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Passivity of nonlinear incremental systems: Application to PI stabilization of nonlinear RLC circuits ☆

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Abstract

It is well known that if the linear time invariant system $\dot{x}=\mathbf{A}x+\mathbf{B}u$, $y=\mathbf{C}x$ is passive the associated incremental system $\dot{\tilde{x}}=\mathbf{A}\tilde{x}+\mathbf{B}\tilde{u}$, $\tilde{y}=\mathbf{C}\tilde{x}$, with $\tilde{(\cdot)}=(\cdot)-(\cdot)^\star$, $u^\star, y^\star$ the constant input and output associated to an equilibrium state $x^\star$, is also passive. In this paper, we identify a class of nonlinear passive systems of the form $\dot{x}=f(x)+gu$, $y=h(x)$ whose incremental model is also passive. Using this result we then prove that a large class of nonlinear RLC circuits with strictly convex electric and magnetic energy functions and passive resistors with monotonic characteristic functions are globally stabilizable with linear PI control.

Keywords: Passivity; Incremental models; Nonlinear systems

1. Problem formulation

In many control applications one is interested in operating the system around a *non-zero* equilibrium point. A standard procedure to describe the dynamics in these cases is to generate a so-called *incremental model* with inputs and outputs the deviations with respect to their value at the equilibrium. A natural question that arises is whether a property of the original system is inherited by its incremental model. In this paper, we explore this question regarding passivity. More specifically, we provide a solution to the following problem.

Passivity of Incremental Systems: Given a nonlinear system of the form:

$$\begin{aligned}\dot{x}&=f(x)+gu,\\ y&=h(x),\end{aligned}\tag{1}$$

where x, u, y are functions of t, $x(t)\in\mathbb{R}^n$, $u(t), y(t)\in\mathbb{R}^m$, with $m\leqslant n$, the functions f, h are locally Lipschitz and the matrix $g\in\mathbb{R}^{n\times m}$ is *constant* and has full rank. Fix an equilibrium state $x^\star\in\mathbb{R}^n$, that is,

$$x^\star\in\mathscr{E}:=\{\bar{x}\in\mathbb{R}^n|g^\perp f(\bar{x})=0\},\tag{2}$$

where $g^\perp\in\mathbb{R}^{(n-m)\times n}$ is a full-rank left-annihilator of g, i.e., $g^\perp g=0$ and $\mathrm{rank}\{g^\perp\}=n-m$, and define the constant input and output vectors associated to $x^\star$ as

$$\begin{aligned}u^\star&:=(g^\top g)^{-1}g^\top f(x^\star),\\ y^\star&:=h(x^\star).\end{aligned}\tag{3}$$

Define the incremental model

$$\begin{aligned}\dot{x}&=f(x)+gu^\star+g\tilde{u},\\ \tilde{y}&=h(x)-h(x^\star),\end{aligned}\tag{4}$$

where $\tilde{(\cdot)}=(\cdot)-(\cdot)^\star$ are the incremental variables.

Assume (1) defines a passive mapping $u\to y$. Under which conditions the mapping $\tilde{u}\to\tilde{y}$, defined by (4), is also passive?

The *main contributions* of this paper are, first, the establishment of a condition on the vector field $f(x)$ to ensure passivity of the mapping $\tilde{u}\to\tilde{y}$. Second, we prove that a large class of nonlinear RLC circuits—with strictly convex electric and

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magnetic energy functions and passive resistors with monotonic characteristic functions—satisfy this condition, hence showing that these circuits can be globally stabilized with linear PI control.

2. Some comments and motivation

(1) The question posed above can be recast without invoking incremental models, but using the more general concept of dissipative systems [9], as follows. Assume (1) is dissipative with supply rate $\boldsymbol{u}^\top \boldsymbol{y}$, (this is, of course, equivalent to passivity of the mapping $\boldsymbol{u} \to \boldsymbol{y}$). Under which conditions (1) is also dissipative with respect to the incremental supply rate $\tilde{\boldsymbol{u}}^\top \tilde{\boldsymbol{y}}$? In view of the ubiquity of incremental models in applications we have opted for the formulation of the problem given above.

(2) Invoking Kalman–Yakubovich–Popov's Lemma [8] it is easy to establish that all passive linear time invariant (LTI) systems have passive incremental models. Indeed, if $H(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^\top \mathbf{P}\boldsymbol{x}$, with $\mathbf{P} \in \mathbb{R}^{n\times n}$, $\mathbf{P} = \mathbf{P}^\top > 0$, is a storage function for the original system, $H(\tilde{\boldsymbol{x}}) = \frac{1}{2}\tilde{\boldsymbol{x}}^\top \mathbf{P}\tilde{\boldsymbol{x}}$ is a storage function for the incremental model as well.

(3) Passivity of incremental models has been explored in [4] for the case when (1) is a port controlled Hamiltonian system [8]. Actually, the storage function constructed here is the one used in [4]—but expressed in the original coordinates of the system, see Remark 1.

(4) Motivations to establish passivity of incremental models are manifold. It has been used in [3] for tracking and disturbance rejection—via internal model principles—in passive systems. Another immediate application concerns energy-balancing stabilization. As defined in [7], see also [6], a system is energy-balancing stabilizable if there exists a static state-feedback that assigns to the closed-loop system a storage function equal to the difference between the (open-loop) systems stored energy and the energy supplied by the controller, i.e., $\int \boldsymbol{u}^\top(s)\boldsymbol{y}(s)\,\mathrm{d}s$. As indicated in [7], energy-balancing stabilization is stymied by the presence of pervasive dissipation. The latter is defined as dissipation that makes the supplied power evaluated at the equilibrium non-zero, that is $(\boldsymbol{u}^\star)^\top \boldsymbol{y}^\star \neq 0$. It is clear that this obstacle is conspicuous by its absence in systems with passive incremental models. Results stemming from this observation will be reported elsewhere.

(5) We have adopted in the paper the standard convention of defining passive systems in terms of the existence of a *non-negative* storage function.[1] As will become clear below, all our derivations remain valid if we relax the non-negativity assumption. These, obviously larger, class of systems are referred in [6] as energy-balancing and in [10,2] as cyclopassive—a name that is motivated by the fact that cyclopassive systems cannot create energy *over closed paths* in the state space, in contrast with passive system that cannot create energy *for all* trajectories.

[1] Actually, as one can always add a constant to the storage function, the question is whether it is bounded from below or not.

3. Passivity of the incremental system

Proposition 1. *Assume*:

(A.1) *System* (1) *defines a passive mapping* $\boldsymbol{u} \to \boldsymbol{y}$ *with a convex twice continuously differentiable storage function* $H : \mathbb{R}^n \to \mathbb{R}_+$.

(A.2) *The condition*

$$[\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}^\star)]^\top [\nabla H(\boldsymbol{x}) - \nabla H(\boldsymbol{x}^\star)] \leqslant 0 \tag{5}$$

is satisfied.[2]

Then, the mapping $\tilde{\boldsymbol{u}} \to \tilde{\boldsymbol{y}}$, *defined by* (4) *is also passive with non-negative storage function* $H_0 : \mathbb{R}^n \to \mathbb{R}_+$,

$$H_0(\boldsymbol{x}) = H(\boldsymbol{x}) - \boldsymbol{x}^\top \nabla H(\boldsymbol{x}^\star) - [H(\boldsymbol{x}^\star) - (\boldsymbol{x}^\star)^\top \nabla H(\boldsymbol{x}^\star)]. \tag{6}$$

Proof. We will verify that (A.1) and (A.2) ensure that (4) satisfies the necessary and sufficient conditions for passivity of Hill–Moylan's nonlinear version of Kalman–Yakubovich–Popov's Lemma [2] with the storage function $H_0(\boldsymbol{x})$. Namely, the condition of stability (with Lyapunov function $H_0(\boldsymbol{x})$) of $\boldsymbol{x}^\star$:

$$[\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}\boldsymbol{u}^\star]^\top \nabla H_0(\boldsymbol{x}) \leqslant 0, \tag{7}$$

and the coupling condition between input and the (new) output mappings, that is,

$$\boldsymbol{h}(\boldsymbol{x}) - \boldsymbol{h}(\boldsymbol{x}^\star) = \boldsymbol{g}^\top \nabla H_0(\boldsymbol{x}). \tag{8}$$

From (6) we have that

$$\nabla H_0(\boldsymbol{x}) = \nabla H(\boldsymbol{x}) - \nabla H(\boldsymbol{x}^\star). \tag{9}$$

Now, because of our assumption of constant $\boldsymbol{g}$, $\boldsymbol{f}(\boldsymbol{x}^\star) + \boldsymbol{g}\boldsymbol{u}^\star = 0$ for all equilibrium points. Replacing the latter in (5) and using (9) clearly yields (7). The second condition, (8), follows immediately from (9) and the fact that passivity of (1) implies

$$\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{g}^\top \nabla H(\boldsymbol{x}).$$

It only remains to prove that $H_0(\boldsymbol{x})$ is non-negative, which will be done showing that $\boldsymbol{x}^\star$ is a minimum point of $H_0(\boldsymbol{x})$. Using convexity of $H(\boldsymbol{x})$ yields

$$\nabla^2 H_0(\boldsymbol{x}) = \nabla^2 H(\boldsymbol{x}) \geqslant 0,$$

which shows convexity of $H_0(\boldsymbol{x})$. From (9) we also obtain $\nabla H_0(\boldsymbol{x}^\star) = 0$. Hence, $\boldsymbol{x}^\star$ is a minimum point of $H_0(\boldsymbol{x})$. This completes the proof. □

Remark 1. The storage function $H_0(\boldsymbol{x})$ can be directly derived from [4]—where the case of port Hamiltonian systems is considered and the analysis is carried out in co-energy coordinates ($\nabla H(\boldsymbol{x})$ in our notation). Indeed, integrating Eq. (10) from that paper and expressing the function in the original (energy) coordinates, denoted $\boldsymbol{x}$ here and called $\boldsymbol{z}$ in [4], yields (6).

[2] All vectors defined in the paper are *column* vectors, even the gradient of a scalar function that we denote with the operator $\nabla := \partial/\partial\boldsymbol{x}$. We also define $\nabla^2 = \partial^2/\partial\boldsymbol{x}^2$.

Remark 2. In the LTI case with quadratic storage function, Eq. (5) reduces to the stability condition $\tilde{x}^\top \mathbf{PA}\tilde{x} \leqslant 0$, while the new storage function is given by $H_0(\tilde{x}) = \frac{1}{2}\tilde{x}^\top \mathbf{P}\tilde{x}$. This appealing downward compatibility makes our result a natural extension, to the nonlinear case, of the well-known property of LTI systems.

Remark 3. If (1) is a port controlled Hamiltonian system we have

$$f(x) = [\mathbf{J}(x) - \mathbf{R}(x)]\nabla H(x),$$

where $\mathbf{J}(x) = -\mathbf{J}^\top(x)$ is the interconnection matrix and $\mathbf{R}(x) = \mathbf{R}^\top(x) \geqslant 0$ captures the dissipation effects. Condition (5) will be then satisfied if $\mathbf{J}$ and $\mathbf{R}$ are *constant* matrices. This corresponds to constant interconnections and linear damping—the former is often the case in physical systems, for instance for nonlinear mechanical systems or nonlinear LC circuits. In the next section, we will prove that the incremental model of passive RLC circuits is passive also in the case when the resistors are nonlinear, but with monotonic characteristic function.

Remark 4. For port-controlled Hamiltonian systems with constant interconnection and damping matrices the storage function for the incremental model given in (6) results from a direct application of the interconnection and damping assignment controller design methodology [7]. Indeed, in its simpler formulation, the objective of this controller is to shape the storage function of the system assigning to the closed-loop the dynamics $\dot{x} = (\mathbf{J} - \mathbf{R})\nabla H_d(x) + gv$, where $H_d(x)$ is the desired storage function and v is a free external signal. *Fixing* $v = \tilde{u}$ we see that the objective will be achieved with $H_d(x) = H_0(x)$; and by definition of the equilibrium set (2), the matching equation

$$-gu^\star = (\mathbf{J} - \mathbf{R})\nabla H(x^\star),$$

always has a solution.

Remark 5. For the purposes of stability analysis of passive systems it is often of interest to investigate whether the storage function has an unique minimum at $x^\star$ and whether the function is proper.[3] (As shown in the proof of Proposition 1, $H_0(x)$ has indeed a minimum at $x^\star$, but this may not be unique nor isolated.) The following proposition, whose proof is given in Appendix A, shows that having a strictly convex $H(x)$ is sufficient to ensure that $H_0(x)$ has a unique minimum at $x^\star$ and, furthermore, that it is proper.

Proposition 2. *Assume the storage function $H(x)$ is strictly convex. Then, for every $x^\star \in \mathbb{R}^n$, the new storage function $H_0(x)$ defined in (6) has a unique global minimum at $x^\star$ and is proper.*

4. PI stabilization of nonlinear RLC circuits

In this section, we prove that a large class of nonlinear RLC circuits—with strictly convex electric and magnetic energy functions and passive resistors with monotonic characteristic functions—satisfy the condition of Proposition 1, hence showing that these circuits can be globally stabilized with linear PI control.

We consider RLC circuits consisting of interconnections of (possibly nonlinear) lumped dynamic (n_L inductors, n_C capacitors) and static (n_R resistors, n_{v_S} voltage sources and n_{i_S} current sources) elements. Capacitors and inductors are defined by the physical laws and constitutive relations [1]:

$$i_\mathrm{C} = \dot{q}_\mathrm{C}, \quad v_\mathrm{C} = \nabla H_\mathrm{C}(q_\mathrm{C}), \tag{10}$$
$$v_\mathrm{L} = \dot{\phi}_\mathrm{L}, \quad i_\mathrm{L} = \nabla H_\mathrm{L}(\phi_\mathrm{L}), \tag{11}$$

respectively, where $i_\mathrm{C}(t)$, $v_\mathrm{C}(t)$, $q_\mathrm{C}(t) \in \mathbb{R}^{n_\mathrm{C}}$ are the capacitors currents, voltages and charges, and $i_\mathrm{L}(t)$, $v_\mathrm{L}(t)$, $\phi_\mathrm{L}(t) \in \mathbb{R}^{n_\mathrm{L}}$ are the inductors current, voltage and flux-linkages, $H_\mathrm{L} : \mathbb{R}^{n_\mathrm{L}} \to \mathbb{R}$ is the magnetic energy stored in the inductors and $H_\mathrm{C} : \mathbb{R}^{n_\mathrm{C}} \to \mathbb{R}$ is the electric energy stored in the capacitors. We also define the total energy as

$$H(\phi_\mathrm{L}, q_\mathrm{C}) := H_\mathrm{L}(\phi_\mathrm{L}) + H_\mathrm{C}(q_\mathrm{C}).$$

For the sake of simplicity and to avoid cluttering (even more) the notation we will consider that all current (resp. voltage) controlled resistors are in series with inductors (resp. in parallel with capacitors). In this way, we can write $v_{\mathrm{R}_{\mathrm{L}_i}} = \hat{v}_{\mathrm{R}_{\mathrm{L}_i}}(i_{\mathrm{L}_i})$, $i = 1, \ldots, n_\mathrm{L}$, for the current controlled resistors and $i_{\mathrm{R}_{\mathrm{C}_i}} = \hat{i}_{\mathrm{R}_{\mathrm{C}_i}}(v_{\mathrm{C}_i})$, $i = 1, \ldots, n_\mathrm{C}$ for the voltage controlled resistors, where $\hat{v}_{\mathrm{R}_{\mathrm{L}_i}}, \hat{i}_{\mathrm{R}_{\mathrm{C}_i}} : \mathbb{R} \to \mathbb{R}$ are their characteristic curves. See Fig. 1.

The dynamics of the circuit can be written as a slight extension—to the case of nonlinear resistors—of the port-controlled Hamiltonian model of LC circuits described in [5][4]

$$\begin{bmatrix} \dot{\phi}_\mathrm{L} \\ \dot{q}_\mathrm{C} \end{bmatrix} = \mathbf{J}\nabla H(\phi_\mathrm{L}, q_\mathrm{C}) - \begin{bmatrix} \hat{v}_{\mathrm{R}_\mathrm{L}}(\nabla H_\mathrm{L}(\phi_\mathrm{L})) \\ \hat{i}_{\mathrm{R}_\mathrm{C}}(\nabla H_\mathrm{C}(q_\mathrm{C})) \end{bmatrix} + gu, \tag{12}$$

where

$$\mathbf{J} = \begin{bmatrix} 0 & -\mathbf{\Gamma} \\ \mathbf{\Gamma}^\top & 0 \end{bmatrix}, \quad g = \begin{bmatrix} -\mathbf{B}_{v_\mathrm{S}} & 0 \\ 0 & \mathbf{B}_{i_\mathrm{S}} \end{bmatrix}, \quad u = \begin{bmatrix} v_{v_\mathrm{S}} \\ i_{i_\mathrm{S}} \end{bmatrix}.$$

$v_{v_\mathrm{S}}(t) \in \mathbb{R}^{n_{v_\mathrm{S}}}$ are the voltage sources (in series with inductors), $i_{i_\mathrm{S}}(t) \in \mathbb{R}^{n_{i_\mathrm{S}}}$ the current sources (in parallel with capacitors), $\mathbf{B}_{v_\mathrm{S}} \in \mathbb{R}^{n_\mathrm{L} \times n_{v_\mathrm{S}}}$, $\mathbf{B}_{i_\mathrm{S}} \in \mathbb{R}^{n_\mathrm{C} \times n_{i_\mathrm{S}}}$ are input (full rank) matrices with $n_{v_\mathrm{S}} \leqslant n_\mathrm{L}$, $n_{i_\mathrm{S}} \leqslant n_\mathrm{C}$ and $\mathbf{\Gamma} \in \mathbb{R}^{n_\mathrm{L} \times n_\mathrm{C}}$, is a constant matrix determined by the circuit topology.

The port variables are completed defining the currents and voltages associated to the sources, which are given by

$$y = g^\top \nabla H(\phi_\mathrm{L}, q_\mathrm{C}) = \begin{bmatrix} -\mathbf{B}_{v_\mathrm{S}}^\top \nabla H_\mathrm{L}(\phi_\mathrm{L}) \\ \mathbf{B}_{i_\mathrm{S}}^\top \nabla H_\mathrm{C}(q_\mathrm{C}) \end{bmatrix}. \tag{13}$$

[3] By properness of the function $H_0(x)$ we mean that for any constant $c > 0$ the set $\{x \in \mathbb{R}^n \,|\, H_0(x) \leqslant c\}$ is compact.

[4] Notice that if the resistors are linear, Eq. (12) takes the more familiar form $\dot{x} = (\mathbf{J} - \mathbf{R})\nabla H(x) + gu$ [8]. The circuit can also be written in the previous form, but with $\mathbf{R}(x)$, for nonlinear resistors whose characteristic functions are continuously differentiable and map zero into zero. We thank the anonymous reviewer for this pertinent observation.

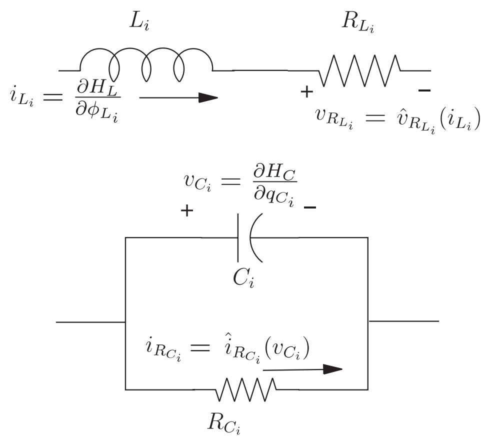

Fig. 1. Current controlled resistors in series with inductors and voltage controlled resistors in parallel with capacitors

Proposition 3. *Consider the dynamics of the nonlinear RLC circuit* (12), (13). *Let* $(\boldsymbol{\phi}_{\mathrm{L}}^{\star}, \boldsymbol{q}_{\mathrm{C}}^{\star})$ *be an equilibrium point with the corresponding constant input* $\boldsymbol{u}^{\star}$ *and output* $\boldsymbol{y}^{\star}$. *Assume*

(B.1) *Inductors and capacitors are passive and their energy functions are twice continuously differentiable and strictly convex.*
(B.2) *The resistors are passive and their characteristic functions are monotone non-decreasing.*
Then, the circuit in closed-loop with the PI controller

$$\dot{\boldsymbol{\xi}} = -\tilde{\boldsymbol{y}},$$

$$\boldsymbol{u} = K_{\mathrm{I}}\boldsymbol{\xi} - K_{\mathrm{P}}\tilde{\boldsymbol{y}}, \tag{14}$$

where $K_{\mathrm{I}} = K_{\mathrm{I}}^{\top} > 0$, $K_{\mathrm{P}} = K_{\mathrm{P}}^{\top} > 0$, *ensures all state trajectories* $(\boldsymbol{\phi}_{\mathrm{L}}(t), \boldsymbol{q}_{\mathrm{C}}(t), \xi(t))$ *are bounded and*

$$\lim_{t\to\infty} \|\tilde{\boldsymbol{y}}(t)\| = 0.$$

If, in addition, the closed-loop system (12)–(14) *satisfies the detectability assumption*
(B.3)

$$\tilde{\boldsymbol{y}}(t) \equiv 0 \quad \Rightarrow \lim_{t\to\infty} \left\| \begin{bmatrix} \tilde{\boldsymbol{\phi}}_{\mathrm{L}}(t) \\ \tilde{\boldsymbol{q}}_{\mathrm{C}}(t) \\ \tilde{\boldsymbol{\xi}}(t) \end{bmatrix} \right\| = 0,$$

where $\boldsymbol{\xi}^{\star} = K_{\mathrm{I}}^{-1}\boldsymbol{u}^{\star}$.
Then,

$$\lim_{t\to\infty} \left\| \begin{bmatrix} \tilde{\boldsymbol{\phi}}_{\mathrm{L}}(t) \\ \tilde{\boldsymbol{q}}_{\mathrm{C}}(t) \\ \tilde{\boldsymbol{\xi}}(t) \end{bmatrix} \right\| = 0.$$

Proof. First, invoking Proposition 1, we will prove that the incremental model of the circuit defines a passive system $\tilde{\boldsymbol{u}} \to \tilde{\boldsymbol{y}}$ with a *proper positive definite* storage function. Since the PI is a passive system, the proof will be then completed with standard passivity-based control arguments.

It is well known, that RLC circuits with passive elements are passive [1] with storage function their total energy. Indeed, computing

$$\dot{H}(\boldsymbol{\phi}_{\mathrm{L}}, \boldsymbol{q}_{\mathrm{C}}) = -\boldsymbol{i}_{\mathrm{L}}^{\top}\hat{\boldsymbol{v}}_{\mathrm{R_L}}(\boldsymbol{i}_{\mathrm{L}}) - \boldsymbol{v}_{\mathrm{C}}^{\top}\hat{\boldsymbol{i}}_{\mathrm{R_C}}(\boldsymbol{v}_{\mathrm{C}}) + \boldsymbol{y}^{\top}\boldsymbol{u} \leqslant \boldsymbol{y}^{\top}\boldsymbol{u},$$

where we have used (10), (11) and (13) to get the identity and passivity of the resistors of Assumption (B.2) to obtain the inequality.[5] Non-negativity of $H(\boldsymbol{\phi}_{\mathrm{L}}, \boldsymbol{q}_{\mathrm{C}})$ follows from passivity of inductors and capacitors of Assumption (B.1).

To prove passivity of the incremental model of (12), (13) we need to verify condition (5) which after some calculations becomes

$$\begin{bmatrix} -\hat{\boldsymbol{v}}_{\mathrm{R_L}}(\nabla H_{\mathrm{L}}(\boldsymbol{\phi}_{\mathrm{L}})) + \hat{\boldsymbol{v}}_{\mathrm{R_L}}(\nabla H_{\mathrm{L}}(\boldsymbol{\phi}_{\mathrm{L}}^{\star})) \\ -\hat{\boldsymbol{i}}_{\mathrm{R_C}}(\nabla H_{\mathrm{C}}(\boldsymbol{q}_{\mathrm{C}})) + \hat{\boldsymbol{i}}_{\mathrm{R_C}}(\nabla H_{\mathrm{C}}(\boldsymbol{q}_{\mathrm{C}}^{\star})) \end{bmatrix}^{\top} \begin{bmatrix} \nabla H_{\mathrm{L}}(\boldsymbol{\phi}_{\mathrm{L}}) - \nabla H_{\mathrm{L}}(\boldsymbol{\phi}_{\mathrm{L}}^{\star}) \\ \nabla H_{\mathrm{C}}(\boldsymbol{q}_{\mathrm{C}}) - \nabla H_{\mathrm{L}}(\boldsymbol{q}_{\mathrm{C}}^{\star}) \end{bmatrix}$$
$$= -(\hat{\boldsymbol{v}}_{\mathrm{R_L}}(\boldsymbol{i}_{\mathrm{L}}) - \hat{\boldsymbol{v}}_{\mathrm{R_L}}(\boldsymbol{i}_{\mathrm{L}}^{\star}))^{\top}(\boldsymbol{i}_{\mathrm{L}} - \boldsymbol{i}_{\mathrm{L}}^{\star})$$
$$- (\hat{\boldsymbol{i}}_{\mathrm{R_C}}(\boldsymbol{v}_{\mathrm{C}}) - \hat{\boldsymbol{i}}_{\mathrm{R_C}}(\boldsymbol{v}_{\mathrm{C}}^{\star}))^{\top}(\boldsymbol{v}_{\mathrm{C}} - \boldsymbol{v}_{\mathrm{C}}^{\star}) \leqslant 0,$$

where we have used Eqs. (10) and (11) for the first identity and the monotonic resistors characteristic condition of Assumption (B.2) for the inequality.

The storage function for the incremental model is computed from (6) as

$$\begin{aligned} H_0(\boldsymbol{\phi}_{\mathrm{L}}, \boldsymbol{q}_{\mathrm{C}}) = {} & H(\boldsymbol{\phi}_{\mathrm{L}}, \boldsymbol{q}_{\mathrm{C}}) - \boldsymbol{\phi}_{\mathrm{L}}^{\top}\nabla H_{\mathrm{L}}(\boldsymbol{\phi}_{\mathrm{L}}^{\star}) - \boldsymbol{q}_{\mathrm{C}}^{\top}\nabla H_{\mathrm{C}}(\boldsymbol{q}_{\mathrm{C}}^{\star}) \\ & - [H(\boldsymbol{\phi}_{\mathrm{L}}^{\star}, \boldsymbol{q}_{\mathrm{C}}^{\star}) - (\boldsymbol{\phi}_{\mathrm{L}}^{\star})^{\top}\nabla H_{\mathrm{L}}(\boldsymbol{\phi}_{\mathrm{L}}^{\star}) \\ & - (\boldsymbol{q}_{\mathrm{C}}^{\star})^{\top}\nabla H_{\mathrm{C}}(\boldsymbol{q}_{\mathrm{C}}^{\star})], \end{aligned} \tag{15}$$

which, under Assumption (B.1) and according to Proposition 2, is strictly convex, has a unique global minimum at the origin and is proper.

To complete the proof of the proposition we note that the incremental model of the closed-loop system takes the form

$$\dot{z} = F(z), \quad \tilde{\boldsymbol{y}} = G(z),$$

where $\boldsymbol{z} = \mathrm{col}(\boldsymbol{\phi}_{\mathrm{L}}, \boldsymbol{q}_{\mathrm{C}}, \boldsymbol{\xi})$ and $F(z)$, $G(z)$ are locally Lipschitz continuous. We, thus, consider the (positive definite and proper) Lyapunov function candidate

$$H_{\mathrm{cl}}(z) = H_0(\boldsymbol{\phi}_{\mathrm{L}}, \boldsymbol{q}_{\mathrm{C}}) + \tfrac{1}{2}\boldsymbol{\xi}^{\top}K_{\mathrm{I}}\boldsymbol{\xi}.$$

Computing the derivative and using (14) we get

$$\dot{H}_{\mathrm{cl}}(z) \leqslant \tilde{\boldsymbol{y}}^{\top}\tilde{\boldsymbol{u}} - \boldsymbol{\xi}^{\top}K_{\mathrm{I}}\tilde{\boldsymbol{y}} = -\tilde{\boldsymbol{y}}^{\top}K_{\mathrm{P}}\tilde{\boldsymbol{y}} \leqslant 0. \tag{16}$$

It follows from (16) that the state $z(t)$ is bounded and $\tilde{\boldsymbol{y}}(t)$ is square integrable. From continuity of $F(z)$, this also implies that $\dot{z}(t)$ is bounded, hence $z(t)$ is uniformly continuous. From continuity of $G(z)$ we also have that $\tilde{\boldsymbol{y}}(t)$ is uniformly continuous, and we conclude $\lim_{t\to\infty}\|\tilde{\boldsymbol{y}}(t)\| = 0$.

Convergence of the incremental state to zero follows using LaSalle's invariance principle and invoking Assumption (B.3) (see, for example, [2]). □

Remark 6. In view of the passivity property of the incremental model of the RLC circuit stabilization can be achieved with any strictly passive controller. For instance, a simple proportional

[5] We recall that a resistor is passive if and only if its characteristic function lives in the first-third quadrant [1].

control $\tilde{u} = -K_P\tilde{y}$—this, however, would require the knowledge of $\boldsymbol{u}^\star$, which is tantamount to knowing the equilibrium point of the system. As seen from (14), this prior knowledge is obviated if we add an integral action, yielding a more robust control law.

5. Conclusions and future research

We have considered general affine passive systems with constant input matrix. We identified a condition on the vector field $\boldsymbol{f}(\boldsymbol{x})$, namely (5), that ensures passivity of the incremental model. Then, we showed that a large class of nonlinear passive RLC circuits—with strictly convex electric and magnetic energy functions and monotonic resistor characteristics—satisfy this condition. Hence, these circuits can be globally stabilized with linear PI control.

Current research is under way along two directions. First, to employ these results for energy-balancing stabilization of physical systems. Second, to derive conditions for passivity of more general error models, for instance, those that appear when tracking feasible trajectories.

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Appendix A. Proof of Proposition 2

It can be checked directly that the function $H_0(\boldsymbol{x})$ is strictly convex, has a unique global minimum at $\boldsymbol{x}^\star$, $H_0(\boldsymbol{x}^\star) = 0$ and $H_0(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in \mathbb{R}^n \backslash \{\boldsymbol{x}^\star\}$. It remains to prove that the function $H_0(\boldsymbol{x})$ is proper.

Denote by $\bar{\mathbf{B}}(\varepsilon, \boldsymbol{x}^\star) := \{\boldsymbol{x} \in \mathbb{R}^n \,|\, \|\boldsymbol{x} - \boldsymbol{x}^\star\| \leqslant \varepsilon\}$ the closed ball with radius ε and $\boldsymbol{x}^\star$ as its center. Let $\mathbf{C} := \{\boldsymbol{x} \in \mathbb{R}^n \,|\, \|\boldsymbol{x} - \boldsymbol{x}^\star\| = 1\}$ denote the sphere with radius 1 and $\boldsymbol{x}^\star$ as its center. Let $\alpha = \min_{\boldsymbol{x}\in\mathbf{C}} H_0(\boldsymbol{x}) > 0$.

$H_0(\boldsymbol{x})$ is proper if the inverse image of every compact set is also compact, that is, if

$$\{\boldsymbol{x} \in \mathbb{R}^n \,|\, H_0(\boldsymbol{x}) \leqslant k\alpha\} \subset \bar{\mathbf{B}}(k, \boldsymbol{x}^\star) \quad \forall k \geqslant 1.$$

The latter is equivalent to

$$\mathbb{R}^n \backslash \bar{\mathbf{B}}(k, \boldsymbol{x}^\star) \subset \{\boldsymbol{x} \in \mathbb{R}^n \,|\, H_0(\boldsymbol{x}) > k\alpha\} \quad \forall k \geqslant 1. \tag{A.1}$$

To prove Eq. (A.1) take any $\boldsymbol{a} \in \mathbb{R}^n \backslash \bar{\mathbf{B}}(k, \boldsymbol{x}^\star)$ and let $\boldsymbol{b}$ be the vector found at the intersection of $\bar{\mathbf{B}}(k, \boldsymbol{x}^\star)$ and the convex combination of $\boldsymbol{x}^\star$ and $\boldsymbol{a}$

$$\boldsymbol{b} = \gamma\boldsymbol{a} + (1-\gamma)\boldsymbol{x}^\star, \quad \gamma = \frac{k}{\|\boldsymbol{a} - \boldsymbol{x}^\star\|} < 1, \tag{A.2}$$

that is

$$\boldsymbol{b} = \boldsymbol{x}^\star + \frac{k}{\|\boldsymbol{a} - \boldsymbol{x}^\star\|}(\boldsymbol{a} - \boldsymbol{x}^\star).$$

By the strict convexity of $H_0(\boldsymbol{x})$ and (A.2) we know that

$$H_0(\boldsymbol{b}) < \gamma H_0(\boldsymbol{a}) + (1-\gamma) H_0(\boldsymbol{x}^\star) = \gamma H_0(\boldsymbol{a}). \tag{A.3}$$

Now let $\boldsymbol{c}$ be the vector found at the intersection of $\mathbf{C}$ and the convex combination of $\boldsymbol{x}^\star$ and $\boldsymbol{b}$

$$\boldsymbol{c} = \lambda\boldsymbol{b} + (1-\lambda)\boldsymbol{x}^\star, \quad \lambda = \frac{1}{k} \leqslant 1,$$

that is

$$\boldsymbol{c} = \boldsymbol{x}^\star + \frac{1}{k}(\boldsymbol{b} - \boldsymbol{x}^\star).$$

From $\boldsymbol{c} \in \mathbf{C}$ and again, from the strict convexity of $H_0(\boldsymbol{x})$ we have that

$$\alpha \leqslant H_0(\boldsymbol{c}) \leqslant \frac{1}{k} H_0(\boldsymbol{b}) + \left(1 - \frac{1}{k}\right) H_0(\boldsymbol{x}^\star) = \frac{1}{k} H_0(\boldsymbol{b}). \tag{A.4}$$

From Eqs. (A.3), (A.4) and noting that $\gamma < 1$ we have that[6]

$$\alpha k \leqslant H_0(\boldsymbol{b}) < H_0(\boldsymbol{a}).$$

The latter inequality is valid for any $k \geqslant 1$ and for all $\boldsymbol{a} \in \mathbb{R}^n \backslash \bar{\mathbf{B}}(k, \boldsymbol{x}^\star)$, which is just an equivalent way to state (A.1).

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[6] The non-strict inequality stems from the fact that at $k = 1$, $\boldsymbol{c} = \boldsymbol{b}$.