C^∞ codimension k sub-manifold of $\mathbb{R}^n$, $1 \leq k \leq n$, possibly with boundary ∂m_α. In this paper, we consider time invariant switching manifolds and hence, each switching manifold is described locally by $m_\alpha = \{x : m_\alpha^1(x) = 0 \wedge \cdots \wedge m_\alpha^k(x) = 0\}$. It is assumed that $m_\alpha \cap m_\beta = \emptyset$, for all $\alpha, \beta \in Q \times Q, \alpha \neq \beta$, except in those cases where m_α is identified with its reverse ordered version $m_{\bar\alpha}$ giving $m_\alpha = m_{\bar\alpha}$. $\square$

A1: The initial state $h_0 := (q_0, x(t_0)) \in H$ is such that $m_{q_0, q_j}(x_0) \neq 0$, for all $q_j \in Q$. $\square$

III. HYBRID OPTIMAL CONTROL PROBLEMS

A2: Let $\{l_q\}_{q \in Q}$, $l_q \in C^{n_l}(\mathbb{R}^n \times U \to \mathbb{R}_+)$, $n_l \geq 1$, be a family of running cost functions; $\{c_\sigma\}_{\sigma \in \Sigma} \in C^{n_c}(\mathbb{R}^n \times \Sigma \to \mathbb{R}_+)$, $n_c \geq 1$, be a family of switching cost functions; and $g \in C^{n_g}(\mathbb{R}^n \to \mathbb{R}_+)$, $n_g \geq 1$, be a terminal cost function satisfying the following:

There exists $K_l < \infty$ and $1 \leq \gamma_l < \infty$ such that $|l_q(x, u)| \leq K_l(1 + \|x\|^{\gamma_l})$, for all $x \in \mathbb{R}^n$, $u \in U$, $q \in Q$.

There exists $K_c < \infty$ and $1 \leq \gamma_c < \infty$ such that $|c_\sigma(x)| \leq K_c(1 + \|x\|^{\gamma_c})$, for all $x \in \mathbb{R}^n$, $\sigma \in \Sigma$.

There exists $K_g < \infty$ and $1 \leq \gamma_g < \infty$ such that $|g(x)| \leq K_g(1 + \|x\|^{\gamma_g})$, for all $x \in \mathbb{R}^n$. $\square$

Consider the initial time t_0, final time $t_f < \infty$, initial hybrid state $h_0 = (q_0, x_0)$, and the upper-bound of maximum number of swithchings $\bar{L} < \infty$. Let

$$S_L = \{(t_0, id), (t_1, \sigma_{q_0 q_1}), \ldots, (t_L, \sigma_{q_{L-1} q_L})\} \\ \equiv \{(t_0, q_0), (t_1, q_1), \ldots, (t_L, q_L)\} \quad (2)$$

be a hybrid switching sequence and let $I_L := (S_L, u), u \in \mathscr{U}$ be a hybrid input trajectory which subject to A0 and A1 results in a (necessarily unique) hybrid state process (see [4]) and is such that L controlled and autonomous switchings occur on the time interval $[t_0, T(I_L)]$, where $T(I_L) \leq t_f$. In this paper, the number of switchings L is held fixed and we denote the corresponding set of inputs by $\{I_L\}$.

Define the *hybrid cost* on $[t_0, t_f]$ as

$$J(t_0, t_f, h_0, L; I_L) := \sum_{i=0}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s), u(s)) ds \\ + \sum_{j=1}^{L} c_{\sigma_{q_{j-1} q_j}}(x_{q_{j-1}}(t_j-)) + g(x_{q_L}(t_f)) \quad (3)$$

subject to

$$\dot{x}_{q_i}(t) = f_{q_i}(x_{q_i}(t), u(t)), \quad a.e.\ t \in [t_i, t_{i+1}), \quad (4)$$

$$h_0 = (q_0, x_{q_0}(t_0)) = (q_0, x_0), \quad (5)$$

$$x_{q_j}(t_j) = \xi\left(x_{q_{j-1}}(t_j-)\right) \equiv \xi\left(\lim_{t \uparrow t_j} x_{q_{j-1}}(t)\right) \quad (6)$$

where $0 \leq i \leq L$, $1 \leq j \leq L$, $t_{L+1} = t_f < \infty$ and $L + 2 \leq \bar{L} < \infty$.

Then the Hybrid Optimal Control Problem (HOCP) is to find the infimum $J^o(t_0, t_f, h_0, L)$ over the family of input trajectories $\{I_L\}$, i.e.

$$J^o(t_0, t_f, h_0, L) = \inf_{I_L} J(t_0, t_f, h_0, L; I_L) \quad (7)$$

$\square$

IV. HYBRID MINIMUM PRINCIPLE

Theorem 1 [29] Consider the hybrid system $\mathbb{H}$ together with the assumptions A0, A1 and A2 as above and the HOCP (7) for the hybrid cost (3). Define the family of system Hamiltonians by

$$H_{q_j}(x, \lambda, u) = \lambda^T f_{q_j}(x, u) + l_{q_j}(x, u) \quad (8)$$

for $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, $u \in U$, $q_j \in Q$. Then for the optimal switching sequence q^o and along the optimal trajectory x^o there exists an adjoint process λ^o such that

$$\dot{x}^o = \frac{\partial H_{q^o}}{\partial \lambda}(x^o, \lambda^o, u^o), \quad (9)$$

$$\dot{\lambda}^o = -\frac{\partial H_{q^o}}{\partial x}(x^o, \lambda^o, u^o) \quad (10)$$

almost everywhere $t \in [t_0, t_f]$ with

$$x^o(t_0) = x_0, \quad (11)$$

$$x^o(t_j) = \xi(x^o(t_j-)), \quad (12)$$

$$\lambda^o(t_f) = \nabla g(x^o(t_f)), \quad (13)$$

$$\lambda^o(t_j-) \equiv \lambda^o(t_j) = \nabla \xi^T \lambda^o(t_j+) + p\hat{n}_m + \nabla c_\sigma, \quad (14)$$

where $p = 0$ when t_j indicates the time of a controlled switching and $p \in \mathbb{R}$ when t_j indicates the time of an autonomous switching, and

$$\hat{n}_m \parallel \underset{\text{span}\{\nabla m^i\}}{\text{PROJ}} f_{q_{j-1}}(x^o(t_j-), u^o(t_j-)) \qquad (15)$$

i.e. $\hat{n}_m$ is a vector in $\mathbb{R}^n$ parallel to the projection of $f_{q_{j-1}}$ in the (generally non-orthogonal) vector space generated by the span of $\left\{\nabla m^i_{q_{j-1}q_j}\right\}$, $i \in \{1, \cdots, k\}$.

Moreover, the Hamiltonian is minimized with respect to the control input

$$H_{q^o}(x^o, \lambda^o, u^o) \leq H_{q^o}(x^o, \lambda^o, u) \qquad (16)$$

for all $u \in U$; and at a switching time t_j the Hamiltonian satisfies

$$H_{q_{j-1}}(t_j-) \equiv H_{q_{j-1}}(t_j) = H_{q_j}(t_j) \equiv H_{q_j}(t_j+) \qquad (17)$$

$\square$

V. HYBRID DYNAMIC PROGRAMMING AND ITS RELATION TO THE HYBRID MIMIMUM PRINCIPLE

In Hybrid Dynamic Programming the value function V evaluated at a time $t \in [t_0, t_f]$ and the state $h = (q, x)$ is defined as the optimal cost-to-go for the hybrid system (1) with the performance function (3). For simplicity of notation, in the statement of Hybrid Dynamic Programming, we use x instead of x^o in order to indicate that x refers to the general solution of the corresponding HOCP passing through it. We adapt the same notation for q^o, t^o_j, etc.

Theorem 2 [29] If at the instant t and the hybrid state (q, x) the value function V for the HOCP (3) is differentiable then it necessarily satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$-\frac{\partial V}{\partial t} - \inf_u \left\{ l_q(x, u) + \langle \nabla_x V, f_q(x, u) \rangle \right\} = 0 \qquad (18)$$

In addition, the value function satisfies the following terminal time condition

$$V(t_f, q_L, x, 0) = g(x) \qquad (19)$$

and the boundary conditions

$$V(t_j, q, x, L - j + 1)$$
$$= \min_\sigma \left\{ V(t_j, \Gamma(q, \sigma), \xi(x), L - j) + c_\sigma(x) \right\} \qquad (20)$$

subject to

$$m^1_\sigma(x) = 0, \cdots, m^k_\sigma(x) = 0 \qquad (21)$$

if t_j is a time of an autonomous switching; and

$$V(\tau, q, x, L - j + 1) \leq V(\tau, \Gamma(q, \sigma), \xi(x), L - j) + c_\sigma(x) \qquad (22)$$

with the equality achieved for $\tau = t_j$, the time of a controlled switching. $\square$

Theorem 3 [29] If in addition to the assumptions A0, A1 and A2 the functions f_q and l_q are continuously differentiable for all $q \in Q$, and the (necessarily Lipschitz) value function V is twice continuously differentiable almost everywhere

in Lebesgue sense on $\mathbb{R} \times \mathbb{R}^n$ then the adjoint process locally describes the gradient of the value function, i.e. at all Lebesgue points and times

$$\lambda^o(t) = \nabla_x V(t, q, x, .))|_{x = x^o} \qquad a.e. \ t \in [t_0, t_f] \qquad (23)$$

where x^o denotes the optimal trajectory passing through the point x and λ^o is the adjoint process corresponding to x^o. $\square$

VI. ILLUSTRATIVE EXAMPLE

A. Problem Formulation

Consider the following mechanical system with two point masses m_1 and m_2 each one attached to separate spring and damper with the configuration depicted in Figure 1. The spring and the damper attached to the mass m_1 have the stiffness and damping coefficients k_1 and c_1 respectively and apply forces to m_1 in the direction of the x axis and the spring and the damper attached to the mass m_2 have the stiffness and damping coefficients k_2 and c_2 respectively and apply forces to m_2 in the direction of the y axis. The neutral positions for the springs k_1 and k_2 have the coordinates $(d_1, 0)$ and $(0, d_2)$ respectively in the coordinate system shown in Figure 1. Denoting $x_1 := x$, $x_2 := \dot{x}$, $x_3 := y$ and $x_4 := \dot{y}$ the dynamics of the system is described as

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k_1}{m_1} x_1 - \frac{c_1}{m_1} x_2 + \frac{1}{m_1} u_1 + \frac{k_1}{m_1} d_1 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\frac{k_2}{m_2} x_3 - \frac{c_2}{m_2} x_4 + \frac{1}{m_2} u_2 + \frac{k_2}{m_2} d_2
\end{aligned} \qquad (24)$$

which has the matrix representation

$$\dot{x} = A_1 x + B_1 u + D_1 \qquad (25)$$

with

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1}{m_1} & \frac{-c_1}{m_1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-k_2}{m_2} & \frac{-c_2}{m_2} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}, D_1 = \begin{bmatrix} 0 \\ \frac{k_1}{m_1} d_1 \\ 0 \\ \frac{k_2}{m_2} d_2 \end{bmatrix} \qquad (26)$$

When both masses pass through the origin at the same time a collision occurs. Denoting the time of the collision by t_s this incident corresponds to a switching manifold in the form of a codimension 2 submanifold of $\mathbb{R}^4$ described by

$$m : \{x_1(t_s-) = 0 \wedge x_3(t_s-) = 0\} \qquad (27)$$

Consider a completely plastic collision in which the masses attach to each other and hence, the speeds after the collision determined by the law of conservation of linear momentum are related to speeds before the collision by

$$\begin{aligned}
(m_1 + m_2) v_x(t_s+) &\equiv (m_1 + m_2) v_x(t_s) = m_1 v_{1x}(t_s-) \\
(m_1 + m_2) v_y(t_s+) &\equiv (m_1 + m_2) v_y(t_s) = m_2 v_{2y}(t_s-)
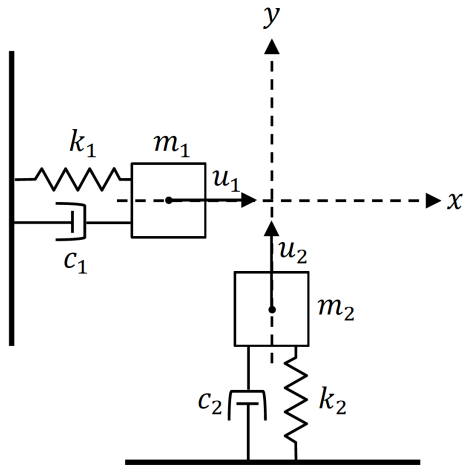\end{aligned} \qquad (28)$$

Fig. 1. The system studied in the example

that determines the corresponding autonomous jump map as

$$
\begin{bmatrix} x_1(t_s) \\ x_2(t_s) \\ x_3(t_s) \\ x_4(t_s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{m_1}{m_1+m_2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{m_2}{m_1+m_2} \end{bmatrix} \begin{bmatrix} x_1(t_s-) \\ x_2(t_s-) \\ x_3(t_s-) \\ x_4(t_s-) \end{bmatrix}
$$
(29)

Assuming decoupled stiffness and damping in the two directions (see e.g. [30], [31]) the dynamics of the system after the collision is described by

$$
\dot{x}_1 = x_2
$$
$$
\dot{x}_2 = -\frac{k_1}{m_1+m_2}x_1 - \frac{c_1}{m_1+m_2}x_2 + \frac{1}{m_1+m_2}u_1 + \frac{k_1}{m_1+m_2}d_1
$$
$$
\dot{x}_3 = x_4
$$
$$
\dot{x}_4 = -\frac{k_2}{m_1+m_2}x_3 - \frac{c_2}{m_1+m_2}x_4 + \frac{1}{m_1+m_2}u_2 + \frac{k_2}{m_1+m_2}d_2
$$
(30)

which has the matrix representation

$$
\dot{x} = A_2 x + B_2 u + D_2
$$
(31)

with

$$
A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1}{m_1+m_2} & \frac{-c_1}{m_1+m_2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-k_2}{m_1+m_2} & \frac{-c_2}{m_1+m_2} \end{bmatrix},
$$

$$
B_2 = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1+m_2} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_1+m_2} \end{bmatrix}, D_2 = \begin{bmatrix} 0 \\ \frac{k_1}{m_1+m_2}d_1 \\ 0 \\ \frac{k_2}{m_1+m_2}d_2 \end{bmatrix}
$$
(32)

For the hybrid system described above consider the optimal control problem

$$
J(x_0, T, u) = \int_0^T l(x,u)\,dt + c(x(t_s-)) + g(x(T))
$$
(33)

with the running costs

$$
l_1(x,u) = l_2(x,u) \equiv l(x,u) = \frac{1}{2}(u_1^2 + u_2^2) = \frac{1}{2}u^T u
$$
(34)

Take the switching cost as the kinetic energy just before switching (i.e. collision) which is

$$
c(x(t_s-)) = \frac{1}{2}m_1(x_2(t_s-))^2 + \frac{1}{2}m_2(x_4(t_s-))^2
$$
(35)

and assume that the terminal cost penalizes the total energy at the final time T, i.e.

$$
g(x(T)) = \frac{1}{2}(m_1+m_2)(x_2(T))^2 + \frac{1}{2}(m_1+m_2)(x_4(T))^2
$$
$$
+ \frac{1}{2}k_1(x_1(T)-d_1)^2 + \frac{1}{2}k_2(x_3(T)-d_2)^2
$$
(36)

Consequently, the hybrid optimal control problem is defined as finding the minimum of J in (33) and the corresponding minimizing control inputs for the given system.

B. The HMP Results

Employing Theorem 1, the Hamiltonian is formed as

$$
H_i(x,\lambda,u) = \lambda^T(A_i x + B_i u + D_i) + \frac{1}{2}u^T u
$$
(37)

The Hamiltonian minimization condition (16) gives

$$
\frac{\partial H_i}{\partial u} = 0 \Rightarrow u^o = -B_i^T \lambda^o
$$
(38)

and hence, from (9) and (10)

$$
\dot{x}^o = A_i x^o - B_i B_i^T \lambda^o + D_i
$$
(39)
$$
\dot{\lambda}^o = -A_i^T \lambda^o
$$
(40)

with the initial condition for x^o given as

$$
x^o(0) = x_0
$$
(41)

and its boundary condition (12) given as

$$
x(t_s) = P x(t_s-)
$$
(42)

where P is defined from (29) as

$$
P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{m_1}{m_1+m_2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{m_2}{m_1+m_2} \end{bmatrix}
$$
(43)

The terminal condition for λ^o is given from (13) as

$$
\lambda^o(T) = \nabla g(x(T)) = G(x - r_f)
$$
(44)

with G and r_f determined from (36) as

$$
G = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & m_1+m_2 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & m_1+m_2 \end{bmatrix}, r_f = \begin{bmatrix} d_1 \\ 0 \\ d_2 \\ 0 \end{bmatrix}
$$
(45)

The boundary condition for λ^o is determined by (14) as

$$
\lambda^o(t_j-) \equiv \lambda^o(t_j) = P^T \lambda^o(t_j+) + p\hat{n}_m + Cx
$$
(46)

with C defined from (35) as

$$
C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 \end{bmatrix}
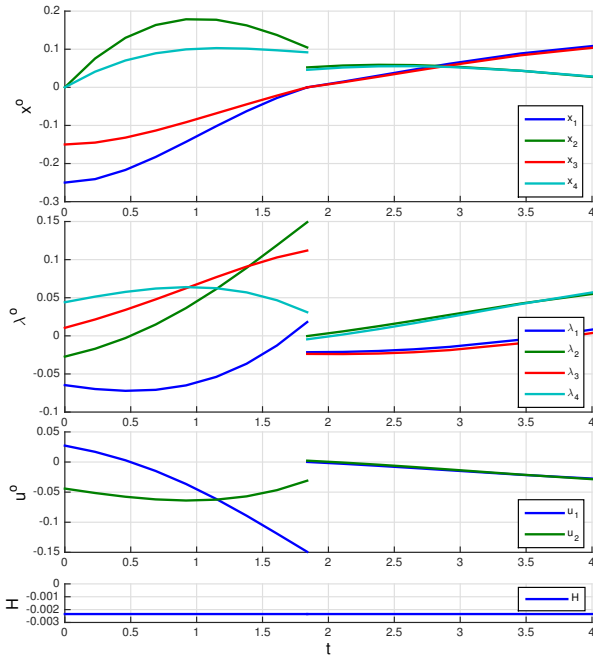$$
(47)

Fig. 2. The results for the parameter values $m_1 = m_2 = 1$, $k_1 = k_2 = 1$, $c_1 = c_2 = 1$, $d_1 = d_2 = 0.1$, the initial condition $x_0 = [-0.25, 0, -0.15, 0]^T$ and the terminal time $T = 4$

and $\hat{n}_m$ determined from (15) as

$$
\hat{n}_m \parallel \underset{\text{span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\}}{\text{PROJ}} \left\{A_1 x^o(t_s-) - B_1 B_1^T \lambda^o(t_s-) + D_1\right\}
$$

$$
= \begin{bmatrix} x_2(t_s-) \\ 0 \\ x_4(t_s-) \\ 0 \end{bmatrix} \quad (48)
$$

Taking $\hat{n}_m$ equal to its defining vector in (48), the boundary condition (46) becomes

$$
\begin{bmatrix} \lambda_1^o(t_s) \\ \lambda_2^o(t_s) \\ \lambda_3^o(t_s) \\ \lambda_4^o(t_s) \end{bmatrix} = \begin{bmatrix} \lambda_1^o(t_s+) + p x_2(t_s-) \\ \frac{m_1}{m_1+m_2}\lambda_2^o(t_s+) + m_1 x_2(t_s-) \\ \lambda_3^o(t_s+) + p x_4(t_s-) \\ \frac{m_2}{m_1+m_2}\lambda_4^o(t_s+) + m_2 x_4(t_s-) \end{bmatrix} \quad (49)
$$

The scalar parameter p and the switching time t_s together with the optimal trajectory and its corresponding adjoint process are determined by solving the differential equations (39) and (40) subject to the initial, terminal and boundary conditions (41), (42), (44) and (49) together with the Hamiltonian continuity condition from (17) as

$$
\lambda^{oT}_{(t_s+)}\left[A_2 x^o_{(t_s+)} - B_2 B_2^T \lambda^o_{(t_s+)} + D_2\right] + \frac{1}{2}\lambda^{oT}_{(t_s+)} B_1 B_1^T \lambda^o_{(t_s+)}
$$
$$
= \lambda^{oT}_{(t_s-)}\left[A_1 x^o_{(t_s-)} - B_1 B_1^T \lambda^o_{(t_s-)} + D_1\right] + \frac{1}{2}\lambda^{oT}_{(t_s-)} B_1 B_1^T \lambda^o_{(t_s-)} \quad (50)
$$

or

$$
\lambda^{oT}_{(t_s+)}\left[A_2 x^o_{(t_s+)} - \frac{1}{2}B_2 B_2^T \lambda^o_{(t_s+)} + D_2\right]
$$
$$
= \lambda^{oT}_{(t_s-)}\left[A_1 x^o_{(t_s-)} - \frac{1}{2}B_1 B_1^T \lambda^o_{(t_s-)} + D_1\right] \quad (51)
$$

The results for the parameter values $m_1 = m_2 = 1$, $k_1 = k_2 = 1$, $c_1 = c_2 = 1$, $d_1 = d_2 = 0.1$, the initial condition $x_0 = \begin{bmatrix} -0.25 & 0 & -0.15 & 0 \end{bmatrix}^T$ and the terminal time $T = 4$ are demonstrated in Figure 2.

C. HDP Results from their Relation to the HMP Results

Employing Theorem 3 and the results of Theorem 1 established in the previous part, we find the value function satisfying the necessary conditions in Theorem 2. To this end we rewrite equations (39) and (40) in the matrix form

$$
\begin{bmatrix} \dot{x}^o \\ \dot{\lambda}^o \end{bmatrix} = \begin{bmatrix} A_i & -B_i B_i^T \\ 0 & -A_i^T \end{bmatrix} \begin{bmatrix} x^o \\ \lambda^o \end{bmatrix} + \begin{bmatrix} D_i \\ 0 \end{bmatrix} \quad (52)
$$

and denote its state transition matrix by ϕ_i. Then the solution of (52) for $t \in (t_s, T]$ can be written as

$$
\begin{bmatrix} x^o(t) \\ \lambda^o(t) \end{bmatrix} = \phi_2(t, t_s)\begin{bmatrix} x^o(t_s) \\ \lambda^o(t_s+) \end{bmatrix} + \int_{t_s}^t \phi_2(t, \tau)\begin{bmatrix} D_2(\tau) \\ 0 \end{bmatrix} d\tau \quad (53)
$$

and also as

$$
\begin{bmatrix} x^o(T) \\ \lambda^o(T) \end{bmatrix} = \phi_2(T, t)\begin{bmatrix} x^o(t) \\ \lambda^o(t) \end{bmatrix} + \int_t^T \phi_2(T, \tau)\begin{bmatrix} D_2(\tau) \\ 0 \end{bmatrix} d\tau \quad (54)
$$

Partitioning ϕ in the form of

$$
\phi_2(T, t) = \begin{bmatrix} \phi_{2,11}(T, t) & \phi_{2,12}(T, t) \\ \phi_{2,21}(T, t) & \phi_{2,22}(T, t) \end{bmatrix} \quad (55)
$$

and denoting

$$
\begin{bmatrix} f_{d2,1}(t) \\ f_{d2,2}(t) \end{bmatrix} := \int_t^T \begin{bmatrix} \phi_{2,11}(T, t) & \phi_{2,12}(T, t) \\ \phi_{2,21}(T, t) & \phi_{2,22}(T, t) \end{bmatrix} \begin{bmatrix} D_2(\tau) \\ 0 \end{bmatrix} d\tau \quad (56)
$$

we can rewrite (54) as

$$
x^o(T) = \phi_{2,11}(T, t)x^o(t) + \phi_{2,12}(T, t)\lambda^o(t) + f_{d2,1}(t) \quad (57)
$$
$$
\lambda^o(T) = \phi_{2,21}(T, t)x^o(t) + \phi_{2,22}(T, t)\lambda^o(t) + f_{d2,2}(t) \quad (58)
$$

Substituting $x^o(T)$ and $\lambda^o(T)$ from (57) and (58) into (44) gives

$$
G\left(\phi_{2,11}(T, t)x^o(t) + \phi_{2,12}(T, t)\lambda^o(t) + f_{d2,1}(t) - r_f\right)
$$
$$
= \phi_{2,21}(T, t)x^o(t) + \phi_{2,22}(T, t)\lambda^o(t) + f_{d2,2}(t) \quad (59)
$$

or

$$
[G\phi_{2,11}(T, t) - \phi_{2,21}(T, t)]x^o(t) + Gf_{d2,1}(t) - Gr_f - f_{d2,2}(t)
$$
$$
= [\phi_{2,22}(T, t) - G\phi_{2,12}(T, t)]\lambda^o(t) \quad (60)
$$

that gives

$$
\lambda^o(t) =
$$
$$
[\phi_{2,22}(T, t) - G\phi_{2,12}(T, t)]^{-1}[G\phi_{2,11}(T, t) - \phi_{2,21}(T, t)]x^o(t)
$$
$$
+ [\phi_{2,22}(T, t) - G\phi_{2,12}(T, t)]^{-1}[Gf_{d2,1}(t) - Gr_f - f_{d2,2}(t)] \quad (61)
$$

The existence of the inverse in the previous equation is provided by a theorem of Kalman [32]. Defining

$$
K_2(t) := [\phi_{2,22}(T, t) - G\phi_{2,12}(T, t)]^{-1}[G\phi_{2,11}(T, t) - \phi_{2,21}(T, t)] \quad (62)
$$

and

$$s_2(t) := [\phi_{2,22}(T,t) - G\phi_{2,12}(T,t)]^{-1} [Gf_{d2,1}(t) - Gr_f - f_{d2,2}(t)] \tag{63}$$

the equation (61) is expressed as

$$\lambda^o(t) = K_2(t)x^o(t) + s_2(t), \qquad t \in (t_s, T] \tag{64}$$

with

$$K_2(T) = G \tag{65}$$

$$s_2(T) = -Gr_f \tag{66}$$

In particular, for the right limit at t_s we have

$$\lambda^o(t_s+) = K_2(t_s)x^o(t_s) + s_2(t_s) \tag{67}$$

Similarly, for the solution of (52) for $t \in [0, t_s)$ we have

$$x^o(t_s-) = \phi_{1,11}(t_s,t)x^o(t) + \phi_{1,12}(t_s,t)\lambda^o(t) + f_{d1,1}(t) \tag{68}$$

$$\lambda^o(t_s) = \phi_{1,21}(t_s,t)x^o(t) + \phi_{1,22}(t_s,t)\lambda^o(t) + f_{d1,2}(t) \tag{69}$$

with the definition of $f_{d1,1}(t)$ and $f_{d1,2}(t)$ for $t \in [0, t_s)$ being

$$\begin{bmatrix} f_{d1,1}(t) \\ f_{d1,2}(t) \end{bmatrix} := \int_t^{t_s} \begin{bmatrix} \phi_{1,11}(t_s,\tau) & \phi_{1,12}(t_s,\tau) \\ \phi_{1,21}(t_s,\tau) & \phi_{1,22}(t_s,\tau) \end{bmatrix} \begin{bmatrix} D_1(\tau) \\ 0 \end{bmatrix} d\tau \tag{70}$$

Using (42) and the boundary condition (46) we may write

$$\begin{aligned} \lambda^o(t_s) &= P^T\lambda^o(t_s+) + p\hat{n}_m + Cx^o(t_s-) \\ &= P^T[K_2(t_s)x^o(t_s) + s_2(t_s)] + p\hat{n}_m + Cx^o(t_s-) \\ &= [P^TK_2(t_s)P + C]x^o(t_s-) + P^Ts_2(t_s) + p\hat{n}_m \end{aligned} \tag{71}$$

Substituting $x^o(t_s-)$ and $\lambda^o(t_s)$ from equations (68) and (69) we get

$$\begin{aligned} \phi_{1,21}(t_s,t)x^o(t) + \phi_{1,22}(t_s,t)\lambda^o(t) + f_{d1,2}(t) = \\ [P^TK_2(t_s)P+C][\phi_{1,11}(t_s,t)x^o(t) + \phi_{1,12}(t_s,t)\lambda^o(t) + f_{d1,1}(t)] \\ + P^Ts_2(t_s) + p\hat{n}_m \end{aligned} \tag{72}$$

or

$$\begin{aligned} &[\phi_{1,22}(t_s,t) - [P^TK_2(t_s)P+C]\phi_{1,12}(t_s,t)]\lambda^o(t) \\ &= ([P^TK_2(t_s)P+C]\phi_{1,11}(t_s,t) - \phi_{1,21}(t_s,t))x^o(t) \\ &+ [P^TK_2(t_s)P+C]f_{d1,1}(t) - f_{d1,2}(t) + P^Ts_2(t_s) + p\hat{n}_m \end{aligned} \tag{73}$$

With the definition of

$$\begin{aligned} K_1(t) := &[\phi_{1,22}(t_s,t) - [P^TK_2(t_s)P+C]\phi_{1,12}(t_s,t)]^{-1} \\ &([P^TK_2(t_s)P+C]\phi_{1,11}(t_s,t) - \phi_{1,21}(t_s,t)) \end{aligned} \tag{74}$$

and

$$\begin{aligned} s_1(t) := &[\phi_{1,22}(t_s,t) - [P^TK_2(t_s)P+C]\phi_{1,12}(t_s,t)]^{-1} \\ &([P^TK_2(t_s)P+C]f_{d1,1}(t) - f_{d1,2}(t) + P^Ts_2(t_s) + p\hat{n}_m) \end{aligned} \tag{75}$$

it is concluded that

$$\lambda^o(t) = K_1(t)x^o(t) + s_1(t), \qquad t \in [0, t_s) \tag{76}$$

Note that the following relations hold by the definitions of $K_i(t)$ and $s_i(t)$:

$$K_1(t_s) = P^TK_2(t_s)P + C \tag{77}$$

$$s_1(t_s) = P^Ts_2(t_s) + p\hat{n}_m \tag{78}$$

Taking the time derivative of (64) and (76) it can be shown that

$$\dot{K}_i = K_iB_iB_i^TK_i - K_iA_i - A_i^TK_i \tag{79}$$

$$\dot{s}_i = -(A_i^T - K_iB_iB_i^T)s_i - K_iD_i \tag{80}$$

From equation (23) and the result of Theorem 3 the gradient of the value function is equal to the adjoint process and hence

$$V(t, q_2, x, 0) = \frac{1}{2}x^TK_2(t)x + s_2(t)^Tx + \alpha_2(t) \tag{81}$$

where from Theorem 2 and the terminal condition (19), $\alpha_2(T)$ should satisfy

$$\alpha_2(T) = \frac{1}{2}D_2^TD_2 \tag{82}$$

From Theorem 2 and the HJB equation (18) we must have

$$\begin{aligned} &\frac{1}{2}x^T\dot{K}_2x + \dot{s}_2^Tx + \dot{\alpha}_2 + \frac{1}{2}(K_2x+s_2)^TB_2B_2^T(K_2x+s_2) \\ &+ (K_2x+s_2)^T(A_2x - B_2B_2^T[K_2x+s_2] + D_2) = 0 \end{aligned} \tag{83}$$

which results in

$$\begin{aligned} &\frac{1}{2}x^T(\dot{K}_2 + K_2A_2 + A_2^TK_2 - K_2B_2B_2^TK_2)x \\ &\qquad (\dot{s}_2 + A_2^Ts_2 - K_2B_2B_2^Ts_2 + K_2D_2)^Tx \\ &\qquad\qquad + \dot{\alpha}_2 - \frac{1}{2}s_2^TB_2B_2^Ts_2 + s_2^TD_2 = 0 \end{aligned} \tag{84}$$

and hence (see also (79) and (80))

$$\dot{\alpha}_2 = \frac{1}{2}s_2^TB_2B_2^Ts_2 - s_2^TD_2, \qquad t \in (t_s, T] \tag{85}$$

Similarly

$$V(t, q_1, x, 1) = \frac{1}{2}x^TK_1(t)x + s_1(t)^Tx + \alpha_1(t) \tag{86}$$

concludes that

$$\dot{\alpha}_1 = \frac{1}{2}s_1^TB_1B_1^Ts_1 - s_1^TD_1, \qquad t \in [0, t_s) \tag{87}$$

which, together with (85), gives

$$\dot{\alpha}_i = \frac{1}{2}s_i^TB_iB_i^Ts_i - s_i^TD_i \tag{88}$$

For determining the boundary condition for $\alpha(t)$ at t_s we consider the boundary condition (20) for V that states

$$V(t_s-, q_1, x, 1) = V(t_s+, q_2, Px, 0) + \frac{1}{2}x^TCx \tag{89}$$

i.e.

$$\begin{aligned} &\frac{1}{2}x^TK_1(t_s-)x + s_1(t_s-)^Tx + \alpha_1(t_s-) \\ &= \frac{1}{2}x^T[P^TK_2(t_s+)P+C]x + s_2(t_s+)^TPx + \alpha_2(t_s+) \end{aligned} \tag{90}$$

From the boundary conditions for K_i and s_i in (77) and (78) we get

$$\alpha_1\left(t_s-\right) + p\hat{n}_m^T x = \alpha_2\left(t_s+\right) \tag{91}$$

but since for all $x \in \{x : m(x) = 0\}$

$$\hat{n}_m^T x = \begin{bmatrix} x_2\left(t_s-\right) & 0 & x_4\left(t_s-\right) & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = 0 \tag{92}$$

the boundary condition for $\alpha(t)$ at t_s becomes

$$\alpha_1\left(t_s-\right) \equiv \alpha_1\left(t_s\right) = \alpha_2\left(t_s\right) \equiv \alpha_2\left(t_s+\right) \tag{93}$$

Hence, the value function is constructed in the form of the equations (81) and (86) where K_i, s_i and α_i are respectively the solutions of (79), (80) and (88) with the terminal conditions (65), (66) and (82) and the boundary conditions (77), (78) and (93). $\qquad\square$

VII. Concluding Remarks

The generalization of the hybrid optimal control theory to include the class of hybrid systems with low dimensional switching manifolds makes possible the modelling and optimal control of a larger class of hybrid systems including collision and rendez-vous problems. In general, for N masses to meet in the same location simultaneously, the rendezvous point corresponds to a codimesion dN submanifold in $\mathbb{R}^{2dN}$, the state space corresponding to the dynamics of the N masses in the d-dimensional physical space, i.e. $3N$ constraints for the 3 dimensional space, $2N$ constraints for planar motions and N constraints for linear motions.

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