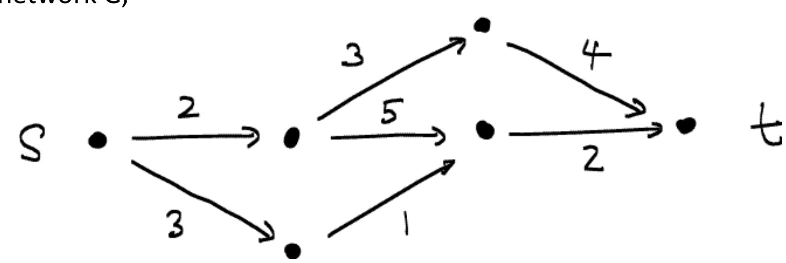
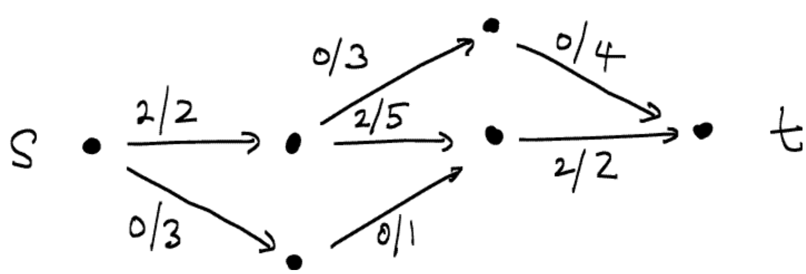
Exercises: Network Flows 2

Questions

1. Consider a flow network G,



and a flow f, where I have written f(e)/ c(e) at each edge. (This is a notation that is commonly used to show both the flow and capacities on a single graph).



Draw the residual graph G_f and find an augmenting path (in the residual graph).

2.
   a. Each maximum flow defines a minimum capacity cut, using the method discussed in class. Is this minimum capacity cut unique? That is, for a given maximum flow, could there be more than one minimum capacity cut for this flow?

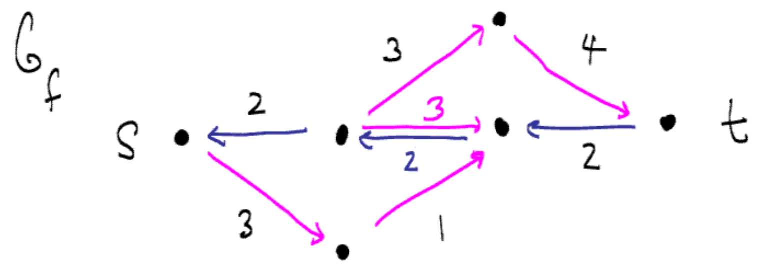
   b. Does each minimum capacity cut define a unique maximum flow?

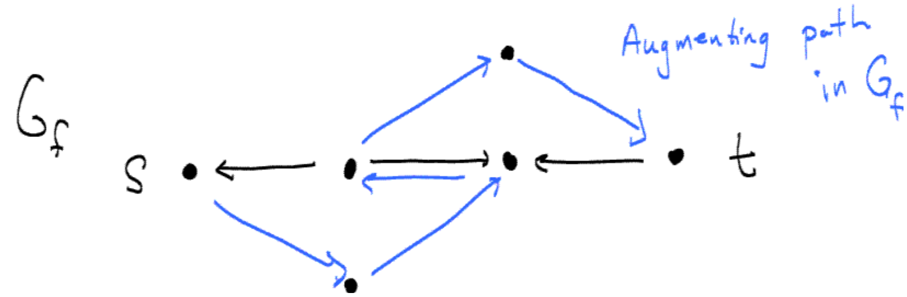
3. Suppose you are organizing a conference where researchers present articles they have written. Researchers who want to present an article send a paper to the conference organizers. The conference organizers have access to a committee of reviewers who are each willing to read up mB articles each. Each paper submission gets reviewed by up to mA reviewers. Moreover, each submission has a particular topic and each reviewer has a specialization for a set of topics, so papers on a given topic only get reviewed by those reviewers who are experts on that topic. The conference organizers need to decide which reviewers will review each article (or equivalently, which articles will be reviewed by which reviewers). Explain how they could use a flow network to solve this problem.

4. Consider the problem defined at the end of the lecture of finding a maximal matching in a bipartite graph. It was claimed that if you augment this graph with vertices s and t (see details in the lecture), and use Ford-Fulkerson to find a maximum flow in this new graph, then the max flow corresponds to a maximal matching in the original bipartite graph. Why ? (Its intuitively obvious but see if you can give a more formal argument.)

Answers

1. Here I have marked the forward and backward edges with coloring scheme used in the lectures.



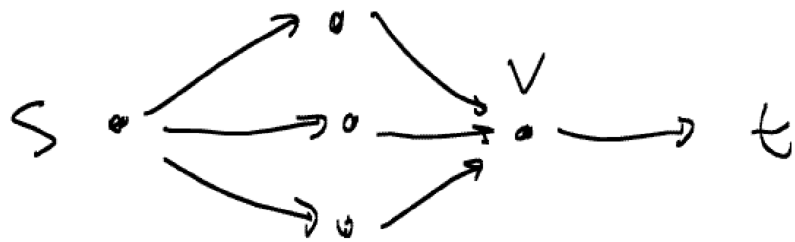


The (bottleneck) for the flow on this augmenting path is $\beta = 1$.

2.

a. The mincut might not be unique. Consider a network flow that is two edges (s,u) and (u, t) both having the same capacity. There is only one max flow, but there are two min cuts.

b. No, the maximum flow might not be unique. An example is shown below, where all edges have the same capacity e.g. 1. The edge (v,t) is a min st cut but there are 3 possible max flows, which correspond to the three edges out of s.



3. There is a set A of articles and a set B of reviewers. (We could define it the other way around.) Add a directed edge (α,β) to the graph if some article α in A could be reviewed by some reviewer β in B, namely the article is on a topic that the reviewer is an expert in. Set the capacity of that edge to be 1. Add a vertex s (source) and a vertex t (terminal). For each α in A, add an edge (s, α) of capacity mA. For each β in B, add an edge (β , t) of capacity mB. Run Ford-Fulkerson to get the maximum flow and compute the corresponding cut(A,B). This gives the set of edges which correspond to the assignment of articles to reviewers.

4. Here is a proof by contradiction. Suppose that the flow found by Ford-Fulkerson on the constructed st graph did not give a maximal matching in the original bipartite graph. That is, suppose there were a matching on the bipartite graph that was bigger (more edges) than the one found by Ford-Fulkerson. But any matching on the bipartite graph defines a flow on the st graph and the amount of that flow is equal to the size of the matching (by construction). So then the flow found by Ford-Fulkerson could not have been the max flow. But this is impossible since Ford Fulkerson computes the max flow.