Preprints, 5th IFAC Conference on Analysis and Design of Hybrid Systems
October 14-16, 2015. Georgia Tech, Atlanta, USA

Time Optimal Hybrid Minimum Principle and the Gear Changing Problem for Electric Vehicles

Ali Pakniyat * **Peter E. Caines** *

* *Centre for Intelligent Machines (CIM) and the Department of Electrical and Computer Engineering (ECE), McGill University*
E-mails: pakniyat@cim.mcgill.ca, peterc@cim.mcgill.ca

Abstract: The statement of the Hybrid Minimum Principle is presented for time optimal control problems where autonomous and controlled state jumps at the switching instants are accompanied by changes in the dimension of the state space. A key aspect of the analysis is the relationship between the Hamiltonians and the adjoint processes before and after the switching instants. As an example application, an electric vehicle equipped with a two-speed seamless transmission, that augments an additional degree of freedom during the transition period, is modelled within this framework. The state-dependant motor torque constraints are converted to state-independent control input constraints via a change of variable and the introduction of auxiliary discrete states. The problem of the minimum acceleration time required for reaching the speed of $100\,km/hr$ is formulated within the presented framework and the Time Optimal Hybrid Minimum Principle is employed in order to find the optimal control inputs and the optimal gear changing instants.

Keywords: Optimal Control, Hybrid Systems, Minimum Principle, Electric Vehicle, Gear Selection

1. INTRODUCTION

There is now an extensive literature on the optimal control of hybrid systems. With the exception of the studies on Hybrid Dynamic Programming (see e.g. [Bensoussan and Menaldi (1997); Branicky et al. (1998); Dharmatti and Ramaswamy (2005); Pakniyat and Caines (2014a,c, 2015); Schöllig et al. (2007)]) and the reachability approach for specific applications (e.g. [Lygeros, Tomlin, and Sastry (1997)]) the majority of research on the optimal control of hybrid systems is focused on the Hybrid Minimum Principle (HMP) [Clarke and Vinter (1989); Garavello and Piccoli (2005); Pakniyat and Caines (2013, 2014a,c, 2015); Passenberg et al. (2011); Shaikh and Caines (2007); Sussmann (1999); Taringoo and Caines (2013); Xu and Antsaklis (2004)] which is the generalization of the fundamental Pontryagin Maximum Principle. The HMP gives necessary conditions for the optimality of the trajectory and the control inputs of a given hybrid system with fixed initial conditions and a sequence of autonomous and controlled switchings. These conditions are expressed in terms of the minimization of the distinct Hamiltonians defined along the hybrid trajectory of the system corresponding to a sequence of discrete states and continuous valued control inputs on the associated time intervals. A feature of special interest is the boundary conditions on the adjoint processes and the Hamiltonian functions at autonomous and controlled switching times and states; these boundary conditions may be viewed as a generalization of the optimal control case of the Weierstrass–Erdmann conditions of the calculus of variations.

In past work of the authors (see [Pakniyat and Caines (2013, 2014a,c)]) the statement of the Hybrid Minimum Principle is presented for the general class of hybrid optimal control problems with autonomous and controlled state jumps and in the presence of a large range of running, terminal and switching costs. The aim of this paper is to formulate the gear changing problem for electric vehicles in the hybrid optimal control framework. To this end, we extend the formulation presented in [Pakniyat and Caines (2014d)] for gear-equipped electric vehicles with the inclusion of the transmission dynamics by considering the model of a seamless dual break transmission system reported in [Rahimi M., Pakniyat, and Boulet (2014)]. Due to the special structure of the transmission system considered, the mechanical degree of freedom and hence, the dimension of the (continuous) state space of the system are dependant on the status of the transmission, i.e. whether a gear number is fixed or the system is undergoing a transition between the two gears. In order to avoid state-dependant input constraints, the torque constraints for the electric motor which have a special type of speed dependence (see Fig. 1) are converted to state-independent control input constraints with a change of variables and the introduction of auxiliary discrete states. With the association of discrete states to these physical and auxiliary statuses of the transmission and the electric motor, a hybrid model is derived that satisfies the basic assumptions required for the statement of the Time Optimal Hybrid Minimum Principle. Employing this theorem, the problem of the minimum acceleration time required for reaching the speed of $100\,km/hr$ (or $60\,mph$) from the stationary state is solved and the results are demonstrated.

2. BASIC ASSUMPTIONS

A *hybrid system (structure)* $\mathbb{H}$ is defined as a septuple

$$\mathbb{H} = \{H := Q \times M, I := \Sigma \times U, \Gamma, A, F, \Xi, \mathscr{M}\} \qquad (1)$$

where the symbols in the expression are defined as below.

A0: $Q = \{1,2,...,|Q|\} \equiv \{q_1, q_2, ..., q_{|Q|}\}, |Q| < \infty$, is a finite set of *discrete states (components)*.

$M = \{\mathbb{R}^{n_q}\}_{q \in Q}$ is a family of finite dimensional continuous state spaces, where $n_q \leq n < \infty$ for all $q \in Q$.

$H := Q \times M$ is called the *(hybrid) state space* of the hybrid system $\mathbb{H}$.

$I := \Sigma \times U$ is the set of system input values, where $|\Sigma| < \infty$ and $U = \{U_q\}_{q \in Q}$ with $U_q \subset \mathbb{R}^{m_q}$ is the set of *admissible input control values*, where U_q is a compact set in $\mathbb{R}^{m_q}$.

The set of admissible (continuous) control inputs $\mathscr{U}(U) := L_\infty([t_0, T_*), U)$, is defined to be the set of all measurable functions that are bounded up to a set of measure zero on $[t_0, T_*), T_* < \infty$. The boundedness property necessarily holds since admissible input functions take values in the compact set U.

$\Gamma : H \times \Sigma \to H$ is a time independent (partially defined) *discrete state transition map*.

$\Xi : H \times \Sigma \to H$ is a time independent (partially defined) *continuous state jump transition map*. All $\xi_\sigma \in \Xi$, $\xi_\sigma : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$, $p \in A(q, \sigma)$ are assumed to be continuously differentiable in the continuous state $x \in \mathbb{R}^{n_q}$.

$A : Q \times \Sigma \to Q$ denotes both a finite automaton and the automaton's associated transition function on the state space Q and event set Σ, such that for a discrete state $q \in Q$ only the discrete controlled and uncontrolled transitions into the q-dependant subset $\{A(q, \sigma), \sigma \in \Sigma\} \subset Q$ occur under the projection of Γ on its Q components: $\Gamma : Q \times \mathbb{R}^n \times \Sigma \to H|_Q$.

F is an indexed collection of *vector fields* $\{f_q\}_{q \in Q}$ such that $f_q \in C^{k_{f_q}}(\mathbb{R}^{n_q} \times U_q \to \mathbb{R}^{n_q})$, $k_{f_q} \geq 1$, satisfies a uniform (in x) Lipschitz condition, i.e. there exists $L_f < \infty$ such that $\|f_q(x_1, u) - f_q(x_2, u)\| \leq L_f \|x_1 - x_2\|$, $x_1, x_2 \in \mathbb{R}^{n_q}$, $u \in U_q$, $q \in Q$.

$\mathscr{M} = \{m_\alpha : \alpha \in Q \times Q,\}$ denotes a collection of *switching manifolds* such that, for any ordered pair $\alpha = (p, q)$, m_α is a smooth, i.e. C^∞, codimension 1 sub-manifold of $\mathbb{R}^{n_q}$, described locally by $m_\alpha = \{x : m_\alpha(x) = 0\}$. □

A1: The initial state $h_0 := (q_0, x(t_0)) \in H$ is such that $m_{q_0, q_j}(x_0) \neq 0$, for all $q_j \in Q$. □

3. TIME OPTIMAL HYBRID MINIMUM PRINCIPLE

Consider the initial time t_0, initial hybrid state $h_0 = (q_0, x_0)$ and the terminal hybrid state $h_f = (q_f, x_f)$ to be reached in a finite time $t_f < \infty$. Let

$$S_L = \left\{(t_0, id), (t_1, \sigma_{q_0 q_1}), \ldots, (t_{L-1}, \sigma_{q_{L-2} q_{L-1}}), \left(t_L, \sigma_{q_{L-1} q_f}\right)\right\} \equiv \left\{(t_0, q_0), (t_1, q_1), \ldots, (t_{L-1}, q_{L-1}), \left(t_L, q_f\right)\right\} \tag{2}$$

be a hybrid switching sequence and let $I_L := (S_L, u), u \in \mathscr{U}$ be a hybrid input trajectory which subject to A0 and A1 results in a (necessarily unique) hybrid state process (see [Shaikh and Caines (2007)]) and is such that L controlled and autonomous switchings occur on the time interval $[t_0, T(I_L)]$, where $T(I_L) \leq t_f$. In this paper, the number of switchings L is held fixed and we denote the corresponding set of inputs by $\{I_L\}$.

Define the *hybrid cost* as

$$J\left(t_0, t_f, h_0, L; I_L\right) := t_f \equiv \sum_{i=0}^{L} \int_{t_i}^{t_{i+1}} dt \tag{3}$$

subject to

$$\dot{x}_{q_i}(t) = f_{q_i}(x_{q_i}(t), u(t)), \; a.e.\, t \in [t_i, t_{i+1}), \tag{4}$$

$$x_{q_0}(t_0) = x_0, \tag{5}$$

$$x_{q_j}(t_j) = \xi\left(x_{q_{j-1}}(t_j-)\right) \equiv \xi\left(\lim_{t \uparrow t_j} x_{q_{j-1}}(t)\right) \tag{6}$$

$$x_{q_f}\left(t_f\right) = x_f \tag{7}$$

where $0 \leq i \leq L$, $1 \leq j \leq L$ and $t_{L+1} = t_f < \infty$.

Then the Hybrid Optimal Control Problem (HOCP) is to find the infimum $J^o\left(t_0, t_f, h_0, L\right)$ over the family of input trajectories $\{I_L\}$, i.e.

$$J^o\left(t_0, t_f, h_0, L\right) = \inf_{I_L} J\left(t_0, t_f, h_0, L; I_L\right) \tag{8}$$

Theorem 1 [Pakniyat and Caines (2014b)] Consider the hybrid system $\mathbb{H}$ together with the assumptions A0 and A1 as above and the HOCP (8) for the hybrid cost (3). Define the family of system Hamiltonians by

$$H_{q_j}(x, \lambda, u) = \lambda^T f_{q_j}(x, u) + 1 \tag{9}$$

$x, \lambda \in \mathbb{R}^{n_{q_j}}, u \in U_{q_j}, q_j \in Q$. Then for an optimal switching sequence q^o and along the corresponding optimal trajectory x^o, there exists an adjoint process λ^o such that

$$\dot{x}^o = \frac{\partial H_{q^o}}{\partial \lambda}(x^o, \lambda^o, u^o), \tag{10}$$

$$\dot{\lambda}^o = -\frac{\partial H_{q^o}}{\partial x}(x^o, \lambda^o, u^o) \tag{11}$$

almost everywhere $t \in \left[t_0, t_f\right]$ with

$$x^o(t_0) = x_0, \tag{12}$$

$$x^o(t_j) = \xi(x^o(t_j-)), \tag{13}$$

$$x^o\left(t_f\right) = x_f, \tag{14}$$

$$\lambda^o(t_j-) \equiv \lambda^o(t_j) = \nabla \xi^T \lambda^o(t_j+) + p \nabla m, \tag{15}$$

where $p \in \mathbb{R}$ when t_j indicates the time of an autonomous switching, and $p = 0$ when t_j indicates the time of a controlled switching. Moreover, the Hamiltonian is minimized with respect to the control input, i.e.

$$H_{q^o}(x^o, \lambda^o, u^o) \leq H_{q^o}(x^o, \lambda^o, u) \tag{16}$$

for all $u \in U_{q^o}$; at the terminal time t_f the Hamiltonian equals zero

$$H_{q_f}\left(t_f\right) = 0 \tag{17}$$

and at a switching time t_j the Hamiltonian satisfies

$$H_{q_{j-1}}(t_j-) = H_{q_j}(t_j+) + p\frac{\partial m}{\partial t} \tag{18}$$

□

4. ELECTRIC VEHICLE WITH TRANSMISSION

4.1 Vehicle Dynamics

According to Newton's second law of motion, with m being the effective mass of the vehicle and z the coordinate along the road, the car's acceleration $a = dv/dt$ depends on the resultant of the traction force F_{tr}, the aerodynamic force $\frac{1}{2}\rho_a C_d A_f v^2$, the gravitational force along the road $mg \sin\gamma(z)$ and the rolling resistance force $mgC_r \cos\gamma(z)$. Thus the system dynamics is described by

$$\begin{aligned} \frac{dz}{dt} &= v \\ \frac{dv}{dt} &= \frac{1}{m}F_{tr} - \frac{1}{2m}\rho_a C_d A_f v^2 - g\sin\gamma(z) - gC_r\cos\gamma(z) \end{aligned} \tag{19}$$

Assuming that the road has zero grading (i.e. $\gamma(z) \equiv 0$) the dynamics for the car speed v becomes decoupled from its position z. The traction force F_{tr} is related to the motor torque T_M through the transmission system consisting of a dual planetary gear set (presented in the following section) and the differential with the gear ratios GR_i and i_{fd} respectively where the index i in GR_i represents the gear number. At each gear number i the dynamics of the car is described as

$$\frac{dv}{dt} = -\frac{\rho_a C_d A_f}{2m} v^2 + \frac{i_{fd} GR_i}{m R_w} T_M - g C_r \tag{20}$$

4.2 Dynamics of the Transmission

The full derivation of the dynamics of the dual planetary transmission system considered in this paper is reported in detail in [Rahimi M., Pakniyat, and Boulet (2014)]. In summary, the two-stage planetary gear sets provide a constant gear ratio when either the common sun gears or the common ring gears are held fixed. At these states, the input and output speeds and torques are geometrically related by the gear ratios of the transmission. In the first gear, the ring gear of the transmission is held fixed i.e. $\omega_R = 0$ resulting in the gear ratio

$$GR_1 = \frac{1+R_2}{1+R_1} \tag{21}$$

and in the second gear, the sun gear is held fixed i.e. $\omega_S = 0$ resulting in the gear ratio

$$GR_2 = \frac{(1+R_2)R_1}{(1+R_1)R_2} \tag{22}$$

During the gear changing, however, the mechanical degree of freedom is increased by one and the number of control inputs is increased by two. Namely, the minimum number of states required to present the dynamics of the car during the gear shifting process is two (selected here to be ω_S and ω_R the angular velocity of the sun gear and the ring gear of the transmission) and, in addition to the motor torque T_M, the brake torque acting on the sun gear T_{BS} and the brake torque acting on the ring gear T_{BR} influence the dynamics during the shifting period.

Considering that the input speed, which is the angular velocity of the motor ω_M, and the output speed, which is the car velocity v, are geometrically related to the considered states according to the equations

$$\omega_M = \frac{\omega_S + R_1 \omega_R}{1+R_1} \tag{23}$$

and

$$v = \frac{R_w(\omega_S + R_2 \omega_R)}{i_{fd}(1+R_2)} \tag{24}$$

the dynamics equation of the vehicle is derived from the transmission system dynamics in [Rahimi M., Pakniyat, and Boulet (2014); Eq. (13)] as

$$\dot{\omega}_S = -A_{SS}\omega_S + A_{SR}\omega_R - A_{SA}(\omega_S + R_2\omega_R)^2 + B_{SS}T_{BS} - B_{SR}T_{BR} + B_{SM}T_M - D_{SL} \tag{25}$$

$$\dot{\omega}_R = A_{RS}\omega_S - A_{RR}\omega_R - A_{RA}(\omega_S + R_2\omega_R)^2 - B_{RS}T_{BS} - B_{RR}T_{BR} + B_{RM}T_M - D_{RL} \tag{26}$$

with $T_{BS} \in [-|T_{BS}|^{max}, 0]$ and $T_{BR} \in [-|T_{BR}|^{max}, 0]$ and where the coefficients are introduced in Appendix A.

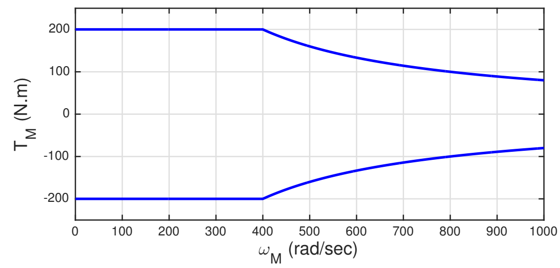

Fig. 1. The motor torque constraint T_M^{max} as a function of the motor speed ω_M

4.3 Electric Motor

The electric motor considered in this paper has specifications similar to the TM4 MOTIVE A® motor whose torque is constrained as a function of its speed according to Figure 1, namely

$$|T_M| \leq T_M^{max} \tag{27}$$

and

$$|T_M \omega_M| \leq P_M^{max} \tag{28}$$

with $T_M^{max} = 200 N.m$ and $P_M^{max} = 80 kW$.

In order to avoid mixed state and input constraints like (28) we define a change of variable by the introduction of

$$u = \frac{T_M}{T_M^{max}}, \qquad \omega_M < \omega^* \tag{29}$$

$$u = \frac{T_M \omega_m}{P_M^{max}}, \qquad \omega_M \geq \omega^* \tag{30}$$

with $\omega^* = 400 \frac{rad}{sec}$. Thus the constraints (27) and (28) will both become $u \in [-1, 1]$ which lies within the assumption A0 requiring U to be an invariant compact set.

4.4 Hybrid System Formulation

In order to present the system dynamics in the hybrid framework presented in section 2, the following discrete states are assigned to each continuous dynamics of the system:

q_1 with $x = [v] \in \mathbb{R}$ corresponds to the dynamics in the first gear and in the maximum torque limit region with the vector field

$$\dot{x} = f_1(x,u) = -A_v x^2 - B_1 u - C_r g \tag{31}$$

where

$$A_v = \frac{\rho_a C_d A_f}{2m} \tag{32}$$

and

$$B_1 = \frac{i_{fd} GR_1}{m R_w} T_M^{max} \tag{33}$$

When the motor speed $\omega_M = \frac{i_{fd} GR_1 v}{R_w}$ reaches $\omega^* = 400 rad/sec$ the system autonomously switches to q_2 with $x = [v] \in \mathbb{R}$ which corresponds to the dynamics in the first gear and in the maximum power limit region possessing the vector field

$$\dot{x} = f_2(x,u) = -A_v x^2 - B_2 \frac{u}{x} - C_r g \tag{34}$$

with

$$B_2 = \frac{P_M^{max}}{m} \tag{35}$$

The switching manifold $m_{q_1 q_2}$ is thus represented as

$$m_{q_1 q_2}(x) \equiv x - \frac{\omega^* R_w}{i_{fd} GR_1} = 0 \tag{36}$$

Due to space limitation and according to the manoeuvre studied in this paper, the dynamics for the maximum torque limit region during the gear changing process and in the second gear are eliminated and thus we assign q_3 with $x = [\omega_S, \omega_R]^T \in \mathbb{R}^2$ to the dynamics in the maximum power limit region during the gear changing with the vector field

$$\dot{x} = f_3(x,u) \tag{37}$$

where

$$\begin{aligned}\dot{x}_1 &= f_3^{(1)}(x,u) = -A_{SS}x_1 + A_{SR}x_2 - A_{SA}(x_1+R_2x_2)^2 \\ &+B_{SS}T_{BS} - B_{SR}T_{BR} + B_{SM}P_M^{max}(1+R_1)\frac{u}{x_1+R_1x_2} - D_{SL}, \\ \dot{x}_2 &= f_3^{(2)}(x,u) = A_{RS}x_1 - A_{RR}x_2 - A_{RA}(x_1+R_2x_2)^2 \\ &-B_{RS}T_{BS} - B_{RR}T_{BR} + B_{RM}P_M^{max}(1+R_1)\frac{u}{x_1+R_1x_2} - D_{RL}\end{aligned} \tag{38}$$

The jump map corresponding to the (controlled) transition between q_2 and q_3 is described by

$$x(t_{s_2}) = \xi_{q_2q_3}(x(t_{s_2}-)) = \frac{i_{fd}(1+R_2)}{R_w}\begin{bmatrix}1\\0\end{bmatrix}x(t_{s_2}-) \tag{39}$$

where the jump map $\xi_{q_2q_3} : \mathbb{R} \to \mathbb{R}^2$ from $x(t_{s_2}-) \in \mathbb{R}$ to $x(t_{s_2}) \in \mathbb{R}^2$ is differentiable.

When the speed of the sun gear ω_S becomes zero the system switches to q_4 with $x = [v] \in \mathbb{R}$ that corresponds to the dynamics in the second gear and in the maximum power limit region and the vector field becomes

$$\dot{x} = f_4(x,u) = -A_vx^2 - B_4\frac{u}{x} - C_rg \tag{40}$$

with

$$B_4 = \frac{P_M^{max}}{m} \tag{41}$$

Note that although q_2 and q_4 has the same dynamics equation in terms of the normalized control input u, the motor torque T_M and its speed ω_M are different in these two dynamics. The switching manifold corresponding to the transition from q_3 to q_4 is described as

$$m_{q_3q_4}(x) \equiv x_1 = 0 \tag{42}$$

and the jump map corresponding to this transition is described by

$$x(t_{s_3}) = \xi_{q_3q_4}(x(t_{s_3}-)) = \frac{R_w}{i_{fd}(1+R_2)}[\,1\;\;R_2\,]\,x(t_{s_3}-) \tag{43}$$

with $\xi_{q_3q_4} : \mathbb{R}^2 \to \mathbb{R}$ differentiable.

5. TIME OPTIMAL ACCELERATION

The hybrid optimal control problem considered in this paper is the minimization of the acceleration period required for reaching the top speed of $100\frac{km}{hr} = 27.78\frac{m}{s} \approx 60mph$ starting from the stationary state in the first gear and terminating in the second gear. Hence, the cost to be minimized is

$$J(u, T_{BS}, T_{BR}; t_{s_1}, t_{s_2}, t_{s_3}) = \int_{t_0}^{t_{s_1}} dt + \int_{t_{s_1}}^{t_{s_2}} dt + \int_{t_{s_2}}^{t_{s_3}} dt + \int_{t_{s_3}}^{t_f} dt \tag{44}$$

with t_f being the first time that $x(t) = 27.78$ is satisfied.

The family of system Hamiltonians are formed as

$$H_1(x,\lambda,u) = 1 + \lambda\left(-A_vx^2 - B_1u - C_rg\right) \tag{45}$$

$$H_2(x,\lambda,u) = 1 + \lambda\left(-A_vx^2 - B_2\frac{u}{x} - C_rg\right) \tag{46}$$

$$\begin{aligned}&H_3(x,\lambda,u,T_{BS},T_{BR}) \\ &= 1 + \lambda_1\Big(-A_{SS}x_1 + A_{SR}x_2 - A_{SA}(x_1+R_2x_2)^2 \\ &+B_{SS}T_{BS} - B_{SR}T_{BR} + B_{SM}P_M^{max}(1+R_1)\frac{u}{x_1+R_1x_2} - D_{SL}\Big) \\ &+\lambda_2\Big(A_{RS}x_1 - A_{RR}x_2 - A_{RA}(x_1+R_2x_2)^2 \\ &-B_{RS}T_{BS} - B_{RR}T_{BR} + B_{RM}P_M^{max}(1+R_1)\frac{u}{x_1+R_1x_2} - D_{RL}\Big)\end{aligned} \tag{47}$$

and

$$H_4(x,\lambda,u) = 1 + \lambda\left(-A_vx^2 - B_4\frac{u}{x} - C_rg\right) \tag{48}$$

Then according to the Time Optimal Hybrid Minimum Principle in section 3, the adjoint process dynamics is determined as

$$\dot{\lambda} = \frac{-\partial H_1}{\partial x} = (2A_vx)\lambda, \qquad t \in [t_0, t_{s_1}] \tag{49}$$

$$\dot{\lambda} = \frac{-\partial H_2}{\partial x} = -\left(-2A_vx + B_2\frac{u^o}{x^2}\right)\lambda, \qquad t \in (t_{s_1}, t_{s_2}] \tag{50}$$

$$\dot{\lambda} = \frac{-\partial H_3}{\partial x}, \qquad t \in (t_{s_2}, t_{s_3}] \tag{51}$$

with

$$\begin{aligned}\dot{\lambda}_1 &= \frac{-\partial H_3}{\partial x_1} = \\ &-\lambda_1\left(-A_{SS} - 2A_{SA}(x_1+R_2x_2) - B_{SM}P_M^{max}(1+R_1)\frac{u^o}{(x_1+R_1x_2)^2}\right) \\ &-\lambda_2\left(A_{RS} - 2A_{RA}(x_1+R_2x_2) - B_{RM}P_M^{max}(1+R_1)\frac{u^o}{(x_1+R_1x_2)^2}\right),\end{aligned} \tag{52}$$

$$\begin{aligned}\dot{\lambda}_2 &= \frac{-\partial H_3}{\partial x_2} = \\ &-\lambda_1\left(A_{SR} - 2R_2A_{SA}(x_1+R_2x_2) - B_{SM}P_M^{max}(1+R_1)\frac{R_1u^o}{(x_1+R_1x_2)^2}\right) \\ &-\lambda_2\left(-A_{RR} - 2R_2A_{RA}(x_1+R_2x_2) - B_{RM}P_M^{max}(1+R_1)\frac{R_1u^o}{(x_1+R_1x_2)^2}\right)\end{aligned} \tag{53}$$

as well as

$$\dot{\lambda} = \frac{-\partial H_4}{\partial x} = -\left(-2A_vx + B_4\frac{u^o}{x^2}\right)\lambda, \qquad t \in (t_{s_3}, t_f] \tag{54}$$

The boundary conditions for λ are determined from Eq. (15) as

$$\begin{aligned}\lambda(t_{s_3}) &= \nabla\xi_{q_3q_4}^T\lambda(t_{s_3}+) + p_3\nabla m_{q_3q_4} \\ &= \frac{R_w}{i_{fd}(1+R_2)}\begin{bmatrix}1\\R_2\end{bmatrix}\lambda(t_{s_3}+) + p_3\begin{bmatrix}1\\0\end{bmatrix}\end{aligned} \tag{55}$$

$$\lambda(t_{s_2}) = \nabla\xi_{q_2q_3}^T\lambda(t_{s_2}+) = \frac{i_{fd}(1+R_2)}{R_w}[\,1\;\;0\,]\,\lambda(t_{s_2}+) \tag{56}$$

$$\lambda(t_{s_1}) = \lambda(t_{s_1}+) + p_1 \tag{57}$$

It can be easily verified that for the above dynamics and boundary conditions, the adjoint process has a negative sign for all $t \in [t_0, t_f]$ (see also Fig. 2) and hence the Hamiltonian minimization condition (16) results in $u^o(t) = 1$ for $t \in [t_0, t_f]$ as well as $T_{BS}^o(t) = -|T_{BS}|^{max}$ and $T_{BR}^o(t) = 0$ for $t \in [t_{s_2}, t_{s_3}]$.

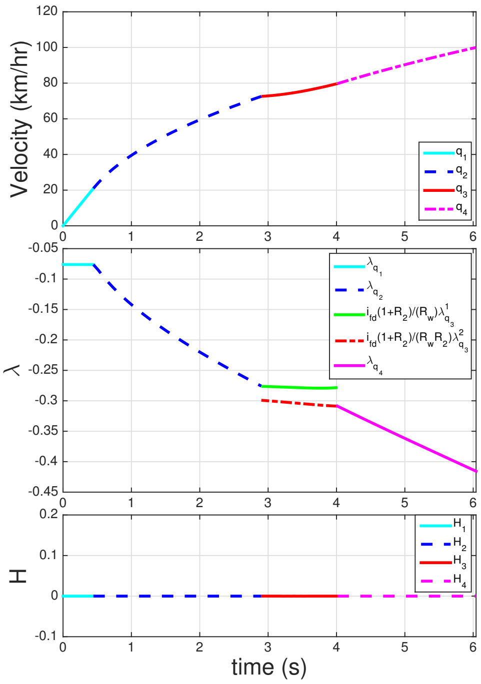

Fig. 2. The car speed, the adjoint processes and the corresponding Hamiltonians for the minimum acceleration period problem

The Hamiltonian terminal condition (17) gives

$$\begin{aligned} &H_4\left(x\left(t_f\right),\lambda\left(t_f\right),u\left(t_f\right)\right) \\ &\quad = 1+\lambda\left(t_f\right)\left(-A_v x\left(t_f\right)^2 - B_4\frac{u\left(t_f\right)}{x\left(t_f\right)} - C_r g\right) = 0 \end{aligned} \tag{58}$$

and the Hamiltonian continuity at switching instants is deduced from (18) as

$$H_3\left(x,\lambda,u\right)_{\left(t_{s_3}-\right)} = H_4\left(x,\lambda,u\right)_{\left(t_{s_3}+\right)} \tag{59}$$

$$H_2\left(x,\lambda,u\right)_{\left(t_{s_2}-\right)} = H_3\left(x,\lambda,u\right)_{\left(t_{s_2}+\right)} \tag{60}$$

$$H_1\left(x,\lambda,u\right)_{\left(t_{s_1}-\right)} = H_2\left(x,\lambda,u\right)_{\left(t_{s_1}+\right)} \tag{61}$$

The results for the parameter values presented in Appendix B are illustrated in Figure 2. For better illustration, the speed of the vehicle is shown in km/hr and, in addition, the components λ_1 and λ_2 of the adjoint process in $t \in \left[t_{s_2}, t_{s_3}\right]$ are multiplied by $i_{fd}\left(1+R_2\right)/R_w$ and $i_{fd}\left(1+R_2\right)/\left(R_w R_2\right)$ respectively so that the boundary conditions (55) and (56) can be verified more easily. The optimal values for the switching and final times are $t_{s_1} = 0.444$, $t_{s_2} = 2.901$, $t_{s_3} = 4.014$, $t_f = 6.042$.

ACKNOWLEDGEMENT

This work is supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) and the Automotive Partnership Canada (APC). The authors would also like to acknowledge the transmission design team (Mir Saman Rahimi Mousavi, Benoit Boulet and other team members at McGill University) and the industrial partners TM4, Linamar, and Infolytica.

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Appendix A. TRANSMISSION DYNAMICS DERIVATION

The dynamics equation for the transmission is presented in [Rahimi M., Pakniyat, and Boulet (2014); Eq. (13)] as

$$\dot{\omega}_S = \frac{1}{a}\left(-C_S\tau\omega_S + C_R\lambda\omega_R + \tau\left[T_{BS}+T_{Sf}\right] - \lambda\left[T_{BR}+T_{Rf}\right] + cT_M - dT_l\right) \quad (A.1)$$

$$\dot{\omega}_R = \frac{1}{a}\left(C_S\lambda\omega_S - C_R\gamma\omega_R - \lambda\left[T_{BS}+T_{Sf}\right] + \gamma\left[T_{BR}+T_{Rf}\right] + eT_M - fT_l\right) \quad (A.2)$$

With this transmission mounted on a vehicle with the dynamics equation (20) the load torque T_l on the transmission is related to the resistance forces on the car and the acceleration term in the form of

$$T_l = \frac{R_w}{i_{fd}}\left(\frac{\rho C_d A_f R_w^2\left(\omega_S + R_2\omega_R\right)^2}{2i_{fd}^2\left(1+R_2\right)^2} + C_r mg + \frac{mR_w\left(\dot{\omega}_S + R_2\dot{\omega}_R\right)}{i_{fd}\left(1+R_2\right)}\right) \quad (A.3)$$

Substituting (A.3) into (A.1) and (A.2) and solving for the explicit equations for $\dot{\omega}_S$ and $\dot{\omega}_R$, the equations (25) and (26) are achieved with the parameters related to the values presented in [Rahimi M., Pakniyat, and Boulet (2014)] and in Table B.2 by

$$\begin{aligned}
A_{SS} &= \frac{C_S\left(a\,i_{fd}^2\left(1+R_2\right)\tau + \left(f\tau + d\lambda\right)R_2 mR_w^2\right)}{a^2 i_{fd}^2\left(1+R_2\right) + a\left(d+R_2 f\right)mR_w^2} \\
A_{SR} &= \frac{C_R\left(a\,i_{fd}^2\left(1+R_2\right)\lambda + \left(f\lambda + d\gamma\right)R_2 mR_w^2\right)}{a^2 i_{fd}^2\left(1+R_2\right) + a\left(d+R_2 f\right)mR_w^2} \\
A_{SA} &= \frac{d\rho C_d A_f R_w^3}{2i_{fd}\left(1+R_2\right)\left(a\,i_{fd}^2\left(1+R_2\right) + \left(d+R_2 f\right)mR_w^2\right)} \\
B_{SS} &= \frac{a\,i_{fd}^2\left(1+R_2\right)\tau + \left(f\tau + d\lambda\right)R_2 mR_w^2}{a^2 i_{fd}^2\left(1+R_2\right) + a\left(d+R_2 f\right)mR_w^2} \\
B_{SR} &= \frac{a\,i_{fd}^2\left(1+R_2\right)\lambda + \left(f\lambda + d\gamma\right)R_2 mR_w^2}{a^2 i_{fd}^2\left(1+R_2\right) + a\left(d+R_2 f\right)mR_w^2} \\
B_{SM} &= \frac{a\,i_{fd}^2\left(1+R_2\right)c + \left(fc - de\right)R_2 mR_w^2}{a^2 i_{fd}^2\left(1+R_2\right) + a\left(d+R_2 f\right)mR_w^2} \\
D_{SL} &= \frac{a\,i_{fd}\left(1+R_2\right)R_w dC_r mg}{a^2 i_{fd}^2\left(1+R_2\right) + a\left(d+R_2 f\right)mR_w^2} \\
&\quad - \frac{\left(a\,i_{fd}^2\left(1+R_2\right)\tau + \left(f\tau + d\lambda\right)R_2 mR_w^2\right)T_{Sf}}{a^2 i_{fd}^2\left(1+R_2\right) + a\left(d+R_2 f\right)mR_w^2} \\
&\quad - \frac{\left(a\,i_{fd}^2\left(1+R_2\right)\lambda + \left(f\lambda + d\gamma\right)R_2 mR_w^2\right)T_{Rf}}{a^2 i_{fd}^2\left(1+R_2\right) + a\left(d+R_2 f\right)mR_w^2}
\end{aligned} \quad (A.4)$$

$$\begin{aligned}
A_{RS} &= \frac{C_S\left(a\,i_{fd}^2\left(1+R_2\right)\lambda + \left(f\tau + d\lambda\right)mR_w^2\right)}{a^2 i_{fd}^2\left(1+R_2\right) + a\left(d+R_2 f\right)mR_w^2} \\
A_{RR} &= \frac{C_R\left(a\,i_{fd}^2\left(1+R_2\right)\gamma + \left(f\lambda + d\gamma\right)mR_w^2\right)}{a^2 i_{fd}^2\left(1+R_2\right) + a\left(d+R_2 f\right)mR_w^2} \\
A_{RA} &= \frac{f\rho C_d A_f R_w^3}{2i_{fd}\left(1+R_2\right)\left(a\,i_{fd}^2\left(1+R_2\right) + \left(d+R_2 f\right)mR_w^2\right)} \\
B_{RS} &= \frac{a\,i_{fd}^2\left(1+R_2\right)\lambda + \left(f\tau + d\lambda\right)mR_w^2}{a^2 i_{fd}^2\left(1+R_2\right) + a\left(d+R_2 f\right)mR_w^2} \\
B_{RR} &= \frac{a\,i_{fd}^2\left(1+R_2\right)\gamma + \left(f\lambda + d\gamma\right)mR_w^2}{a^2 i_{fd}^2\left(1+R_2\right) + a\left(d+R_2 f\right)mR_w^2} \\
B_{RM} &= \frac{a\,i_{fd}^2\left(1+R_2\right)e + \left(de - fc\right)mR_w^2}{a^2 i_{fd}^2\left(1+R_2\right) + a\left(d+R_2 f\right)mR_w^2} \\
D_{SL} &= \frac{a\,i_{fd}\left(1+R_2\right)R_w f C_r mg}{a^2 i_{fd}^2\left(1+R_2\right) + a\left(d+R_2 f\right)mR_w^2} \\
&\quad - \frac{\left(a\,i_{fd}^2\left(1+R_2\right)\lambda + \left(f\tau + d\lambda\right)mR_w^2\right)T_{Sf}}{a^2 i_{fd}^2\left(1+R_2\right) + a\left(d+R_2 f\right)mR_w^2} \\
&\quad - \frac{\left(a\,i_{fd}^2\left(1+R_2\right)\gamma + \left(f\lambda + d\gamma\right)mR_w^2\right)T_{Rf}}{a^2 i_{fd}^2\left(1+R_2\right) + a\left(d+R_2 f\right)mR_w^2}
\end{aligned} \quad (A.5)$$

Appendix B. PARAMETER VALUES

The car parameters considered in this paper are presented in Table B.1 and the values for the parameters of the transmission are brought from [Rahimi M., Pakniyat, and Boulet (2014)] and are presented in Table B.2.

Parameter	Value	Unit
m	1000	kg
R_w	0.3	m
ρ	1.2	$\frac{kg}{m^3}$
A_f	2	m^2
C_d	0.3	–
C_r	0.02	–
g	9.81	$\frac{m}{s^2}$
i_{fd}	12	–

Table B.1. The parameters considered for the electric vehicle

Parameter	Value
a	1.1098×10^{-4}
c	1.0182×10^{-2}
d	1.7858×10^{-3}
e	-6.913×10^{-4}
f	1.3320×10^{-3}
C_S	1×10^{-3}
C_R	1×10^{-3}
τ	5.2162×10^{-2}
λ	1.0808×10^{-2}
γ	4.3671×10^{-3}

Table B.2. The parameters considered for the transmission