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The Hybrid Minimum Principle in the Presence of Switching Costs

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***Abstract*—Hybrid optimal control problems are studied for systems where, in addition to running costs, switching between discrete states incurs costs. A key aspect of the analysis is the relationship between the Hamiltonian and the adjoint process before and after the switching instants. In this paper, the analysis is performed for systems for which autonomous and controlled state jumps are not permitted. First the results are established in the hybrid Mayer optimal control problem setup using the needle variation technique, and then the results for the hybrid Bolza optimal control problem are established via the calculus of variations methodology.**

I. Introduction

There is now an extensive literature on the optimal control of hybrid systems (see e.g. [1], [2], [3], [4], [5], [6], [7], [8], [9]). With the exception of the variational inequality in [8] and the work in [9], [10], the results and methods in this body of work consist of generalizations of the Pontryagin Maximum Principle (PMP). A feature of special interest in these analyses is the boundary conditions on adjoint processes and the Hamiltonian function at autonomous and controlled switching times and states; these boundary conditions may be viewed as a generalization to the optimal control case of the Erdmann-Weierstrass conditions of the calculus of variations. As is well known, Dynamic Programming (DP) provides sufficient conditions for optimality based upon the Principle of Dynamic Programming, which in the standard non-hybrid case, and under the assumption of smoothness of the value function, results in the celebrated Hamilton-Jacobi-Bellman (HJB) equation [11]. In the case of non-smooth value functions, the so-called viscosity solutions [9] give a general class of solutions to the HJB equation. Those hybrid optimal control problems (with autonomous or controlled switchings) where switching incurs costs constitute a class of problems which have been the subject of only limited study. In fact the value function for hybrid systems with switching costs will not in general be smooth at the switching instants, and hence viscosity solutions are studied in [9], [10], where switching costs are a function of switching state. In this paper, the relationship between the Hamiltonian and the adjoint process before and after switching instants with costs is determined for hybrid systems which are general except for the restriction that autonomous and controlled state jumps are not permitted. These results are expressed in the Hybrid Minimum Principle (HMP) framework in this paper and a consecutive work will be studied in the Dynamic Programming framework.

II. Problem Formulation

To simplify the analysis, necessary optimality conditions are only presented in this paper for trajectories with a single switching event; however the results may be generalized (as for instance in [3]) to optimal trajectories with several switching events by iterating the proof procedure backwards in time along the trajectory from the terminal instant.

A. Basic Assumptions

Consider a *hybrid system (structure)* $\mathbb{H}$

$$\mathbb{H} = \{H := Q \times \mathbb{R}^n, I, \Gamma, A, F, \mathcal{M}\} \tag{1}$$

with the following properties:

$Q = \{1, 2, \dots, |Q|\} \equiv \{q_j\}_{j \in Q}$ is the finite set of *discrete states (components)*.

$H = Q \times \mathbb{R}^n$ is the *(hybrid) state space* of the hybrid system $\mathbb{H}$.

$I = \Sigma \times U$ is the set of system input values with Σ being the set of autonomous and controlled transition labels extended with the identity element such that $\sigma_{i,j} \in \Sigma$ for $i \in Q$ only if $j \in A(i)$

$U \subset \mathbb{R}^m$ is the set of *admissible input control values*, where U is an open bounded set in $\mathbb{R}^m$.

The set of *admissible input control functions* is taken to be $\mathcal{U}(U) := L_\infty([t_0, T_*), U)$, which is the set of all measurable functions that are bounded up to a set of measure zero on $[t_0, T_*), T_* < \infty$. The boundedness property necessarily holds since admissible input functions take values in the open bounded set U which has compact closure $\bar{U}$.

$\Gamma : H \times \Sigma \to H$ is a time independent (partially defined) *discrete (state) transition map* which is the identity on the second ($\mathbb{R}^n$) component of H.

$A : Q \times \Sigma \to Q$ is such that $A(q_i, \sigma_{i,j}) = q_j$.

$F = \{f_j\}_{j \in Q}$ is the collection of *vector fields* such that $f_j \in C^k(\mathbb{R}^n \times U \to \mathbb{R}^n), k \geq 1$ satisfies a uniform (in

x) Lipschitz condition, i.e. there exists $L_f < \infty$ such that $\|f_j(x_1, u) - f_j(x_2, u)\| \leq L_f \|x_1 - x_2\|$, $x_1, x_2 \in \mathbb{R}^n$, $u \in U$, $j \in Q$. We also assume that there exists $K_f < \infty$ such that $\max_{j \in Q} \left(\sup_{u \in U} (\|f_j(0, u)\|) \right) \leq K_f$.

$\mathcal{M} = \{m_{i,j}(x) = 0 : i, j \in Q\}$ is a *switching manifold*, also called *guard*, such that $m_{i,j}$ is a smooth, i.e. C^∞ codimension 1 sub-manifold of $\mathbb{R}^n$.

The initial state $h_0 := (q_0, x(t_0)) \in H$ is such that $m(x_0) \neq 0$.

A hybrid system with a single switch from $q_1 \in Q$ to $q_2 \in Q$ at the time $t_s \in (t_0, t_f)$ has a representation in the form

$$\dot{x}_{q_1}(t) = f_{q_1}(x_{q_1}(t), u(t)), \quad a.e.\, t \in [t_0, t_s)$$

$$\dot{x}_{q_2}(t) = f_{q_2}(x_{q_2}(t), u(t)), \quad a.e.\, t \in [t_s, t_f]$$

subject to

$$h_0 = (q_1, x_{q_1}(t_0)) = (q_1, x_0)$$

$$x_{q_2}(t_s) = \lim_{t \uparrow t_s} x_{q_1}(t)$$

$$u(t) \in U \subset \mathbb{R}^m$$

$$u(.) \in L_\infty([t_0, t_f), U)$$

If the discrete control switching input σ_{q_1, q_2} is a controlled switching, the time t_s may be selected on $[t_0, t_f]$ without any constraint, while in the autonomous switching case it must occur when the condition $m(x_s(t_s)) = 0$ is satisfied.

B. The Hybrid Optimal Control Problem

Let $\{l_j\}_{j \in Q}, l_j \in C^{n_l}(\mathbb{R}^n \times U \to \mathbb{R}_+), n_l \geq 1$, be a family of cost functions; $c \in C^{n_c}(\mathbb{R}^n \to \mathbb{R}_+), n_c \geq 1$, be the switching cost function; and $g \in C^{n_g}(\mathbb{R}^n \to \mathbb{R}_+), n_g \geq 1$, be a terminal cost function satisfying the following:

There exists $K_l < \infty$ and $1 \leq \gamma_l < \infty$ such that $|l_j(x, u)| \leq K_l (1 + \|x\|^{\gamma_l})$, $x \in \mathbb{R}^n, u \in U, j \in Q$.

There exists $K_c < \infty$ and $1 \leq \gamma_c < \infty$ such that $|c(x)| \leq K_c (1 + \|x\|^{\gamma_c})$, $x \in \mathbb{R}^n$.

There exists $K_g < \infty$ and $1 \leq \gamma_g < \infty$ such that $|g(x)| \leq K_g (1 + \|x\|^{\gamma_g})$, $x \in \mathbb{R}^n$.

Consider the initial time t_0, final time $t_f < \infty$, and initial hybrid state $h_0 = (q_0, x_0)$. Let $S = (t_s, \sigma_{i,j})$ be the hybrid switching input and let $\mathcal{I} := (S, u), u \in \mathcal{U}$, be the hybrid input which, subject to the assumptions above, results in a (necessarily unique) hybrid state process [3]. Define the *hybrid cost function* as

$$\begin{aligned} J(t_0, t_f, h_0; \mathcal{I}) &:= \int_{t_0}^{t_s} l_{q_1}(x_{q_1}(s), u(s))\, ds \\ &+ \int_{t_s}^{t_f} l_{q_2}(x_{q_2}(s), u(s))\, ds + c(x(t_s)) + g(x_{q_2}(t_f)) \end{aligned} \tag{2}$$

Then the Hybrid Optimal Control Problem (HOCP) is to find the infimum $J^o(t_0, t_f, h_0)$ over the family of input trajectories $\{\mathcal{I}\}$.

III. HMP Solution to the HOCP

In this section, we first develop the results of the Hybrid Minimum Principle (HMP) for the *Mayer problem* (i.e. with $l_1(x, u) = l_2(x, u) \equiv 0$) and then extend the results for the general case in Eq. (2) (also called the *Bolza problem*). The theorems below are proved only in the the case of a single switching hybrid system, but the results can easily be extended to a general hybrid system with several switching events by iterating the proof procedure backwards in time along the trajectory from the terminal instant.

A. HMP for Mayer HOCPs

Theorem 1. Consider the hybrid system $\mathbb{H}$ with a single switching on the switching manifold m with the Mayer HOCP

$$J(t_0, t_f, h_0; \mathcal{I}) := c(x(t_s)) + g(x_{q_2}(t_f)) \tag{3}$$

and define

$$H_{q_i}(x, \lambda, u) = \lambda^T f_{q_i}(x, u), \; x, \lambda \in \mathbb{R}^n, u \in U^{cpt}, q_i \in Q \tag{4}$$

1) Let $J^o(t_0, t_f, h_0) = \underset{\{\mathcal{I}\}}{inf} J^o(t_0, t_f, h_0; \mathcal{I})$ be realized at an infimizing control $\mathcal{I}^o$ and trajectory $\left\{ (q_1^o, x_{q_1}^o), (q_2^o, x_{q_2}^o) \right\}$.
2) Let t_s, denote the switching time along the optimal trajectory.
3) Assume that x^o meets the switching manifold m transversally subject to one switching

Then

(i) There exists a (continuous from the left), piecewise absolutely continuous adjoint process λ^o satisfying

$$\dot{\lambda}_1^o = -\frac{\partial H_{q_1}(x^o, \lambda^o, u^o)}{\partial x}, \; a.e.\, t \in (t_0, t_s) \tag{5}$$

$$\dot{\lambda}_2^o = -\frac{\partial H_{q_2}(x^o, \lambda^o, u^o)}{\partial x}, \; a.e.\, t \in (t_s, t_f) \tag{6}$$

with $\lambda^o(t_0)$ free and the following boundary value conditions

- At the final time t_f

$$\lambda^o(t_f) = \nabla g(x^o(t_f)) \tag{7}$$

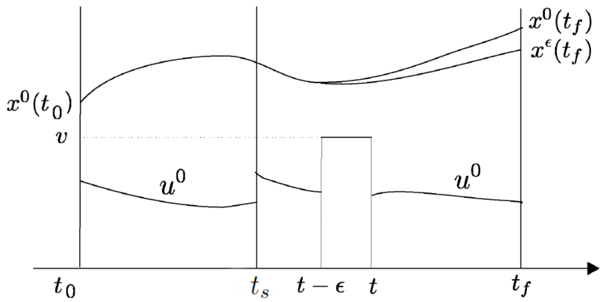

Fig. 1. Variation in u^o and the corresponding perturbed trajectory

- If t_s is a controlled switching time, then

$$\lambda^o(t_s-) \equiv \lambda^o(t_s) = \lambda^o(t_s+) + \nabla c(x^o(t_s)) \tag{8}$$

- If t_s is an autonomous switching time satisfying $m(x(t_s)) = 0$, then for some $p \in \mathbb{R}$

$$\begin{aligned} \lambda^o(t_s-) &\equiv \lambda^o(t_s) \\ &= \lambda^o(t_s+) + \nabla c(x^o(t_s)) + p\nabla m(x^o(t_s)) \end{aligned} \tag{9}$$

(ii) Along the optimal trajectory, the Hamiltonian minimization conditions hold:

$$\begin{aligned} &H_{q^o(t)}(x^o(t), \lambda^o(t), u^o(t)) \\ &\leq H_{q^o(t)}(x^o(t), \lambda^o(t), v) \\ &a.e.\, t \in [t_0, t_f],\ \forall v \in U \end{aligned} \tag{10}$$

Proof: The proof is based on the fact that the cost functional is minimized on the optimal trajectory and hence, every needle variation results in an equal or a higher value for it;

(i) First, consider a needle variation in the time interval (t_s, t_f).

$$u^\epsilon(\tau) = \begin{cases} u^o(\tau) & if\ t_s \leq \tau < t-\epsilon \\ v & if\ t-\epsilon \leq \tau < t \\ u^o(\tau) & if\ t \leq \tau \leq t_f \end{cases} \tag{11}$$

This corresponds to a perturbed trajectory $x^\epsilon(\tau); \tau \in [t_0, t_f]$. The variation and the corresponding state response are depicted in Fig. 1. Define

$$\delta x^\epsilon(\tau) := x^\epsilon(\tau) - x^o(\tau) \tag{12}$$

then

$$\begin{aligned} \delta x^\epsilon(\tau) &= x^\epsilon(t-\epsilon) + \textstyle\int_{t-\epsilon}^{\tau} f_{q_2}(x^\epsilon(s), u^\epsilon(s))\, ds \\ &\quad - x^o(t-\epsilon) + \textstyle\int_{t-\epsilon}^{\tau} f_{q_2}(x^o(s), u^o(s))\, ds \\ &= \textstyle\int_{t-\epsilon}^{\tau} [f_{q_2}(x^\epsilon(s), u^\epsilon(s)) - f_{q_2}(x^o(s), u^o(s))]\, ds \end{aligned}$$

The corresponding deviation δx^ϵ right after the perturbation (at time t) can be computed as

$$\begin{aligned} \delta x^\epsilon(t) &= x^\epsilon(t) - x^o(t) \\ &= \textstyle\int_{t-\epsilon}^{t} [f_{q_2}(x^\epsilon(s), v) - f_{q_2}(x^o(s), u^o(s))]\, ds \end{aligned}$$

Define

$$y^\epsilon(\tau) := \frac{1}{\epsilon}\delta x^\epsilon(\tau)$$

and

$$y(\tau) := \lim_{\epsilon \to 0} \frac{1}{\epsilon}\delta x^\epsilon(\tau) \tag{13}$$

Hence

$$y(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t-\epsilon}^{t} [f_{q_2}(x^\epsilon(s), v) - f_{q_2}(x^o(s), u^o(s))]\, ds$$

This gives (see [3])

$$y(t) = f_{q_2}(x^o(t), v) - f_{q_2}(x^o(t), u^o(t)) \tag{14}$$

After the perturbation, i.e. in the time interval $\tau \in [t, t_f]$

$$\delta x^\epsilon(\tau) = \delta x^\epsilon(t) + \int_t^\tau \left[f_{q_2}\left(x^\epsilon_{(s)}, u^o_{(s)}\right) - f_{q_2}\left(x^o_{(s)}, u^o_{(s)}\right)\right] ds$$

For small enough ϵ, the following approximation holds

$$\delta x^\epsilon(\tau) = \Phi_2(\tau, t)\, \delta x^\epsilon(t) + H.O.T$$

where $\Phi_j(\tau, \tau_0)$ is the state transition matrix corresponding to the system

$$\frac{d}{d\tau} z(\tau) = \frac{\partial}{\partial x} f_{q_j}(x^o(\tau), u^o(\tau))\, z(\tau)$$

.

This gives

$$\begin{aligned} y(t_f) &= \lim_{\epsilon \to 0} y^\epsilon(t_f) \\ &= \Phi_2(t_f, t)\, [f_{q_2}(x^o(t), v) - f_{q_2}(x^o(t), u^o(t))] \end{aligned} \tag{15}$$

Now, since x^o is an optimal trajectory, we have

$$g(x^\epsilon(t_f)) + c(x^\epsilon(t_s)) \geq g(x^o(t_f)) + c(x^o(t_s)) \tag{16}$$

But since the perturbation at $t \in (t_s, t_f)$ does not change the trajectory in $[t_0, t_s]$ and the switching point $x^\epsilon(t_s) = x^o(t_s)$, we can write

$$g(x^\epsilon(t_f)) = g(x^o(t_f) + \epsilon y^\epsilon(t_f)) \geq g(x^o(t_f))$$

Dividing by ϵ

$$\frac{1}{\epsilon}[g(x^o(t_f) + \epsilon y^\epsilon(t_f)) - g(x^o(t_f))] \geq 0$$

Taking the limit as $\epsilon \to 0$

$$\frac{1}{\epsilon}\left[g(x^o(t_f)) + \epsilon(\nabla g(x^o(t_f)))^T y^\epsilon(t_f) - g(x^o(t_f))\right] \geq 0$$

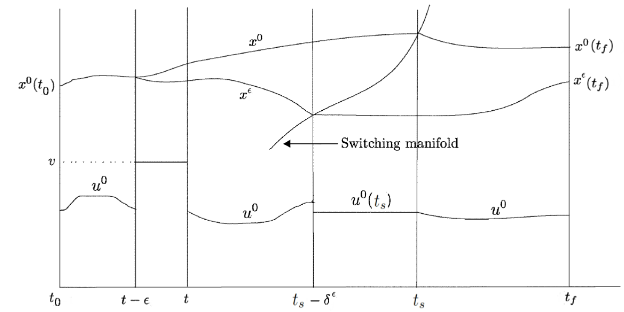

Fig. 2. Autonomous switching case: variation in u^o and the corresponding perturbed trajectory

or

$$\left(\nabla g\left(x^o\left(t_f\right)\right)\right)^T y\left(t_f\right) \geq 0 \tag{17}$$

Replacing $y\left(t_f\right)$ by its value, we get

$$\begin{aligned} &\left(\nabla g\left(x^o\left(t_f\right)\right)\right)^T \Phi_2\left(t_f, t\right) f_{q_2}\left(x^o\left(t\right), v\right) \\ &\geq \left(\nabla g\left(x^o\left(t_f\right)\right)\right)^T \Phi_2\left(t_f, t\right) f_{q_2}\left(x^o\left(t\right), u^o\left(t\right)\right) \end{aligned} \tag{18}$$

Setting

$$\lambda_2^T\left(t\right) = \left(\nabla g\left(x^o\left(t_f\right)\right)\right)^T \Phi_2\left(t_f, t\right) \quad t \in \left[t_s, t_f\right]$$

we obtain

$$\lambda_2\left(t_f\right) = \nabla g\left(x^o\left(t_f\right)\right) \tag{19}$$

and

$$\begin{aligned} \dot{\lambda_2}\left(t\right) &= -\left(\frac{\partial f_{q_2}}{\partial x}\left(x^o\left(t\right), u^o\left(t\right)\right)\right)^T \Phi_2^T\left(t_f, t\right) \nabla g\left(x^o\left(t_f\right)\right) \\ &= -\left(\frac{\partial f_{q_2}}{\partial x}\left(x^o\left(t\right), u^o\left(t\right)\right)\right)^T \lambda_2\left(t\right) = -\frac{\partial H_2}{\partial x} \end{aligned} \tag{20}$$

(ii) Now consider a needle variation in the time interval (t_0, t_s). This causes a change in the switching time as the perturbed trajectory does not necessarily hit the switching manifold at time t_s. If the trajectory arrives on the switching manifold earlier (say $t_s - \delta^\epsilon$), then the perturbed system switches to the discrete state q_2 earlier while the optimal trajectory is still at q_1. Hence, the control input u^ϵ for the perturbed trajectory is taken to be as (also see Fig. 2)

$$u^\epsilon\left(\tau\right) = \begin{cases} u^o\left(\tau\right) & if\ t_0 \leq \tau < t - \epsilon \\ v & if\ t - \epsilon \leq \tau < t \\ u^o\left(\tau\right) & if\ t \leq \tau < t_s - \delta^\epsilon \\ u^o\left(t_s\right) & if\ t_s - \delta^\epsilon \leq \tau < t_s \\ u^o\left(\tau\right) & if\ t_s \leq \tau \leq t_f \end{cases} \tag{21}$$

The case where the trajectory hits the switching manifold at a later time can be handled similarly.

Similar to the previous part

$$\delta x^\epsilon\left(t\right) = \int_{t-\epsilon}^{t} \left[f_{q_1}\left(x^\epsilon\left(s\right), v\right) - f_{q_1}\left(x^o\left(s\right), u^o\left(s\right)\right)\right] ds$$

results in

$$y\left(t\right) = f_{q_1}\left(x^o\left(t\right), v\right) - f_{q_1}\left(x^o\left(t\right), u^o\left(t\right)\right) \tag{22}$$

and hence, at $t_s - \delta^\epsilon$ we may write

$$\delta x^\epsilon\left(t_s - \delta^\epsilon\right) = \Phi_1\left(t_s - \delta^\epsilon, t\right) \delta x^\epsilon\left(t\right) + H.O.T$$

giving

$$y\left(t_s-\right) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \delta x^\epsilon\left(t_s - \delta^\epsilon\right) = \Phi_1\left(t_s, t\right) y\left(t\right) \tag{23}$$

Then at t_s

$$\begin{aligned} \delta x^\epsilon\left(t_s\right) &= \delta x^\epsilon\left(t_s - \delta^\epsilon\right) \\ &+ \int_{t_s-\delta^\epsilon}^{t_s} \left[f_{q_2}\left(x^\epsilon\left(s\right), u^o\left(t_s\right)\right) - f_{q_1}\left(x^o\left(s\right), u^o\left(s\right)\right)\right] \end{aligned}$$

gives

$$\begin{aligned} y\left(t_s\right) &= y\left(t_s-\right) \\ &+ \lim_{\epsilon \to 0} \tfrac{\delta^\epsilon}{\epsilon} \left[f_{q_2}\left(x^o\left(t_s\right), u^o\left(t_s\right)\right) - f_{q_1}\left(x^o\left(t_s\right), u^o\left(t_s\right)\right)\right] \end{aligned}$$

or (see [3])

$$y\left(t_s\right) = y\left(t_s-\right) + q_s \nabla m\left(x^o\left(t_s\right)\right)^T y\left(t_s-\right) f_1^2 \tag{24}$$

with

$$f_1^2 = \left[f_{q_2}\left(x^o\left(t_s\right), u^o\left(t_s\right)\right) - f_{q_1}\left(x^o\left(t_s\right), u^o\left(t_s\right)\right)\right]$$

and

$$q_s = \frac{1}{\nabla m\left(x^o\left(t_s\right)\right)^T f_{q_1}\left(x^o\left(t_s\right), u^o\left(t_s\right)\right)}$$

Note that the denominator in the above expression is nonzero, due to the transversality assumption. From there, $y\left(t_f\right)$ is computed by

$$y\left(t_f\right) = \Phi_2\left(t_f, t_s\right) y\left(t_s\right)$$

which gives

$$y\left(t_f\right) = \Phi_2\left(t_f, t_s\right)\left[\Phi_1\left(t_s, t\right) y\left(t\right) + \gamma_s\right] \tag{25}$$

with

$$\gamma_s = q_s\left(\nabla m\left(x^o\left(t_s\right)\right)\right)^T y\left(t_s-\right) f_1^2$$

Now Eq. (16) becomes

$$\begin{aligned} &\tfrac{1}{\epsilon}\left[g\left(x^o\left(t_f\right) + \epsilon y^\epsilon\left(t_f\right)\right) + c\left(x^o\left(t_s\right) + \epsilon y^\epsilon\left(t_s - \delta^\epsilon\right)\right)\right. \\ &\left. - g\left(x^o\left(t_f\right)\right) - c\left(x^o\left(t_s\right)\right)\right] \geq 0 \end{aligned}$$

Note that the perturbed system switches at the time $t_s-\delta^\epsilon$ and not t_s. Expanding the above equation and taking the limit as $\epsilon \to 0$ we get

$$(\nabla g(x^o(t_f)))^T y(t_f)+(\nabla c(x^o(t_s)))^T y(t_s-) \geq 0 \quad (26)$$

Replacing the values of $y(t_f)$ and $y(t_s-)$, the above equation becomes

$$\begin{gathered}(\nabla g(x^o(t_f)))^T \Phi_2(t_f,t_s)\Phi_1(t_s,t) f_{q_1}(x^o(t),v)\\ +p(\nabla m(x^o(t_s)))^T \Phi_1(t_s,t) f_{q_1}(x^o(t),v)\\ +(\nabla c(x^o(t_s)))^T \Phi_1(t_s,t) f_{q_1}(x^o(t),v)\\ \geq\\ (\nabla g(x^o(t_f)))^T \Phi_2(t_f,t_s)\Phi_1(t_s,t) f_{q_1}(x^o(t),u^o(t))\\ +p(\nabla m(x^o(t_s)))^T \Phi_1(t_s,t) f_{q_1}(x^o(t),u^o(t))\\ +(\nabla c(x^o(t_s)))^T \Phi_1(t_s,t) f_{q_1}(x^o(t),u^o(t))\end{gathered} \quad (27)$$

with

$$p = q_s(\nabla g(x^o(t_f)))^T \Phi_2(t_f,t_s) f_1^2$$

For $t \in [t_0,t_s)$ define

$$\begin{aligned}\lambda_1(t) := {}&\Phi_1^T(t_s,t)\Phi_2^T(t_f,t_s)\nabla g(x^o(t_f))\\ &+p\Phi_1^T(t_s,t)\nabla m(x^o(t_s))\\ &+\Phi_1^T(t_s,t)\nabla c(x^o(t_s))\end{aligned} \quad (28)$$

We obtain

$$\begin{aligned}\lambda_1(t_s-) &= \Phi_2^T(t_f,t_s)\nabla g(x^o(t_f))\\ &\quad +p\nabla m(x^o(t_s))+\nabla c(x^o(t_s))\\ &= \lambda_2(t_s+)+p\nabla m(x^o(t_s))+\nabla c(x^o(t_s))\end{aligned} \quad (29)$$

Inserting (28) into (27) and remembering that $H_{q_j}(x_{q_j},\lambda_j,u) = \lambda^T f_{q_j}(x_{q_j},u)$, the Hamiltonian minimization (10) is proved. Also, by taking the derivative of (28) and noting that

$$\frac{d}{dt}\Phi_j^T(t_j,t) = -\left(\frac{\partial f_j}{\partial x}(x^o(t))\right)^T \Phi_j^T(t_j,t)$$

we get

$$\dot\lambda_1 = -\left(\frac{\partial f_{q_1}}{\partial x}(x^o(t))\right)^T \lambda_1 = -\frac{\partial H_1}{\partial x} \quad (30)$$

The proof of the controlled switching relation (8) is obtained as a special case of the proof of the autonomous switching relation (9) by a parallel argument where there is no switching manifold, δ^ϵ takes the value 0 and so do q_s and p. ■

B. HMP for Bolza HOCPs

For an explicit presentation of the main result in the case where there exist running costs $l_{q_j}(x,u) \geq 0$, we shall establish the corresponding HOCP using the calculus of variation methodology with the assumption of Small Time Tubular Fountain (STTF) controllability property (see [3]).

Theorem 2. Consider the hybrid system $\mathbb{H}$ with the Bolza HOCP with the same assumptions as in Theorem 1. In addition, assume that almost every continuous state x on the optimal trajectory $x^o(.)$ is a Small Time Tubular Fountain (STTF) with respect to $x^o(.)$, and define the Hamiltonian as

$$H_j(x,\lambda,u) = \lambda^T f_{q_j}(x,u) + l_{q_j}(x,u) \quad (31)$$

$x,\lambda \in \mathbb{R}^n, u \in U, q_j \in Q$ where U is an open bounded set in $\mathbb{R}^m$. Assume that the optimal control u^o is such that $u^o(t) \in U$ *a.e.* $t \in [t_0,t_f]$ and consider the optimal cost function $J^o = J(u^o,t_s)$

$$\begin{aligned}J(u^o,t_s) = {}&\textstyle\int_{t_0}^{t_s} l_1(x^o,u^o)\,dt + \int_{t_s}^{t_f} l_2(x^o,u^o)\,dt\\ &+c(x^o(t_s)) + g(x^o(t_f))\end{aligned} \quad (32)$$

Then for the optimal input and the corresponding optimal trajectory x^o, there exists an adjoint process λ^o for which

$$\dot\lambda^o = -\frac{\partial H_i}{\partial x}(x^o,\lambda^o,u^o),\ a.e.\ t \in [t_0,t_f], i \in Q \quad (33)$$

$$\lambda^o(t_f) = \nabla g(x^o(t_f)) \quad (34)$$

$$\lambda^o(t_s-) \equiv \lambda^o(t_s) = \lambda^o(t_s+) + p\,\nabla m|_{x(t_s)} + \nabla c|_{x(t_s)} \quad (35)$$

such that

$$\left.\frac{\partial H_{q^o}}{\partial u}(x^o,\lambda^o,u)\right|_{u=u^o} = 0\ a.e.\ t \in [t_0,t_f] \quad (36)$$

and the Hamiltonian is continuous at the switching time, i.e.

$$H_1(t_s-) = H_1(t_s) = H_2(t_s) = H_2(t_s+) \quad (37)$$

Proof: Assume the adjoint process $\lambda(.)$ is as defined in equations (33), (34) and (35). Since $f_1(x^o,u^o)-\dot x^o \equiv 0$ for all $t \in (t_0,t_s)$, $f_2(x^o,u^o)-\dot x^o \equiv 0$ for all $t \in (t_s,t_f)$ and $m(x^o(t_s)) = 0$, the optimal cost function (32) is equal to

$$\begin{aligned}J(u^o,t_s) = {}&\textstyle\int_{t_0}^{t_s}\left(l_1(x^o,u^o) + \lambda^T(f_1(x^o,u^o)-\dot x^o)\right)dt\\ &+\textstyle\int_{t_s}^{t_f}\left(l_2(x^o,u^o)+\lambda^T(f_2(x^o,u^o)-\dot x^o)\right)dt\\ &+p\,m(x^o(t_s)) + c(x^o(t_s)) + g(x^o(t_f))\end{aligned}$$

which gives

$$
\begin{array}{c} J\left(u^{o},t_{s}\right)=\int_{t_{0}}^{t_{s}}\left(H_{1}\left(x_{1}^{o},\lambda_{1},u^{o}\right)-\lambda_{1}^{T}\dot{x}_{1}^{o}\right)dt \\ +\int_{t_{s}}^{t_{f}}\left(H_{2}\left(x_{2}^{o},\lambda_{2},u^{o}\right)-\lambda_{2}^{T}\dot{x}_{2}^{o}\right)dt \\ +p\,m\left(x^{o}\left(t_{s}\right)\right)+c\left(x^{o}\left(t_{s}\right)\right)+g\left(x^{o}\left(t_{f}\right)\right) \end{array} \tag{38}
$$

Now consider a variational input δu such that $u^o+\delta u^o \in \mathcal{U}$ and $ess\,sup_{[t_0,t_f]}\left|\delta u^o\right| < \delta$. This will result in a new state trajectory $x^o+\delta x^o$ with a new switching time $t_s-\delta t_s$ and switching state $x_s+\delta x_s$. Employing the standard calculus of variation argument one obtains

$$
\begin{array}{c} 0\leq\delta J^{o}=\int_{t_{0}}^{t_{s}-\delta t_{s}}\frac{\partial H_{1}}{\partial u^{o}}\delta u^{o}dt+\int_{t_{s}}^{t_{f}}\frac{\partial H_{2}}{\partial u^{o}}\delta u^{o}dt \\ +\left(\left.H_{1}\right|_{t=t_{s}-\delta t_{s}}-\left.H_{2}\right|_{t=t_{s}}\right)\delta t_{s} \\ -\left.\lambda_{1}^{T}\right|_{t_{s}-\delta t_{s}}\left(\left.\delta x_{1}^{o}\right|_{t_{s}-\delta t_{s}}+\left.\dot{x}_{1}^{o}\right|_{t_{s}-\delta t_{s}}\delta t_{s}\right) \\ +\left.\lambda_{2}^{T}\right|_{t_{s}}\left(\left.\delta x_{2}^{o}\right|_{t_{s}}+\left.\dot{x}_{2}^{o}\right|_{t_{s}}\delta t_{s}\right) \\ +p\left.\nabla m\right|_{t_{s}}\delta x_{s}+\left.\nabla c\right|_{t_{s}}\delta x_{s}+H.O.T \end{array}
$$

Noting the continuity relations and the definitions:

$$
\left.x_{1}^{o}\right|_{t_{s}}=\left.x_{2}^{o}\right|_{t_{s}} \tag{39}
$$

$$
\left.\left(x_{1}^{o}+\delta x_{1}^{o}\right)\right|_{t_{s}-\delta t_{s}}=\left.\left(x_{2}^{o}+\delta x_{2}^{o}\right)\right|_{t_{s}-\delta t_{s}} \tag{40}
$$

$$
\delta x_{s}:=\left.x_{2}^{o}\right|_{t_{s}}-\left.x_{1}^{o}\right|_{t_{s}-\delta t_{s}} \tag{41}
$$

$$
\left.\delta x_{1}^{o}\right|_{t_{s}-\delta t_{s}}=\left.\left(x_{1}^{o}+\delta x_{1}^{o}\right)\right|_{t_{s}-\delta t_{s}}-\left.x_{1}^{o}\right|_{t_{s}-\delta t_{s}} \tag{42}
$$

$$
\left.\delta x_{2}^{o}\right|_{t_{s}}=\left.\left(x_{2}^{o}+\delta x_{2}^{o}\right)\right|_{t_{s}}-\left.x_{2}^{o}\right|_{t_{s}} \tag{43}
$$

and using the relations

$$
\delta x_{s}=\left.\delta x_{1}^{o}\right|_{t_{s}-\delta t_{s}}+\left.\dot{x}_{1}\right|_{t_{s}-\delta t_{s}}\delta t_{s} \tag{44}
$$

and

$$
\delta x_{s}=\left.\delta x_{2}^{o}\right|_{t_{s}}+\left.\dot{x}_{2}\right|_{t_{s}}\delta t_{s} \tag{45}
$$

(see [3]), the expression for δJ^o becomes

$$
\begin{array}{c} 0\leq\delta J^{o}=\int_{t_{0}}^{t_{s}-\delta t_{s}}\frac{\partial H_{1}}{\partial u^{o}}\delta u^{o}dt+\int_{t_{s}}^{t_{f}}\frac{\partial H_{2}}{\partial u^{o}}\delta u^{o}dt \\ +\left(\left.H_{1}\right|_{t=t_{s}-\delta t_{s}}-\left.H_{2}\right|_{t=t_{s}}\right)\delta t_{s} \\ +\left(-\left.\lambda_{1}^{T}\right|_{t_{s}-\delta t_{s}}+\left.\lambda_{2}^{T}\right|_{t_{s}}+p\nabla m+\nabla c\right)\delta x_{s}+H.O.T \end{array} \tag{46}
$$

and this must hold for arbitrary variations δu^0. Now by (35) the coefficient of δx_s vanishes to $\mathcal{O}\left(\delta\right)$. Furthermore, by the STTF hypothesis, the resulting variations can be made to yield independent variations in δt_s and δu (see [3]), and hence all the coefficients of the variational terms must be equal to zero, which yields the result in the theorem for the autonomous case.

The result in the controlled switching case is obtained as a special case of the autonomous one by a parallel proof method where there is no switching manifold and hence $p=0$. ∎

Remark: Although the Small Time Tubular Fountain (STTF) property was assumed in this theorem, the results in [7] permits one to omit this condition in the absence of switching costs.

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