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On the Relation between the Minimum Principle and Dynamic Programming for Hybrid Systems

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***Abstract*— Hybrid optimal control problems are studied for systems where autonomous and controlled state jumps are allowed at the switching instants and in addition to running costs, switching between discrete states incurs costs. A key aspect of the analysis is the relationship between the Hamiltonian and the adjoint process in the Minimum Principle before and after the switching instants as well as the relationship between adjoint processes in the Minimum Principle and the gradient of the value function. In this paper we prove that under certain assumptions the adjoint process in the Hybrid Minimum Principle and the gradient of the value function in Hybrid Dynamic Programming are governed by the same dynamic equation and have the same boundary conditions and hence are identical to each other.**

I. INTRODUCTION

There is now an extensive literature on the optimal control of hybrid systems (see e.g. [1]–[11]). On one hand, the generalizations of the Pontryagin Maximum Principle (PMP), which is a necessary condition for optimality, results in the Hybrid Minimum Principle (HMP) [1]–[3], [6], [7], [12]–[14]. Namely, given the initial conditions and a sequence of autonomous or controlled switchings, the HMP gives necessary conditions for the optimality of the trajectory and the control inputs of a given hybrid system. These conditions are expressed in terms of the minimization of the distinct Hamiltonians defined along the sequence of the discrete states of the hybrid trajectory. A feature of special interest is the boundary conditions on the adjoint processes and the Hamiltonian functions at autonomous and controlled switching times and states; these boundary conditions may be viewed as a generalization of the optimal control case of the Erdmann-Weierstrass conditions of the calculus of variations [15].

On the other hand, Dynamic Programming (DP) provides sufficient conditions for optimality based upon the Dynamic Programming Principle [16], [17]. With the exception of Hybrid Dynamic Programming (HDP) for regional dynamic systems [18], [19], the discretized version of HDP for continuous systems [20], [21] and the verification theorem in [22], the current generalizations of Dynamic Programming to hybrid systems are formulated for systems that undergo jumps at autonomous and controlled switching times [8]–[11]. However, the assumed HDP jump condition [8]–[11], which apparently is restrictive due to the requirement of the system to jump to a certain set, does not appear in the HMP formulation. In past works of the authors (see [23], [24]) the results of the HMP were given in the general case where autonomous and controlled state jumps are allowed at the switching instants and, in addition to running costs, it is assumed that switching between discrete states incurs costs. Moreover, it was stated in [24] that under certain conditions, the adjoint process in the Minimum Principle and the gradient of the value function in Dynamic Programming are equal and an analytic example was provided. In this paper, we give a proof for this relationship by showing that the adjoint process in the Hybrid Minimum Principle and the gradient of the value function in Hybrid Dynamic Programming are governed by the same dynamic equation and have the same boundary conditions and hence are identical to each other.

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II. HYBRID SYSTEMS

A *hybrid system (structure)* $\mathbb{H}$ is a septuple

$$\mathbb{H} = \{H := Q \times \mathbb{R}^n, I := \Sigma \times U, \Gamma, A, F, \Xi, \mathscr{M}\} \qquad (1)$$

where the symbols in the expression are defined as below.

A0: $Q = \{1, 2, ..., |Q|\} \equiv \{q_1, q_2, ..., q_{|Q|}\}, |Q| < \infty$, is a finite set of *discrete states (components)*.

$H := Q \times \mathbb{R}^n$ is called the *(hybrid) state space* of the hybrid system $\mathbb{H}$.

$I := \Sigma \times U$ is the set of system input values, where $|\Sigma| < \infty$.

$\Gamma : H \times \Sigma \to H$ is a time independent (partially defined) *discrete state transition map* which is the identity on the second ($\mathbb{R}^n$) component.

$\Xi : H \times \Sigma \to H$ is a time independent (partially defined) *continuous state jump transition map* which is the identity on the first (Q) component. All $\xi_\sigma \in \Xi$ are assumed to be injective and continuously differentiable in the continuous valued state x.

$A : Q \times \Sigma \to Q$ denotes both a finite automaton and the automaton's associated transition function on the state space Q and event set Σ, such that for a discrete state $q \in Q$ only the discrete controlled and uncontrolled transitions into the q-dependant subset $\{A(q, \sigma), \sigma \in \Sigma\} \subset Q$ occur under the projection of Γ on its Q components: $\Gamma : Q \times \mathbb{R}^n \times \Sigma \to H|_Q$. In other words, Γ can only make a discrete state transition in a hybrid state (q, x) if the automaton A can make the corresponding transition in q.

$U \subset \mathbb{R}^m$ is the set of *admissible input control values*, where U is an open bounded set in $\mathbb{R}^m$ which necessarily has compact closure $\bar{U}$.

$\mathscr{U}(U) := L_\infty([t_0, T_*), U)$, which is the set of all measurable functions that are bounded up to a set of measure zero on $[t_0, T_*), T_* < \infty$. The boundedness property necessarily holds since admissible input functions take values in the open bounded set U.

F is an indexed collection of *vector fields* $\{f_q\}_{q\in Q}$ such that $f_q \in C^k(\mathbb{R}^n \times U \to \mathbb{R}^n), k \geq 1$, satisfies a uniform^x Lipschitz condition, i.e. there exists $L_f < \infty$ such that $\|f_q(x_1,u) - f_q(x_2,u)\| \leq L_f \|x_1 - x_2\|,\ x_1,x_2 \in \mathbb{R}^n,\ u \in U$, $j \in Q$. We also assume that there exists $K_f < \infty$ such that $\max_{q\in Q}\left(\sup_{u\in U}(\|f_q(0,u)\|)\right) \leq K_f$.

$\mathscr{M} = \{\tilde{m}_\alpha^k : \alpha \in Q\times Q, k \in \mathbb{Z}_+\}$ denotes a collection of *switching manifold components*, also called *guard components*, such that, for any ordered pair $\alpha = (p,q)$, $\tilde{m}_\alpha^k$ is a smooth, i.e. C^∞ codimension 1 sub-manifold of $\mathbb{R}^n$, possibly with boundary $\partial\tilde{m}_\alpha^k$, described locally by $\tilde{m}_\alpha^k = \{x : \tilde{m}_\alpha^k(x) = 0\}$. It is assumed that $\tilde{m}_\alpha^k \cap \tilde{m}_\beta^k = \emptyset$, for all $\alpha,\beta \in Q\times Q, \alpha \neq \beta$, $k,l \in \mathbb{Z}_+$, except in those cases where, for some j, $\tilde{m}_\alpha^j$ is identified with its reverse ordered version $\tilde{m}_{\bar{\alpha}}^j$ giving $\tilde{m}_\alpha^j = \tilde{m}_{\bar{\alpha}}^j$. □

A1: The initial state $h_0 := (q_0, x(t_0)) \in H$ is such that $m_{q_0,q_j}(x_0) \neq 0$, for all $q_j \in Q$. □

III. Hybrid Optimal Control Problem

A2: Let $\{l_q\}_{q\in Q}, l_q \in C^{n_l}(\mathbb{R}^n \times U \to \mathbb{R}_+), n_l \geq 1$, be a family of cost functions; $\{c_\sigma\}_{\sigma\in\Sigma} \in C^{n_c}(\mathbb{R}^n \times \Sigma \to \mathbb{R}_+), n_c \geq 1$, be a family of switching cost functions; and $g \in C^{n_g}(\mathbb{R}^n \to \mathbb{R}_+), n_g \geq 1$, be a terminal cost function satisfying the following:

There exists $K_l < \infty$ and $1 \leq \gamma_l < \infty$ such that $|l_q(x,u)| \leq K_l\left(1 + \|x\|^{\gamma_l}\right)$, $x \in \mathbb{R}^n, u \in U, q \in Q$.

There exists $K_c < \infty$ and $1 \leq \gamma_c < \infty$ such that $|c_\sigma(x)| \leq K_c\left(1 + \|x\|^{\gamma_c}\right)$, $x \in \mathbb{R}^n, \sigma \in \Sigma$.

There exists $K_g < \infty$ and $1 \leq \gamma_g < \infty$ such that $|g(x)| \leq K_g\left(1 + \|x\|^{\gamma_g}\right)$, $x \in \mathbb{R}^n$. □

Consider the initial time t_0, final time $t_f < \infty$, initial hybrid state $h_0 = (q_0, x_0)$, and the upper-bound of maximum number of swithchings $\bar{L} < \infty$. Let

$$\begin{aligned} S_L &= \left\{(t_0, id), \left(t_1, \sigma_{q_0 q_1}\right), \ldots, \left(t_L, \sigma_{q_{L-1} q_L}\right)\right\} \\ &\equiv \left\{(t_0, q_0), (t_1, q_1), \ldots, (t_L, q_L)\right\} \end{aligned}$$

be a hybrid switching sequence and let $I_L := (S_L, u), u \in \mathscr{U}$, where $\mathscr{U} = \mathscr{U}^o$ or $\mathscr{U} = \mathscr{U}^{cpt}$, be a hybrid input trajectory which subject to A0 and A1 results in a (necessarily unique) hybrid state process (see [3]) and is such that $L + 2 < \bar{L}$ controlled and autonomous switchings occur on the time interval $[t_0, T(I_L)]$, where $T(I_L) \leq t_f$. In this paper, the number of switchings L is held fixed and we denote the corresponding set of inputs by $\{I_L\}$. Define the *hybrid cost function* on $[t_0, t_f]$ as

$$\begin{aligned} &J\left(t_0, t_f, h_0, L; I_L\right) := \\ &\textstyle\sum_{i=0}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}\left(x_{q_i}(s), u(s)\right) ds \\ &+ \textstyle\sum_{j=1}^{L} c_{\sigma_{q_{j-1} q_j}}\left(t_j, x_{q_{j-1}}(t_j-)\right) + g\left(x_{q_L}(t_f)\right) \end{aligned} \tag{2}$$

subject to

$$\begin{aligned} &\dot{x}_{q_i}(t) = f_{q_i}\left(x_{q_i}(t), u(t)\right), a.e. t \in [t_i, t_{i+1}), \\ &h_0 = \left(q_0, x_{q_0}(t_0)\right) = (q_0, x_0), \\ &x_{q_j}(t_j) = \xi\left(x_{q_{j-1}}(t_j-)\right) \equiv \xi\left(\lim_{t\uparrow t_j} x_{q_{j-1}}(t)\right) \end{aligned} \tag{3}$$

where $0 \leq i \leq L$, $1 \leq j \leq L$, $t_{L+1} = t_f < \infty$ and $L + 2 \leq \bar{L} < \infty$.

Then the Hybrid Optimal Control Problem (HOCP) is to find the infimum $J^o\left(t_0, t_f, h_0, L\right)$ over the family of input trajectories $\{I_L\}$, i.e.

$$J^o\left(t_0, t_f, h_0, L\right) = \inf_{I_L} J\left(t_0, t_f, h_0, L; I_L\right) \tag{4}$$

IV. Hybrid Minimum Principle

Theorem 1 [25] Consider the hybrid system $\mathbb{H}$ together with the assumptions A0, A1 and A2 as above and the HOCP (4). In addition, assume that in any discrete state $q_j \in Q$ the system (3) is locally state-to-state controllable on any time interval. Define the family of system Hamiltonians by

$$H_{q_j}(x, \lambda, u) = \lambda^T f_{q_j}(x, u) + l_{q_j}(x, u) \tag{5}$$

$x, \lambda \in \mathbb{R}^n, u \in U, q_j \in Q$. Assume that the optimal control u^o is such that $u^o(t) \in U$ *a.e.* $t \in [t_0, t_f]$ and consider the optimal value for the cost function $J\left(t_0, t_f, h_0, L; I_L\right)$

$$\begin{aligned} &J^o\left(t_0, t_f, h_0, L\right) = \\ &\textstyle\sum_{i=0}^{L} \int_{s=t_i}^{t_{i+1}} l_{q_i}\left(x_{q_i}^o(s), u^o(s)\right) ds \\ &+ \textstyle\sum_{i=1}^{L} c_{\sigma_{q_{i-1} q_i}}\left(t_i, x_{q_{i-1}}^o(t_i-)\right) + g\left(x_{q_L}^o(t_f)\right) \end{aligned} \tag{6}$$

Then along the optimal trajectory q^o, x^o, there exists an adjoint process λ^o for which

$$\dot{\lambda}^o = -\frac{\partial H_{q^o}}{\partial x}\left(x^o, \lambda^o, u^o\right), \ a.e.\ t \in [t_0, t_f] \tag{7}$$

$$\lambda^o(t_f) = \nabla g\left(x^o(t_f)\right), \tag{8}$$

and

$$\lambda^o(t_j-) \equiv \lambda^o(t_j) = \nabla\xi^T \lambda^o(t_j+) + p\nabla m + \nabla c_\sigma, \tag{9}$$

with $p \in \mathbb{R}$ when t_j indicates the time of an autonomous switching, and $p = 0$ when t_j indicates the time of a controlled switching. Moreover, the Hamiltonian is minimized with respect to the control input

$$H\left(q^o, x^o, \lambda^o, \sigma^o, u^o\right) \leq H\left(q^o, x^o, \lambda^o, \sigma^o, u\right) \tag{10}$$

for all $u \in U$; and at a switching time t_j the Hamiltonian satisfies

$$H_{q_{j-1}}(t_j-) = H_{q_j}(t_j+) + p\frac{\partial m}{\partial t} + \frac{\partial c_\sigma}{\partial t} \tag{11}$$

□

V. HYBRID DYNAMIC PROGRAMMING

In Hybrid Dynamic Programming the value function V evaluated at a time $\tau \in [t_0, t_f]$ and the state $h = (q, x)$ is defined as the optimal cost-to-go for the hybrid system (1) with the performance function (6). For simplicity of notation, in the rest of the paper and unless otherwise stated, we use x instead of x^o in order to indicate that x refers to the general solution of the corresponding HOCP passing through it. Then the Hybrid Dynamic Programming Theorem [25] states that for $q_j \in Q$ which corresponds to a time interval $(t_{j-1}, t_j]$, and for any $\tau \in (t_{j-1}, t_j]$, it is the case that

$$\begin{aligned} V(\tau, q, x(\tau), L-j) = \inf_u \{ & \textstyle\int_\tau^{t_j} l_{q_{j-1}}(x,u)\,ds \\ & + \textstyle\sum_{i=j}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}(x,u)\,ds \\ & + \textstyle\sum_{i=j}^{L} c_{\sigma_{q_{i-1}q_i}}\left(t_i, x_{q_{i-1}}(t_i-)\right) + g\left(x_{q_L}(t_f)\right)\} \end{aligned} \quad (12)$$

Along an optimal trajectory q, x, u the left limit of the value function will be denoted by

$$V(t_j-, q_{j-1}, x, L-j+1) = \lim_{t \uparrow t_j} V(t, q_{j-1}, x, L-j+1) \quad (13)$$

VI. THE RELATIONSHIP BETWEEN THE MINIMUM PRINCIPLE AND DYNAMIC PROGRAMMING

It is known that in classical optimal control the gradient of the value function in Dynamic programming is equal to the adjoint process in the Minimum Principle (or the negative of the co-state in the Pontryagin Maximum Principle) under certain assumptions [17], [26]. In this section, we prove in Theorem 2 that this relationship holds almost everywhere between the gradient of the value function in Hybrid Dynamic Programming and the adjoint process in the Hybrid Minimum Principle. Before proceeding to Theorem 2, we shall give a formal definition of Mayer HOCP.

Definition 1. Mayer Hybrid Optimal Control Problem (i): Define the extended continuous state

$$\hat{x}_q := \begin{bmatrix} x_q \\ z_q \end{bmatrix} \quad (14)$$

such that the extended vector fields become

$$\dot{\hat{x}}_q = \hat{f}_q(\hat{x}, u) := \begin{bmatrix} f_q(x,u) \\ l_q(x,u) \end{bmatrix} \quad (15)$$

with the initial condition

$$\hat{h}_0 = (q_0, \hat{x}_{q_0}(t_0)) = \left(q_0, \begin{bmatrix} x_0 \\ 0 \end{bmatrix}\right) \quad (16)$$

and the switching boundary condition governed by the extended jump function defined as

$$\hat{x}_{q_j}(t_j) = \hat{\xi}_{(i)}\left(\hat{x}_{q_{j-1}}(t_j-)\right) := \begin{bmatrix} \xi\left(x_{q_{j-1}}(t_j-)\right) \\ z_{q_{j-1}}(t_j-) \end{bmatrix} \quad (17)$$

Then the Bolza HOCP (2) turns into the Mayer HOCP (i) with

$$J(t_0, t_f, \hat{h}_0, L; I_L) := \hat{g}\left(\hat{x}_{q_L}(t_f)\right) + \sum_{i=1}^{L} c_\sigma\left(t_i, x_{q_{i-1}}(t_i-)\right) \quad (18)$$

where $\hat{g}\left(\hat{x}_{q_L}(t_f)\right) = z(t_f) + g\left(x(t_f)\right)$ □

Definition 2. Mayer Hybrid Optimal Control Problem (ii): Define the extended state $\hat{x}_q$ as (14) and the extended vector field $\hat{f}_q$ as (15) with the initial condition (16); and define the extended jump function as

$$\hat{x}(t_j) = \hat{\xi}_{(ii)}(\hat{x}(t_j-)) := \begin{bmatrix} \xi(x(t_j-)) \\ z(t_j-) + c(x(t_j-)) \end{bmatrix} \quad (19)$$

Then the Bolza HOCP (2) turns into the Mayer HOCP (ii) with

$$J(t_0, t_f, \hat{h}_0, L; I_L) := \hat{g}\left(\hat{x}_{q_L}(t_f)\right) \quad (20)$$

where $\hat{g}\left(\hat{x}_{q_L}(t_f)\right) = z(t_f) + g\left(x(t_f)\right)$ □

Lemma 1 The extended adjoint processes in definitions (i) and (ii) of the Mayer HOCP are both equal to

$$\hat{\lambda}(t) = \begin{bmatrix} \lambda(t) \\ 1 \end{bmatrix} \quad (21)$$

where $\lambda(t)$ is the adjoint process for the Bolza HOCP (2). □

Proof: For both definitions the extended Hamiltonian is

$$\hat{H} = \hat{\lambda}^T \hat{f}_q(\hat{x}, u) = \begin{bmatrix} \lambda^T & \lambda_{n+1} \end{bmatrix} \begin{bmatrix} f_q(x,u) \\ l_q(x,u) \end{bmatrix} \quad (22)$$

The extended adjoint process dynamics is then

$$\begin{aligned} \dot{\hat{\lambda}} &= -\frac{\partial \hat{H}}{\partial \hat{x}} = \begin{bmatrix} -\frac{\partial \hat{H}}{\partial x} \\ -\frac{\partial \hat{H}}{\partial z} \end{bmatrix} = \begin{bmatrix} -\frac{\partial \hat{H}}{\partial x} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{\partial}{\partial x}\left(\lambda^T f_q(x,u) + l_q(x,u)\right) \\ 0 \end{bmatrix} \end{aligned} \quad (23)$$

with the terminal condition

$$\hat{\lambda}(t_f) = \nabla_{\hat{x}} \hat{g}\left(\hat{x}(t_f)\right) = \begin{bmatrix} \nabla_x \hat{g}\left(\hat{x}(t_f)\right) \\ \nabla_z \hat{g}\left(\hat{x}(t_f)\right) \end{bmatrix} = \begin{bmatrix} \nabla_x g\left(x(t_f)\right) \\ 1 \end{bmatrix} \quad (24)$$

For definition (i) the boundary conditions for the extended adjoint process is

$$\begin{aligned} \hat{\lambda}(t_j-) \equiv \hat{\lambda}(t_j) = & \nabla_{\hat{x}} \hat{\xi}_{(i)}\Big|_{\hat{x}(t_j-)}^T \hat{\lambda}(t_j+) \\ & + p\, \nabla_{\hat{x}} \hat{m}\big|_{x(t_j-)} + \nabla_{\hat{x}} \hat{c}\big|_{x(t_j-)} \end{aligned} \quad (25)$$

which is

$$\begin{aligned} \hat{\lambda}(t_j-) \equiv \hat{\lambda}(t_j) &= \begin{bmatrix} \frac{\partial \hat{\xi}_{(i)}}{\partial x} \\ \frac{\partial \hat{\xi}_{(i)}}{\partial z} \end{bmatrix}^T \hat{\lambda}(t_j+) + p \nabla_{\hat{x}} \hat{m} + \nabla_{\hat{x}} \hat{c} \\ &= \begin{bmatrix} \frac{\partial \xi_1}{\partial x_1} & \cdots & \frac{\partial \xi_1}{\partial x_n} & \frac{\partial \xi_1}{\partial z} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial \xi_n}{\partial x_1} & \cdots & \frac{\partial \xi_n}{\partial x_n} & \frac{\partial \xi_n}{\partial z} \\ \frac{\partial z}{\partial x_1} & \cdots & \frac{\partial z}{\partial x_n} & \frac{\partial z}{\partial z} \end{bmatrix}^T \hat{\lambda}(t_j+) + p \nabla_{\hat{x}} \hat{m} + \nabla_{\hat{x}} \hat{c} \\ &= \begin{bmatrix} \frac{\partial \xi_1}{\partial x_1} & \cdots & \frac{\partial \xi_1}{\partial x_n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial \xi_n}{\partial x_1} & \cdots & \frac{\partial \xi_n}{\partial x_n} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}^T \hat{\lambda}(t_j+) + \begin{bmatrix} p \nabla m + \nabla c \\ 0 \end{bmatrix} \end{aligned} \quad (26)$$

This gives

$$\hat{\lambda}(t_j-) = \begin{bmatrix} \lambda(t_j-) \\ \lambda_{n+1}(t_j-) \end{bmatrix} \equiv \begin{bmatrix} \lambda(t_j) \\ \lambda_{n+1}(t_j) \end{bmatrix}$$
$$= \begin{bmatrix} \nabla\xi^T & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda(t_j+) \\ \lambda_{n+1}(t_j+) \end{bmatrix} + \begin{bmatrix} p\nabla m + \nabla c \\ 0 \end{bmatrix} \tag{27}$$
$$= \begin{bmatrix} \nabla\xi|_{x(t_j-)}^T \lambda(t_j+) + \nabla c + p\nabla m \\ \lambda_{n+1}(t_j+) \end{bmatrix}$$

From the zero dynamics for λ_{n+1} (Eq. (23)) with the terminal condition being equal to 1 (Eq. (24)) and continuity conditions at switching times (Eq. (27)) it is concluded that $\lambda_{n+1}(t) = 1$ for all $t \in [t_0, t_f]$.

Since in definition (ii) switching costs are embedded in the extended jump functions, extended switching costs are zero giving the boundary conditions (9) as

$$\hat{\lambda}(t_j-) \equiv \hat{\lambda}(t_j) = \nabla\hat{\xi}_{(ii)}\Big|_{\hat{x}(t_j-)}^T \hat{\lambda}(t_j+) + p\,\nabla\hat{m}|_{\hat{x}(t_j-)} \tag{28}$$

which is

$$\hat{\lambda}(t_j-) \equiv \hat{\lambda}(t_j) = \begin{bmatrix} \frac{\partial\hat{\xi}_{(ii)}}{\partial x} \\ \frac{\partial\hat{\xi}_{(ii)}}{\partial z} \end{bmatrix}^T \hat{\lambda}(t_j+) + p\nabla\hat{m}$$
$$= \begin{bmatrix} \frac{\partial\xi_1}{\partial x_1} & \cdots & \frac{\partial\xi_1}{\partial x_n} & \frac{\partial\xi_1}{\partial z} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial\xi_n}{\partial x_1} & \cdots & \frac{\partial\xi_n}{\partial x_n} & \frac{\partial\xi_n}{\partial z} \\ \frac{\partial[z+c]}{\partial x_1} & \cdots & \frac{\partial[z+c]}{\partial x_n} & \frac{\partial[z+c]}{\partial z} \end{bmatrix}^T \hat{\lambda}(t_j+) + p\nabla\hat{m}$$
$$= \begin{bmatrix} \frac{\partial\xi_1}{\partial x_1} & \cdots & \frac{\partial\xi_1}{\partial x_n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial\xi_n}{\partial x_1} & \cdots & \frac{\partial\xi_n}{\partial x_n} & 0 \\ \frac{\partial c}{\partial x_1} & \cdots & \frac{\partial c}{\partial x_n} & 1 \end{bmatrix}^T \hat{\lambda}(t_j+) + p\begin{bmatrix} \nabla m \\ 0 \end{bmatrix} \tag{29}$$

This gives

$$\hat{\lambda}(t_j-) = \begin{bmatrix} \lambda(t_j-) \\ 1 \end{bmatrix} \equiv \begin{bmatrix} \lambda(t_j) \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} \nabla\xi^T & \nabla c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda(t_j+) \\ 1 \end{bmatrix} + p\begin{bmatrix} \nabla m \\ 0 \end{bmatrix} \tag{30}$$
$$= \begin{bmatrix} \nabla\xi|_{x(t_j-)}^T \lambda(t_j+) + \nabla c + p\nabla m \\ 1 \end{bmatrix}$$

which is the same as Eq. (27). The adjoint processes for both definitions of the Mayer HOCP are governed by the same dynamics (23) and specified by the same terminal condition (24) and the same boundary conditions (27) and (30); hence the lemma is proved. ■

Theorem 2 Under the assumptions of Theorem 1, with the additional assumptions that the value function V, f_q and l_q are twice continuously differentiable for all $q \in Q$, the adjoint process for the optimal switching sequence and the gradient of the value function are governed by the same differential equations, i.e.

$$\frac{d}{dt}\lambda^o = -\left(\lambda^{oT}\frac{\partial f_{q^o}}{\partial x} + \frac{\partial l_{q^o}}{\partial x}\right)(x^o, u^o) \tag{31}$$

and

$$\frac{d}{dt}\nabla_x V = -\left(\nabla_x V^T\frac{\partial f_{q^o}}{\partial x} + \frac{\partial l_{q^o}}{\partial x}\right)(x^o, u^o) \tag{32}$$

and have the same boundary conditions, i.e.

$$\lambda^o(t_f) = \nabla g(x^o(t_f)) \tag{33}$$

and

$$\lambda^o(t_j-) = \nabla\xi|_{x(t_j-)}^T \lambda^o(t_j+) + p\,\nabla m|_{x(t_j-)} + \nabla c|_{x(t_j-)} \tag{34}$$

for the adjoint process, and

$$\nabla_x V(t_f, q^o, x(t_f), 0) = \nabla g(x^o(t_f)) \tag{35}$$

and

$$\begin{aligned} &\nabla V(t_j-, q_{j-1}, x(t_j-), L-j+1) \\ &= \nabla\xi|_{x(t_j-)}^T \nabla V(t_j+, q_j, x(t_j+), L-j) \\ &\quad + p\,\nabla m|_{x(t_j-)} + \nabla c|_{x(t_j-)} \end{aligned} \tag{36}$$

for the gradient of the value function. Hence, from the uniqueness resulting from continuous differentiability of the differential equations (31) and (32), the adjoint process and the gradient of the value function are almost everywhere identical on an optimal trajectory, i.e.

$$\lambda^o = \nabla_x V \quad a.e.\ t \in [t_0, t_f] \tag{37}$$

□

Proof: Eq. (31) is a direct result of the HMP in Theorem 1. The proof of Eq. (32) follows the same procedure as in [26]. Under the assumption of the theorem, the value function V is continuously differentiable locally around $x^o_{q^o}(t)$ a.e. $t \in [t_j, t_{j+1}]$ for $j = 0, 1, \cdots, L$, thus it satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$\nabla_t V + \min_u \{l_{q^o}(x,u) + \langle\nabla_x V, f_{q^o}(x,u)\rangle\} = 0 \tag{38}$$

with the terminal condition

$$V(t_f, q^o, x(t_f), 0) = g(x^o(t_f)) \tag{39}$$

In particular, on an optimal hybrid trajectory (q^o, x^o) which is minimized locally with the optimal control input (σ^o, u^o); for all $u \in U$

$$\begin{aligned} &(\nabla_t V + l_{q^o} + \langle\nabla_x V, f_{q^o}\rangle)(x^o, u^o) \\ &\le (\nabla_t V + l_{q^o} + \langle\nabla_x V, f_{q^o}\rangle)(x^o, u) \end{aligned} \tag{40}$$

must hold, and (see [26]) for all x in a neighborhood of x^o we have

$$\begin{aligned} &(\nabla_t V + l_{q^o} + \langle\nabla_x V, f_{q^o}\rangle)(x^o, u^o) \\ &\le (\nabla_t V + l_{q^o} + \langle\nabla_x V, f_{q^o}\rangle)(x, u^o) \end{aligned} \tag{41}$$

i.e.

$$x^o = \arg\min_x (\nabla_t V + l_{q^o} + \langle\nabla_x V, f_{q^o}\rangle)(x, u^o) \tag{42}$$

Assuming that the argument in (42) is continuously differentiable in x, Eq. (42) is equivalent to

$$\frac{\partial}{\partial x}(\nabla_t V + l_{q^o} + \langle\nabla_x V, f_{q^o}\rangle)(x, u^o) = 0 \tag{43}$$

on the optimal trajectory x^o. Since V, f_{q^o} and l_{q^o} are assumed to be twice continuously differentiable we may take the partial derivative inside and write

$$\frac{\partial^2 V}{\partial x \partial t} + \frac{\partial l_{q^o}}{\partial x} + \left\langle \frac{\partial^2 V}{\partial x^2}, f_{q^o} \right\rangle + \left\langle \frac{\partial V}{\partial x}, \frac{\partial f_{q^o}}{\partial x} \right\rangle = 0 \quad (44)$$

But from the definition of the total derivative, we have

$$\frac{d}{dt}\frac{\partial V}{\partial x} = \frac{\partial^2 V}{\partial t \partial x} + \left\langle \frac{\partial^2 V}{\partial x^2}, f_{q^o} \right\rangle = \frac{\partial^2 V}{\partial x \partial t} + \left\langle \frac{\partial^2 V}{\partial x^2}, f_{q^o} \right\rangle \quad (45)$$

where the second equality holds due to the fact that V is twice continuously differentiable. Thus from (44) and (45) we get

$$\frac{d}{dt}\frac{\partial V}{\partial x} = -\left(\frac{\partial l_{q^o}}{\partial x} + \left\langle \frac{\partial V}{\partial x}, \frac{\partial f_{q^o}}{\partial x} \right\rangle \right) \quad (46)$$

which gives (32).

For the terminal and boundary conditions, we note that Eq. (33) and Eq. (34) are results of the HMP and Eq. (35) is simply derived by taking the gradient of (39). To show the boundary condition (36) for the value function, consider the Mayer HOCP equivalent to the HOCP for (6) with the switching cost included in the extended jump function as in Definition 2 and Eq. (19). For simplicity of notation we remove the hat ($\hat{}$) symbol from the letter symbols in the Mayer representation of the system and simply write x, λ, ξ, etc. instead of $\hat{x}, \hat{\lambda}, \hat{\xi}$, etc.

For t_j, the j^{th} switching time, assume an autonomous switching event with its corresponding switching manifold $m \equiv m_{q_{j-1}q_j}(x) = 0$. The analysis for the controlled switching case is simply performed by removing the constraint condition given by m. Consider a reference trajectory $x(t)$ that intersects the switching manifold at t_j and an adjacent trajectory $x'(t)$ that intersects the switching manifold at time $t_j + \delta t$ as depicted in Fig. 1. The choice of x' is such that $\|\delta x(t)\| := \|x'(t) - x(t)\| < \varepsilon$, $t \in [t_0, t_j) \cup [t_j + \delta t, t_f]$ with ε being arbitrarily small. The existence of such choices are insured by the property of continuous dependence on initial conditions [3]. Notice that the choice of ε and δt are independent.

Because the switching costs are embedded in the extended jump functions, the following equalities hold at the times of switching

$$\begin{aligned} &V\left(t_j-, q_{j-1}, x(t_j-), L-j+1\right) \\ &= V\left(t_j+, q_j, x(t_j+), L-j\right) \end{aligned} \quad (47)$$

and

$$\begin{aligned} &V\left(t_j+\delta t-, q_{j-1}, x'(t_j+\delta t-), L-j+1\right) \\ &= V\left(t_j+\delta t+, q_j, x'(t_j+\delta t+), L-j\right) \end{aligned} \quad (48)$$

The following equations hold due to representation of the system in the Mayer format

$$\begin{aligned} &V\left(t_j+, q_j, x(t_j+), L-j\right) \\ &= V\left(t_j+\delta t+, q_j, x(t_j+\delta t+), L-j\right) \end{aligned} \quad (49)$$

and

$$\begin{aligned} &V\left(t_j-, q_{j-1}, x'(t_j-), L-j+1\right) \\ &= V\left(t_j+\delta t-, q_{j-1}, x'(t_j+\delta t-), L-j+1\right) \end{aligned} \quad (50)$$

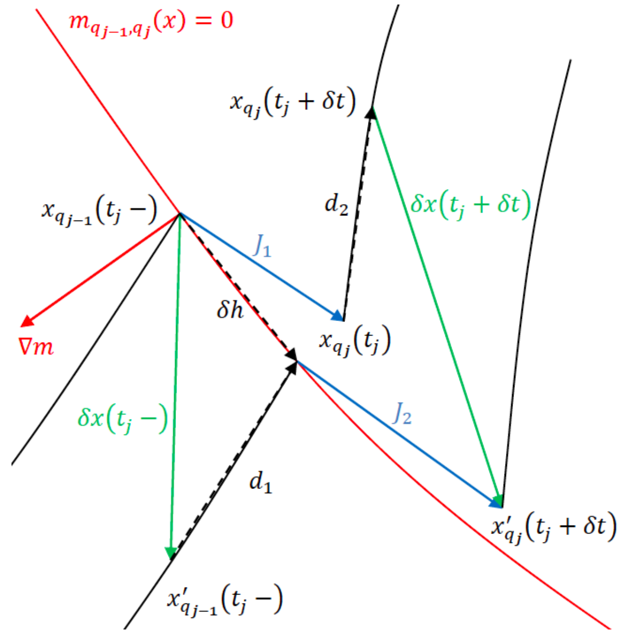

Fig. 1. The choice of trajectories for variation of the value function

Eq. (47) and (49) give

$$\begin{aligned} &V\left(t_j-, q_{j-1}, x(t_j-), L-j+1\right) \\ &= V\left(t_j+\delta t+, q_j, x(t_j+\delta t+), L-j\right) \end{aligned} \quad (51)$$

and Eq. (48) and (50) give

$$\begin{aligned} &V\left(t_j-, q_{j-1}, x'(t_j-), L-j+1\right) \\ &= V\left(t_j+\delta t+, q_j, x'(t_j+\delta t+), L-j\right) \end{aligned} \quad (52)$$

Subtracting (51) from (52) and an application of Taylor series expansion yields

$$\begin{aligned} &\left\langle \nabla V\left(t_j-, q_{j-1}, x(t_j-), L-j+1\right), \delta x(t_j-) \right\rangle \\ &= \left\langle \nabla V\left(t_j+\delta t+, q_j, x(t_j+\delta t+), L-j\right), \delta x(t_j+\delta t) \right\rangle \\ &\quad + O\left(\varepsilon^2\right) \end{aligned} \quad (53)$$

The following exact relations hold according to the dynamics and jump maps governing the system's trajectory (see also Fig. 1)

$$\begin{aligned} x_{q_j}(t_j+\delta t+) &= x_{q_j}(t_j) + d_2 \\ &= \xi\left(x_{q_{j-1}}(t_j-)\right) + d_2 \end{aligned} \quad (54)$$

and

$$\begin{aligned} x'_{q_j}(t_j+\delta t+) &= \xi\left(x'_{q_{j-1}}(t_j+\delta t-)\right) \\ &= \xi\left(x'_{q_{j-1}}(t_j-) + d_1\right) \end{aligned} \quad (55)$$

These equations give

$$\begin{aligned} \delta x(t_j+\delta t+) &= x'_{q_j}(t_j+\delta t) - x_{q_j}(t_j+\delta t) \\ &= \nabla \xi\left(\delta x(t_j-) + d_1\right) - d_2 + O\left(\varepsilon^2\right) \end{aligned} \quad (56)$$

where

$$d_1 = \frac{-\left\langle \delta x(t_j-), \nabla m \right\rangle}{\left\langle f_1, \nabla m \right\rangle} f_1 + O\left(\varepsilon^2\right)$$

and

$$d_2 = \frac{-\left\langle \delta x\left(t_j-\right), \nabla m\right\rangle}{\left\langle f_1, \nabla m\right\rangle} f_2 + O\left(\varepsilon^2\right)$$

This gives Eq. (56) as

$$\begin{aligned} \delta x\left(t_j+\delta t+\right) &= \nabla\xi\, \delta x\left(t_j-\right) \\ &+ \tfrac{\left\langle \delta x\left(t_j-\right), \nabla m\right\rangle}{\left\langle f_1, \nabla m\right\rangle}\left(f_2 - \nabla\xi\, f_1\right) + O\left(\varepsilon^2\right) \end{aligned} \tag{57}$$

Substituting (57) in (53) gives

$$\begin{aligned} &\left\langle \nabla V\left(t_j-, q_{j-1}, x\left(t_j-\right), L-j+1\right), \delta x\left(t_j-\right)\right\rangle \\ &= \left\langle \nabla V\left(t_j+\delta t, q_j, x\left(t_j+\delta t\right), L-j\right), \nabla\xi\, \delta x\left(t_j-\right)\right\rangle \\ &+ \Big\langle \nabla V\left(t_j+\delta t, q_j, x\left(t_j+\delta t\right), L-j\right) \\ &, \tfrac{\left\langle \delta x\left(t_j-\right), \nabla m\right\rangle}{\left\langle f_1, \nabla m\right\rangle}\left(f_2 - \nabla\xi f_1\right)\Big\rangle \end{aligned} \tag{58}$$

Denoting

$$\gamma = \frac{\left\langle \delta x\left(t_j-\right), \nabla m\right\rangle}{\left\langle f_1, \nabla m\right\rangle} \tag{59}$$

and noting that

$$\begin{aligned} &\left\langle \nabla V\left(t_j+\delta t, q_j, x\left(t_j+\delta t\right), L-j\right), \nabla\xi\, \delta x\left(t_j-\right)\right\rangle = \\ &\left\langle \nabla\xi^T \nabla V\left(t_j+\delta t, q_j, x\left(t_j+\delta t\right), L-j\right), \delta x\left(t_j-\right)\right\rangle \end{aligned} \tag{60}$$

and

$$\begin{aligned} &\left\langle \nabla V\left(t_j+\delta t, q_j, x\left(t_j+\delta t\right), L-j\right), \gamma\left(f_2 - \nabla\xi f_1\right)\right\rangle \\ &= \left\langle \nabla V\left(t_j+\delta t, q_j, x\left(t_j+\delta t\right), L-j\right), \left(f_2 - \nabla\xi f_1\right)\right\rangle \gamma \\ &= p\left\langle \nabla m, \delta x\left(t_j-\right)\right\rangle \end{aligned} \tag{61}$$

with

$$p := \left\langle \nabla V\left(t_j+\delta t, q_j, x\left(t_j+\delta t\right), L-j\right), \frac{\left(f_2 - \nabla\xi f_1\right)}{\left\langle f_1, \nabla m\right\rangle}\right\rangle \tag{62}$$

we can write Eq. (58) as

$$\begin{aligned} &\left\langle \nabla V\left(t_j-, q_{j-1}, x\left(t_j-\right), L-j+1\right), \delta x\left(t_j-\right)\right\rangle \\ &= \left\langle \nabla\xi^T \nabla V\left(t_j+\delta t, q_j, x\left(t_j+\delta t\right), L-j\right), \delta x\left(t_j-\right)\right\rangle \\ &+ \left\langle p\nabla m, \delta x\left(t_j-\right)\right\rangle \end{aligned} \tag{63}$$

Since (63) holds for every choice of $\delta x\left(t_j-\right) \in \hat{X}$ we must have

$$\begin{aligned} &\nabla V\left(t_j-, q_{j-1}, [x+\delta x]\left(t_j-\right), L-j+1\right) = \\ &\nabla\xi^T \nabla V\left(t_j+\delta t, q_j, x\left(t_j+\delta t\right), L-j\right) + p\nabla m \end{aligned} \tag{64}$$

Letting $\delta t \to 0$, Eq. (64) becomes equivalent to the boundary condition (28) for the adjoint process. Thus the boundary conditions for $\nabla V \equiv \nabla_{\hat{x}} V$ and $\hat{\lambda}^o$ are the same. Employing Lemma 1 the boundary condition (64) can be restated as in (36). ∎

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