Lecture 22: Decomposition into Controllable and Uncontrollable Parts and Realization Theory

5.8.4 Decomposition into Controllable and Uncontrollable Parts

For the purpose of simulating a given system's input-output behavior, we may have to look for state-space realizations that have the same input-output behavior, but not necessarily the same state-space matrices.

More precisely, two systems are *zero-state equivalent* if they have identical input-output maps assuming initial rest. The two systems are *equivalent* if, for each initial state of either system, there exists another initial state for the other such that their input-output maps are identical.

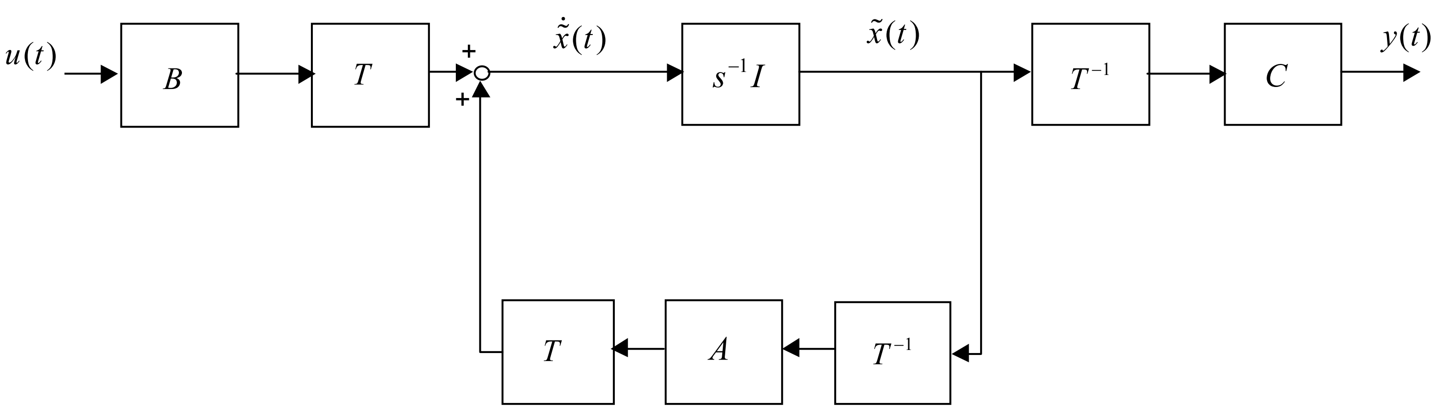
In the linear time-invariant case, equivalent systems differ only by a change of basis of the state. In fact, it is possible to find a basis relative to which the system is decomposed into a zero-state equivalent CC part, and a part that is completely disconnected from the input and which can be termed *completely uncontrollable*.

Consider the LTI system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \qquad (5.105)$$

Let $\tilde{x} = Tx$ be a new state obtained via the similarity transformation T. Then,

$$\begin{aligned} \dot{\tilde{x}}(t) &= \underbrace{TAT^{-1}}_{\tilde{A}} \tilde{x}(t) + \underbrace{TB}_{\tilde{B}} u(t) \\ y(t) &= \underbrace{CT^{-1}}_{\tilde{C}} \tilde{x}(t) + Du(t) \end{aligned} \qquad (5.106)$$



We use the above form to separate the controllable part from the uncontrollable part. To find such a decomposition, we note that a change of basis mapping A into TAT^{-1} via the nonsingular transformation T maps (A,B,C) into the equivalent realization $(\tilde{A},\tilde{B},\tilde{C})=(TAT^{-1},TB,CT^{-1})$.

Let Q be the controllability matrix, and assume $\operatorname{rank}\{Q\}=:q<n$. Then the system is not CC. Choose an invertible matrix $T\in\mathbb{R}^{n\times n}$ such that

$$TQ=:\tilde{Q}=\begin{bmatrix}\tilde{Q}_1\\ \theta_{(n-q)\times nm}\end{bmatrix},\ \tilde{Q}_1\in\mathbb{R}^{q\times nm} \tag{5.107}$$

e.g., T can be the row equivalent transformation that maps Q into its echelon form. The resulting $(\tilde{A},\tilde{B})$ have the block form:

$$\tilde{A}=\begin{bmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ \theta_{(n-q)\times q} & \tilde{A}_{22}\end{bmatrix},\ \tilde{B}=\begin{bmatrix}\tilde{B}_1\\ \theta_{(n-q)\times m}\end{bmatrix},\ \tilde{A}_{11}\in\mathbb{R}^{q\times q},\tilde{A}_{12}\in\mathbb{R}^{q\times(n-q)},\tilde{A}_{22}\in\mathbb{R}^{(n-q)\times(n-q)},\tilde{B}_1\in\mathbb{R}^{q\times m} \tag{5.108}$$

If $\tilde{C}$ is partitioned as

$$\tilde{C}=\begin{bmatrix}\tilde{C}_1 & \tilde{C}_2\end{bmatrix},\ \tilde{C}_1\in\mathbb{R}^{p\times q},\tilde{C}_2\in\mathbb{R}^{p\times(n-q)}. \tag{5.109}$$

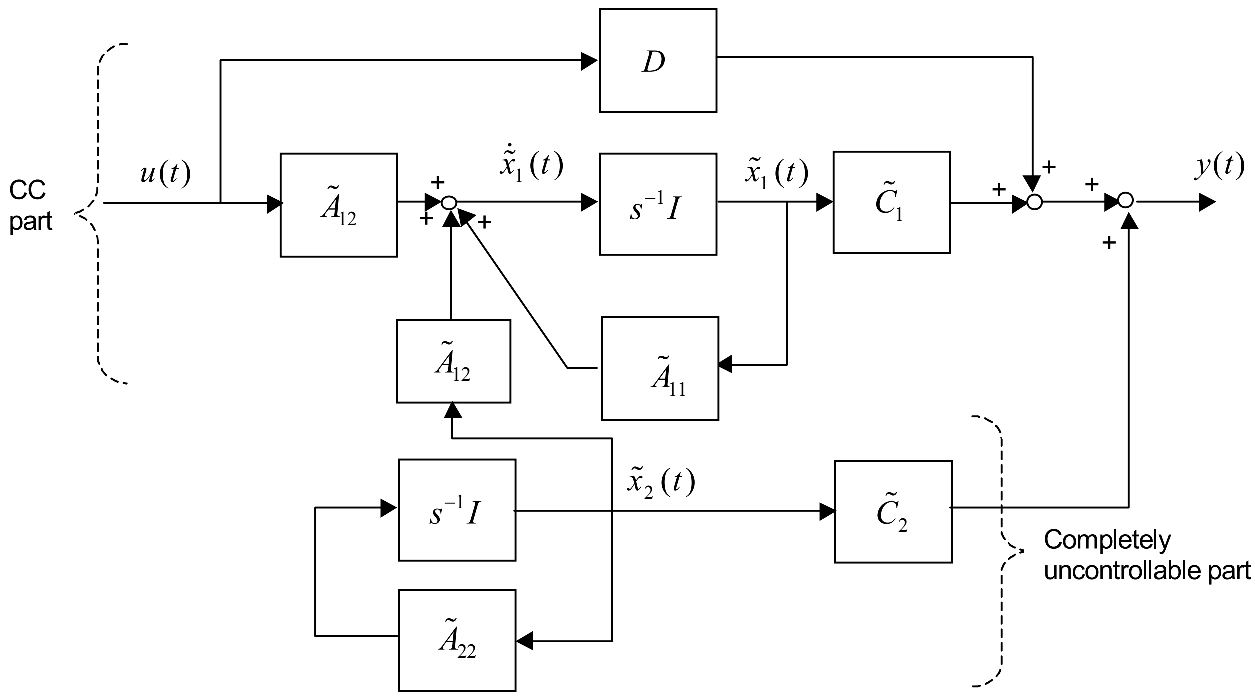
Then, the transformed system consists of the sum of

$$(\tilde{A}_{11},\tilde{B}_1,\tilde{C}_1), \tag{5.110}$$

which is CC as its controllability matrix $\tilde{Q}_1$ is onto by construction, and the system

$$\left(\begin{bmatrix}\theta & \tilde{A}_{12}\\ \theta & \tilde{A}_{22}\end{bmatrix},\theta,\begin{bmatrix}\theta & \tilde{C}_2\end{bmatrix}\right) \tag{5.111}$$

which is disconnected from the input. If the initial state of the transformed system is the zero vector, then the subsystem of (5.111) has no effect on the output, and therefore its CC part (5.110) is zero-state equivalent to the original system.



We will need the definition of an invariant subspace.

<u>Definition: Invariant Subspace</u>

Given:

- A vector space $(\mathcal{V},\mathcal{F})$
- A subspace $\mathcal{M}\subset\mathcal{V}$
- A linear transformation $\mathcal{A}:\mathcal{V}\to\mathcal{V}$

Subspace $\mathcal{M}$ is said to be *invariant* under $\mathcal{A}$ if $\mathcal{A}m\subset\mathcal{M},\ \forall m\in\mathcal{M}$.

<u>Lemma:</u>

If $x\in\mathcal{R}\{Q\}$, then $Ax\in\mathcal{R}\{Q\}$, i.e., $\mathcal{R}\{Q\}$ is an A-invariant subspace.

<u>Proof:</u> Left as an exercise (use the CHT.)

Remarks:

(1) Matrix T can be found by adjoining an $n \times n$ identity matrix to Q and row-reducing $\left[Q \quad I \right]$ to obtain $\left[\begin{array}{c|c} \tilde{Q}_1 & T \\ \theta & \end{array} \right]$

(2) The controllability matrix of the CC part is $\tilde{Q}_1 := \left[\tilde{B}_1 \quad \tilde{A}_{11}\tilde{B}_1 \quad \cdots \quad \tilde{A}_{11}{}^{n-1}\tilde{B}_1 \right]$ (5.112)

(3) The CC part can be written as

$$\begin{aligned} \dot{\tilde{x}}_1(t) &= \tilde{A}_{11}\tilde{x}_1(t) + \tilde{B}_{11}u_1(t) + \tilde{A}_{12}\tilde{x}_2(t) \\ \dot{\tilde{x}}_2(t) &= \tilde{A}_{22}\tilde{x}_2(t) \end{aligned} \tag{5.113}$$

which shows that u does not affect $\tilde{x}_2(t)$ at all.

The q eigenvalues of $\tilde{A}_{11}$ and the corresponding modes are called the controllable eigenvalues and controllable modes of the pair (A, B).

The $n-q$ eigenvalues of $\tilde{A}_{22}$ and the corresponding modes are called the uncontrollable eigenvalues and uncontrollable modes of the pair (A, B).

In the zero-state response of the system, the controllable modes are completely absent.

In particular, in the solution

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{5.114}$$

notice that

$$e^{A(t-\tau)}B = \left(T^{-1}e^{\tilde{A}(t-\tau)}T\right)T^{-1}\tilde{B} = T^{-1}\begin{bmatrix} e^{\tilde{A}_{11}(t-\tau)}\tilde{B}_{11} \\ \theta \end{bmatrix} \tag{5.115}$$

which shows that the input cannot influence the uncontrollable modes.

Example:

Let $A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$.

We wish to reduce the system to the standard form with a CC part and an uncontrollable part. Here,

$$Q = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 1 \end{array}\right]$$

and hence $\operatorname{rank}(Q) = q = 2 < 3 = n$.

Thus, $\dim\left(\mathcal{R}\{Q\}\right) = 2$ and a basis $\{v_1, v_2\}$ for the range is found by taking two linearly independent columns of Q, say the first two, to obtain:

$$T^{-1} = \begin{bmatrix} & & 0 \\ v_1 & v_2 & 0 \\ & & 1 \end{bmatrix}$$

where the last column is selected to make T^{-1} nonsingular. Note that with this choice we will get $T^{-1}\begin{bmatrix} \tilde{Q}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}\tilde{Q}_1 = Q$. We have,

$$\tilde{A} = TAT^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & -1 & 0 \\ \hline 0 & 0 & -2 \end{array}\right] =: \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}$$

$$\tilde{B} = TB = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} =: \begin{bmatrix} \tilde{B}_{11} \\ \hline 0 \end{bmatrix}$$

where $(\tilde{A}_{11}, \tilde{B}_1)$ is CC. The matrix A has three eigenvalues at 0,-1,-2, and from $(\tilde{A}_{11}, \tilde{B}_1)$ the eigenvalues 0,-1 are controllable, while -2 is an uncontrollable eigenvalue.

6 Realization Theory and Algorithms

It is easy to find the input-output relationship in the form of a convolution or a transfer function describing the behavior of an LTI system given by an internal (state-space) description.

The inverse problem, called the *realization problem*, is not as straightforward:

Given an input-output description of a linear system (impulse response or transfer function), determine a state-space model for the system that has the same input-output model.

There are infinitely many possible state-space realizations of a given system. We typically seek the ones that have the least number of first-order differential equations.

6.1 Realizations

Consider the LTV system

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0 \\ y(t) &= C(t)x(t) + D(t)u(t)\end{aligned} \tag{6.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, and A, B, C, D are continuous over $[t_0, t_1]$.

The response of the system is given by:

$$y(t) = C(t)\Phi(t,t_0)x_0 + \int_{t_0}^{t} H(t,\tau)u(\tau)d\tau \ , \tag{6.2}$$

where $H(t,\tau)$ is the $p \times m$ impulse response of the system, given by:

$$H(t,\tau) = \begin{cases} C(t)\Phi(t,\tau)B(\tau) + D(t)\delta(t-\tau), & t \geq \tau \\ 0 & t < \tau \end{cases} \tag{6.3}$$

In the LTI case, the impulse response of the system simplifies to:

$$H(t) = \begin{cases} Ce^{At}B + D\delta(t), & t \geq 0 \\ 0, & t < 0 \end{cases}. \tag{6.4}$$

It also has a Laplace transform:

$$H(s) = C(sI - A)^{-1}B + D, \ \ \mathrm{Re}\{s\} > \max_{i=1,\ldots,n} \mathrm{Re}\{\lambda_i(A)\} \ . \tag{6.5}$$

Definition: Realization of a Linear System

LTV case: A *realization* of $H(t,\tau)$ is any state-space system $\left(A(t),B(t),C(t),D(t)\right)$ whose impulse response is $H(t,\tau)$.

Note that in general, it is not necessary for any $H(t,\tau)$ to have a realization.

LTI case: A *realization* of $H(s)$ is any state-space system $\left(A,B,C,D\right)$ whose transfer function is $H(s)$.

Note again that in general, it is not necessary for any $H(s)$ to have a realization. It can be shown that a necessary condition for $H(s)$ to have a realization is that all of its entries are proper rational functions.

Definition: Markov Parameters

Suppose $H(s)$ is given as a Laurent series:

$$H(s) = H_0 + H_1 s^{-1} + H_2 s^{-2} + \cdots \tag{6.6}$$

The constant matrices H_i are called the *Markov parameters* of the system, and they can be computed as follows:

$$H_0 = \lim_{s\to\infty} H(s), \quad H_1 = \lim_{s\to\infty} s\left(H(s) - H_0\right), \quad H_2 = \lim_{s\to\infty} s^2\left(H(s) - H_0 - H_1 s^{-1}\right), \ldots \tag{6.7}$$

Theorem:

The matrices $\left(A,B,C,D\right)$ form a realization of $H(s)$ iff

$$H_0 = D, \quad H_i = CA^{i-1}B, \quad i = 1,2,\ldots \tag{6.8}$$

Proof:

$$\begin{aligned} H(s) &= C(sI - A)^{-1}B + D, \ \ \mathrm{Re}\{s\} > \max_{i=1,\ldots,n} \mathrm{Re}\{\lambda_i(A)\} \\ &= Cs^{-1}(I - s^{-1}A)^{-1}B + D \\ &= Cs^{-1}\left(\sum_{i=0}^{+\infty}\left(s^{-1}A\right)^i\right)B + D \\ &= \sum_{i=1}^{+\infty} CA^{i-1}Bs^{-i} + D \end{aligned}$$

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Remarks:

(a) The impulse response of a linear system contains no information about the initial conditions. This implies that different state-space realizations of $H(t,\tau)$ will yield the same zero-state response, but may have different zero-input responses.

(b) If a realization exists for $H(t,\tau)$, then infinitely many realizations exists (similarity transformations.)

6.2 Existence of Realizations

Theorem:

The LTV system $H(t,\tau)$ is realizable iff $H(t,\tau)$ can be decomposed as

$$H(t,\tau) = M(t)N(\tau) + D(t)\delta(t-\tau). \tag{6.9}$$

Proof:

(sufficiency) Assume (6.9) holds. Consider the realization: $\left(\theta, N(t), M(t), D(t)\right)\ldots$

(necessity) Let the system $H(t,\tau)$ have a state-space realization. Then,

$$\begin{aligned} H(t,\tau) &= C(t)\Phi(t,\tau)B(\tau) + D(t)\delta(t-\tau),\ \ t \ge \tau \\ &= \underbrace{C(t)\Phi(t,a)}_{M(t)}\underbrace{\Phi(a,\tau)B(\tau)}_{N(\tau)} + D(t)\delta(t-\tau),\ \ t \ge \tau \\ &= M(t)N(\tau) + D(t)\delta(t-\tau),\ \ t \ge \tau \end{aligned}$$

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Theorem:

The LTI system $H(s)$ is realizable iff Theorem:

The LTV system $H(t,\tau)$ is realizable iff $H(t,\tau)$ can be decomposed as

is a matrix of proper rational functions.

Proof:

(necessity) If the system is a realization of $H(s)$, then $H(s) = C(sI - A)^{-1}B + D$ is rational, and $\lim_{s\to\infty} H(s) = D$.

(sufficiency) If $H(s)$ is a proper rational transfer matrix, then we can use the controllable canonical form discussed next to obtain a state-space realization.

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Corollary:

$H(t)$ is realizable as the impulse response of an LTI state-space system iff all entries of $H(t)$ are sums of terms of the form:

$$\alpha t^k e^{\lambda t},\ \ \beta\delta(t),\ \ \alpha,\beta,\lambda \in \mathbb{C},\ \ k \in \{0,1,2,\ldots\}.$$