Geodesic Thin Plate Splines for Image Segmentation

Herve Lombaert, Farida Cheriet
École Polytechnique de Montréal, Canada
herve.lombaert@polymtl.ca

Abstract

Thin Plate Splines are often used in image registration to model deformations. Its physical analogy involves a thin lying sheet of metal that is deformed and forced to pass through a set of control points. The Thin Plate Spline equation minimizes that thin plate bending energy. Rather than using Euclidean distances between control points for image deformation, we are using geodesic distances for image segmentation. Control points become seed points and force the thin plate to pass through given heights. Intuitively, the thin plate surface in the vicinity of a seed point within a region should have similar heights. The minimally bended thin plate actually gives a "confidence" map telling what the closest seed point is for every surface point. The Thin Plate Spline has a closed-form solution which is fast to compute and global optimal. This method shows comparable results to the Graph Cuts method.

1 Introduction

Image segmentation is a crucial part when trying to understand and analyze images. The main goal is to localize different segments of an image. Various methods exist and they are often categorized into either intensity or contour based. The latter category usually involves variational approaches ([1]), and solutions are typically limited to local minima. Intensity based approaches can however offer global minimum solutions. Among these methods, the Graph Cut ([2, 3]) relies on maxflow algorithms to find minimal cuts in an image, and the Random Walker ([4]) solves the Laplace's equation ($\nabla^2 f = 0$) to find unknown potentials in an electrical circuit. The biharmonic equation ($\Delta^2 f = 0$) also has interesting smoothing properties. A classifier as been developed in [5] and used for image segmentation, its solution approximates a inhomogeneous biharmonic equation with Dirichlet boundary conditions. The Thin Plate Splines ([6]) solve the biharmonic equation to find a minimally bended surface passing through a set of control points. Although no application has been used in image segmentation so far, it is largely used to model image deformation. Other deformation models exist but they are defined in a local manner whereas the Thin Plate Splines are motivated by an underlying physical explanation. Many image registration techniques ([7]) rely on such deformation. Image smoothing ([8]) is also possible. In our method, we use geodesic distances in Thin Plate Splines for multi-label image segmentation. The control points are used to mark an image, and the minimally bended surface height gives the segmentation labeling. First, Thin Plate Splines are explained, followed by how the use of geodesic distances permits image segmentation. Results are later shown with a discussion comparing this method with current state-of-the-art algorithms.

2 Thin Plate Splines for Image Segmentation

Finding the surface that bends the least while passing through all the control points (Fig. 1) could be done by minimizing its second order derivatives, $\nabla^2 f = 0$ (or $\Delta f = 0$). In the Thin Plate Splines, the biharmonic equation, $\Delta^2 f = 0$, is minimized, and its fundamental solution is $U(r) = r^2 \log r$ in 2D. The Thin Plate Spline equation gives the surface height $f(x, y)$ for any point (x, y) and is defined by

$$f(x,y) = a_1 + a_x x + a_y y + \sum_{i=1}^{i=N} w_i U\left(||P_i - (x,y)||\right), \tag{1}$$

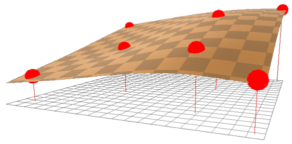


Figure 1. A Thin Plate Spline passing through 7 control points.

where $||P_i - (x, y)||$ is the distance from the point (x, y) to the ith control point, and a_1, a_x, a_y, w_i's are the spline coefficients. The first three terms correspond to the linear part, a plane fitting (in a least square sense). The final summation term corresponds to the bending forces exercised by the N control points.

It can be shown [6] that $f(x, y)$ is also a solution to the biharmonic equation $\Delta^2 f = 0$. The function $f(x, y)$ minimizes the bending energy defined by

$$J_f = \int \left(\left(\frac{\partial^2 f}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 f}{\partial y^2} \right)^2 \right) dxdy.$$

Coefficients $[\mathbf{W} \,|\, a_1\, a_x\, a_y]$ from Eq. 1 (here $\mathbf{W} = [w_1\, w_2 \ldots w_N]$) can all be found by solving the linear equation

$$Y = [W \,|\, a_1\, a_x\, a_y]^T\, L,$$

where Y is the surface elevation at the given control points, $Y = [f(x_1, y_1)\, f(x_2, y_2) \ldots f(x_N, y_N) \,|\, 0\, 0\, 0]$, and L is the matrix specifying the relations among the control points,

$$L = \begin{bmatrix} K & P \\ P^T & 0 \end{bmatrix},$$

K and P are defined as,

$$P = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ \ldots & \ldots & \ldots \\ 1 & x_N & y_N \end{bmatrix}, K = \begin{bmatrix} 0 & U_{12} & \ldots & U_{1N} \\ U_{21} & 0 & \ldots & U_{2N} \\ \ldots & \ldots & \ldots & \ldots \\ U_{N1} & U_{N2} & \ldots & 0 \end{bmatrix}$$

with $U_{ij} = U(||P_i - P_j||)$. Notice here that matrix L is symmetric, and the Thin Plate Spline equation can be efficiently solved with an LU decomposition.

To summarize, the Thin Plate Spline defines a surface that passes through given control points while minimizing its bending energy. The figure 1 shows a Thin Plate Spline passing through seven control points.

Figure 2. Geodesic distance map from a pixel (white cross) to all other pixels.

2.1 Geodesic Thin Plate Splines

Thin Plate Splines are commonly used with an L_2 metric, i.e., distances are Euclidean $||P_i - (x, y)||_2 = \left((x_i - x)^2 + (y_i - y)^2\right)^{\frac{1}{2}}$. We will rather use geodesic distances based on an underlying image. A weighted undirected graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ is constructed from an image, where each node corresponds to a pixel

$$||P_i - (x, y)|| = \sum_{(p,q) \in Q^*} d(p, q),$$

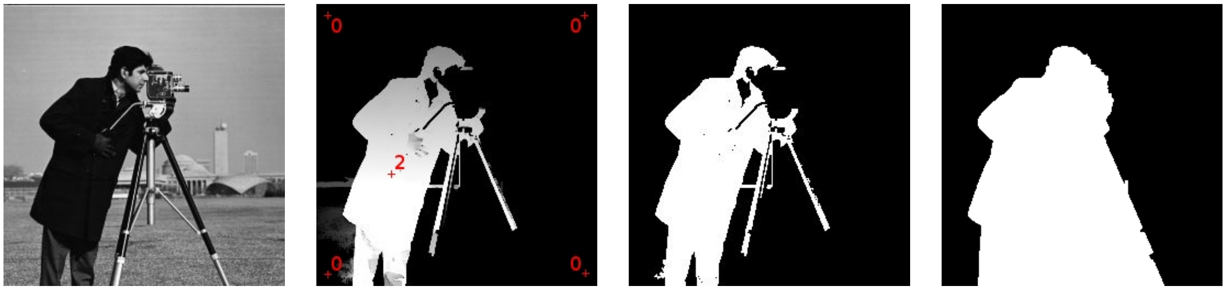
where Q^* is the shortest path in the graph between the nodes corresponding to the control point P_i and the point (x, y). The edge weight $d(p, q)$ is an arbitrary distance function between two neighboring nodes. In our experiments, it is based on the image intensity difference

$$d(p, q) = 1 - exp\left(-\frac{(I_p - I_q)^2}{2\sigma^2} \right)$$

A long distance between two points means there are large intensity variations (i.e., structure boundaries) along the shortest path between these two points. The constant parameter σ controls this disparity penalization. Figure 2 shows an example of such a distance map, pixel intensities give the distance from the pixel to the seed point (i.e., dark means pixel is close to the seed point, brighter means farther).

2.2 Segmentation

Interactive segmentation is performed by marking the image regions with seed points, each region is associated with a certain label. We propose to use these seed points as control points bending a Thin Plate Spline. The spline surface covers the



(a) Original Image (b) Surface Elevation (c) Segmentation (d) Graph Cuts

Figure 3. (a) an original image, (b) the Geodesic Thin Plate Spline with the seed points marking the foreground/background (one seed point in the cameraman coat set at elevation 2, and four others in each corner set at elevation 0), and (c) a segmentation separating all pixels above and below elevation 1, and (d) a comparison with Graph Cuts.

whole image, and its elevation is governed by the heights of the control points. Each region is associated with a label, for instance, there is a control point with elevation 1 in image region 1, a control point with elevation 2 in image region 2, and so on.

The Thin Plate Spline elevation is still defined by equation 1. The only difference is that now the distances depend on intensity variations along a shortest path. This causes the spline surface to have similar elevation within the same image region. For instance, if a seed point in region 1 has elevation 1, the spline surface covering the pixels in this region would have an elevation close to 1, if a seed point in region 2 has elevation 2, that region would have elevations close to 2, and so on. Figure 3(b) shows an example of such surface elevations. A Thin Plate Spline using geodesic distances tells how close a pixel is to another seed point, e.g., a pixel with elevation 1.2 is 0.2 apart from the seed point at elevation 1, and 0.8 apart from the seed point at elevation 2.

Segmentation is performed by finding iso-contours separating seed points on the Thin Plate Spline, e.g., all points below elevation 1.5 are considered part of region 1, and points above it are considered part of region 2. Iso-values are the mid-values between seed point elevations. Figure 3(c) shows an example of a segmented region.

3 Results

The method has been applied on synthetic and real images. The method shows promising results and are comparable to the Graph Cuts method. Seed points are placed manually in different regions and they control a Geodesic Thin Plate Spline. Geodesic distances from the seed points are precomputed with region growths using priority queues, where only boundary pixels closest to the seed points are grown first.

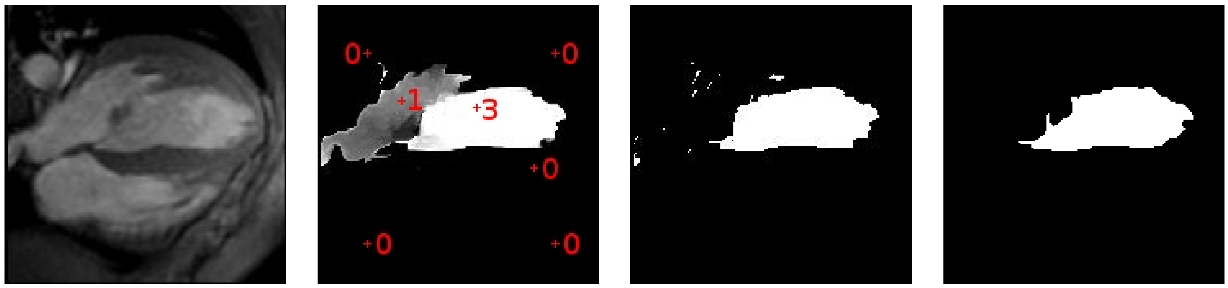
In the first experiment presented here (Fig. 3), a foreground in a 256×256 is extracted from its background. One seed point is in the cameraman coat with elevation 2, and four others are in each image corner with elevation 0 marking the background. The iso-value chosen for the segmentation is 1. The segmentation process took 0.42 sec in a 2.40GHz Core 2 CPU. The same experiment with the Graph Cuts algorithm failed because one camera leg yielded a smaller cut.

In the second experiment (Fig. 4), a heart picture depicts an unclear boundary between the left atrium (LA) and left ventricle (LV). One seed point is placed in the LA with elevation 1, one seed point in the LV with elevation 3, and five other points in the background with elevation 0, one in each corner and one in the myocardium (middle of the picture). The iso-value chosen for the final segmentation is 2. The segmentation process took 0.25 sec.

4 Conclusion

This paper introduces a new method for image segmentation. It brings a well known technique widely used to model deformations to the problem of image segmentation. It is to our knowledge the first attempt of using Thin Plate Splines in image segmentation.

The surface equation is defined in the continuous domain, thus, compared to other state-of-the-art algorithms such as the graph cuts, the method has a sub-pixel accuracy. Furthermore, as opposed



(a) Original Image (b) Surface Elevation (c) Segmentation (d) Graph Cuts

Figure 4. (a) a heart with an unclear boundary between the left atrium (LA) and left ventricle (LV), (b) the Geodesic Thin Plate Spline with one seed point in the LA at elevation 1, one in the LV at elevation 3, and five other seed points at elevation 0, (c) the LV is segmented by taking all pixels whose elevation is above 2, and (d) a Graph Cuts comparison.

to the graph cuts where global optimum is guaranteed only for the binary case, our method can handle multiple labels as long as there is an ordinal structure. Recently, [9] proposes a new graph construction to handle multi labels where each region/boundary is enclosed within each other.

Fast linear algebra is used to find a global optimal solution of the biharmonic equation. Similarly to the Random Walker method, essentially an LU decomposition is used to find the segmentation. However, the number of unknowns is quite small in our case. With K seeds, in our method, there are $K+3$ unknown coefficients (e.g., with 4 seed points there are 7 unknowns ($a_{1,x,y}$ and w_i's)), and in the Random Walker, there are $|I| - K$ unknowns (i.e., the potentials of the unseeded pixels).

In this paper we thus propose an alternative to the current state-of-the-art segmentation algorithm. The method also yields global optimal solutions, is also fast, and also interactive (more latitude would even be possible by relaxing control points ([10]). Possible extensions also include the use of polyharmonic splines for N-D applications ([7]). The results presented earlier already show promising segmentations. There are still open questions, essentially how to handle multi labels in non ordinal structures, what are the effects of a collapsing surface (a possible cause yielding white dots in Fig. 4(c)), what is the robustness to seed point locations, and how to generalize our formulation in a Riemannian metric. Future work will also conduct exhaustive comparisons with the Random Walker.

Acknowledgement

We wish to thank Leo Grady for sharing interesting discussions and comments, as well as Christopher Pal for providing meaningful citations. Funding has been supported by the National Science and Environment Research Council (NSERC).

References

[1] S. Osher and J. A. Sethian. Fronts propagating with curvature-dependent speed: Algorithms based on hamilton-jacobi formulations. *J. of Comp. Physics*, 79:12–49, 1988.

[2] Y. Boykov and V. Kolmogorov. An experimental comparison of min-cut/max-flow algorithms for energy minimization in vision. *TPAMI*, 26(9):1124–1137, 2004.

[3] Y. Boykov and M. Jolly. Interactive graph cuts for optimal boundary/region segmentation of objects in n-d images. In *ICCV*, pages 105–112, 2001.

[4] L. Grady. Random walks for image segmentation. *TPAMI*, 28(11):1768–1783, 2006.

[5] J. Wang, F. Wang, C. Zhang, H. C. Shen, and L. Quan. Linear neighborhood propagation and its applications. *TPAMI*, 31(9):1600–1615, 2008.

[6] F. L. Bookstein. Principal warps: thin-plate splines and the decomposition of deformations. *TPAMI*, 11(6):567–585, 1989.

[7] R. Sprengel, K. Rohr, and H. S. Stiehl. Thin-plate spline approximation for image registration. In *EMBS*, volume 3, pages 1190–1191, 1996.

[8] X. Xie, M. Chung, and G. Wahba. Magnetic resonance image segmentation with thin plate spline thresholding. Technical Report 1105, U. of Wisconsin, Dept. of Statistics, 2006.

[9] A. Delong and Y. Boykov. Globally optimal segmentation of multi-region objects. In *ICCV*, 2009.

[10] G. Wahba. *Spline models for observational data*, volume 59 of *CBMS-NSF Reg. Conf. Series in App. Math.* SIAM, 1990.