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Displacement

Abstract

An attempt is made to present, with some semblance of unity, various representations and notions of *rigid body displacement*. Topics touched on include • equivalence between Euler parameters and real quaternions; homogeneity and norm, inversion and populating a homogeneous rotation matrix by rote, • dual quaternions, the *Study condition* and multiplication, • point, plane, octonian and subgroup transformation, • comparing planar mapping conventions and • an appendix containing a Grassmannian demonstration of transformative dual-adjoint equivalence and some simple examples using the general 8×8 transformation, its adjoint to compute "reciprocals" and its inverse. It is believed that these are revealed for the first time.

Introduction

Various treatments of rigid body motion have been carried out and these efforts span more than two centuries of history. In this regard we cite Study, [9], Blaschke, [1], and Bottema and Roth, [2], and the copious references included in these works. It is the intention here to relate familiar concepts with some not so familiar in order to stimulate formulation and solution of important kinematic problems, both new and old, using image space parameters. When applied with skill, Husty, [5,6,7], Schröcker, [8] and Hayes, [3,4], this often leads to new results or to old ones, vastly simplified so as to enhance computational efficiency and provide greater insight when tackling original problems or investigating interesting "special cases". In some ways this work develops ideas introduced in Zsombor-Murray, [11]. Notwithstanding modern literature, that inevitably contains a review of the underlying theory and effectively demonstrates important applications, there has been no widespread acceptance of Study's *soma* in the kinematics community. If nothing else piques the reader's interest he/she is invited to examine Fig. 4 and investigate the transformation of points, planes, lines and screws using either the comprehensive 8×8 dual quaternion transformation or its inverse. See if the adjoint transformation will generate the reciprocal screw. Below, for convenience, we see from left to right dual quaternions* representing point, plane, line (expressed with Plücker coordinates), a general screw and the screw that will be reciprocated by adjoint premultiplication. Here and elsewhere (*) will be used to denote existence of additional explanation in **Notes**, appended to the last section, **Appendix**. This is to answer criticism of sometimes unconventional notation and notions used herein. The authors understand that their answers may nevertheless fail to satisfy the inquisitors. Further dialogue will be necessary before it is decided whether the former will burn or can be redeemed.

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix}, \begin{bmatrix} 0 \\ p_{23} \\ p_{31} \\ p_{12} \\ 0 \\ p_{01} \\ p_{02} \\ p_{03} \end{bmatrix}, \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix}, \begin{bmatrix} 2c_0 \\ d_1 \\ d_2 \\ d_3 \\ d_0 \\ 2c_1 \\ 2c_2 \\ 2c_3 \end{bmatrix} \tag{1}$$

1 Rotation

A quaternion is given by four numbers; a scalar and three *imaginary* mutually orthogonal vector elements. The homogeneous four-tuple* $\mathbf{q}_r$ represents the *real* -as opposed to dual- quaternion, a rotation of the entire three dimensional frame, attached to some rigid body, about a fixed point origin. It is shown conveniently normed, *i.e.*, $\mathbf{q}_r^2 = 1$, by its first metamorphosis ($\rightarrow$) and its second renders $c_0 = 1$.

$$\mathbf{q}_r \equiv \begin{bmatrix} c_0 \\ c_1\mathbf{i} \\ c_2\mathbf{j} \\ c_3\mathbf{k} \end{bmatrix} \rightarrow \begin{bmatrix} \cos\frac{\phi}{2} \\ \cos\alpha\sin\frac{\phi}{2}\mathbf{i} \\ \cos\beta\sin\frac{\phi}{2}\mathbf{j} \\ \cos\gamma\sin\frac{\phi}{2}\mathbf{k} \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ \cos\alpha\tan\frac{\phi}{2}\mathbf{i} \\ \cos\beta\tan\frac{\phi}{2}\mathbf{j} \\ \cos\gamma\tan\frac{\phi}{2}\mathbf{k} \end{bmatrix} \tag{2}$$

Elements of the normed form are commonly called *Euler parameters.* Rotation is represented as a single displacement of the rigid three-space. It is caused by right hand screw rotation* about a fixed origin through angle ϕ. The fixed axis of rotation is specified by the positive sense unit vector given by direction cosines of angles $\alpha,\ \beta,\ \gamma$ with respect to a fixed Cartesian reference frame with principal axes $x_1,\ x_2,\ x_3$. Dividing the normed form by $\cos\frac{\phi}{2}$ imposes a variable, non-unit magnitude of $\sec\frac{\phi}{2}$ upon the quaternion. Multiple rotations, according to Euler parameter specification, can be carried out by a succession of quaternion multipliers. The first to be imposed is the rightmost*. The last is the leftmost. Multiplication convention requires that every element of the multiplier pre-multiply every element of the multiplicand, subject to the following rules.

• $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are mutually orthogonal vectors of unit *imaginary* magnitude (If one finds the term "real" to be confusing when applied to an entity containing imaginary elements then assume that $\mathbf{q}_r$ stands for "purely-rotational".), so

• $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and the heterogenous products of these are

• $\mathbf{ij} = -\mathbf{ji} = \mathbf{k},\ \ \mathbf{jk} = -\mathbf{kj} = \mathbf{i},\ \ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}$.

Applying the conventions outlined above, the inverse of a quaternion $\mathbf{q}_r^{-1}$ and the quaternion that represents no displacement $\mathbf{q}_0$ are given by the following product.

$$\mathbf{q}_r\mathbf{q}_r^{-1} = \mathbf{q}_r^{-1}\mathbf{q}_r = \mathbf{q}_0 \equiv \begin{bmatrix} c_0 \\ c_1\mathbf{i} \\ c_2\mathbf{j} \\ c_3\mathbf{k} \end{bmatrix} \begin{bmatrix} c_0 \\ -c_1\mathbf{i} \\ -c_2\mathbf{j} \\ -c_3\mathbf{k} \end{bmatrix} = \begin{bmatrix} c_0^2 + c_1^2 + c_2^2 + c_3^2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

The point transformation matrix given may be derived by setting up an arbitrary but judiciously chosen set of four points, the origin and the three points that close the principal Cartesian axes, and their rotationally transformed positions. Upon examining Fig. 1, that shows the circular trajectories of points on a unit sphere and the axis triad, the latter four vectors are seen to be the respective columns of matrix (4) below.

$$\begin{bmatrix} c_0^2+c_1^2+c_2^2+c_3^2 & 0 & 0 & 0 \\ 0 & c_0^2+c_1^2-c_2^2-c_3^2 & 2(c_1c_2-c_0c_3) & 2(c_1c_3+c_0c_2) \\ 0 & 2(c_2c_1+c_0c_3) & c_0^2-c_1^2+c_2^2-c_3^2 & 2(c_2c_3-c_0c_1) \\ 0 & 2(c_3c_1-c_0c_2) & 2(c_3c_2+c_0c_1) & c_0^2-c_1^2-c_2^2+c_3^2 \end{bmatrix} \tag{4}$$

Row and column indices give us the cues to remember and write the elements.

- The diagonal elements are easy; the sum of the squares of all coordinates first, then just c_0^2 and c_i^2 are positive, where i is the row index.
- The other elements of row and column $i = j = 0$ are all zero.
- The off-diagonal elements have the form $2(c_ic_j \pm c_0c_k)$ where $k \neq 0, i, j$.
- To choose the sign ($\pm$) remember the "knight's move" in chess. Start at the bottom middle of the populated 3×3 array and mark it $(+)$. Move to $i = 1, j = 3$ and mark it $(+)$ also.
- The other two elements above the diagonal are marked $(-)$ and we note that these signs are inserted skew-symmetrically beneath the diagonal.
- Geometric duality shows that the *point* rotation transformation matrix that results from rotation of a Cartesian frame about its origin is identical to the matrix that rotates *planes*.

2 Dual Quaternions

A *dual* quaternion $\mathbf{q}$, Eq. 5, represents a general displacement. Note* $|\epsilon| = 1$ and $\epsilon^2 = 0$ and $\mathbf{a}$ is the position vector of the displaced frame origin.

$$\mathbf{q} = \mathbf{q}_r + \mathbf{q}_d = \mathbf{q}_r + \frac{\epsilon}{2}\mathbf{a}\mathbf{q}_r = \begin{bmatrix} c_0 \\ c_1\mathbf{i} \\ c_2\mathbf{j} \\ c_3\mathbf{k} \end{bmatrix} + \frac{\epsilon}{2}\begin{bmatrix} 0 \\ a_1\mathbf{i} \\ a_2\mathbf{j} \\ a_3\mathbf{k} \end{bmatrix}\begin{bmatrix} c_0 \\ c_1\mathbf{i} \\ c_2\mathbf{j} \\ c_3\mathbf{k} \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1\mathbf{i} \\ c_2\mathbf{j} \\ c_3\mathbf{k} \end{bmatrix} + \frac{\epsilon}{2}\begin{bmatrix} -a_1c_1-a_2c_2-a_3c_3) \\ (a_1c_0+a_2c_3-a_3c_2)\mathbf{i} \\ (a_2c_0+a_3c_1-a_1c_3)\mathbf{j} \\ (a_3c_0+a_1c_2-a_2c_1)\mathbf{k} \end{bmatrix} \tag{5}$$

Homogeneous 8-vectors may represent rigid body displacement only if the product,

$$\text{called the } \textit{Study condition}^*,\ [c_0\ \ c_1\ \ c_2\ \ c_3]\begin{bmatrix} -a_1c_1-a_2c_2-a_3c_3 \\ a_1c_0+a_2c_3-a_3c_2 \\ a_2c_0+a_3c_1-a_1c_3 \\ a_3c_0+a_1c_2-a_2c_1 \end{bmatrix} = 0 \tag{6}$$

is satisfied and this is indeed the case.

Multiplication of dual quaternions is illustrated with the pair $\mathbf{Q}$ and $\mathbf{q}$.

$$\mathbf{Q} = \mathbf{Q}_r + \mathbf{Q}_d = \begin{bmatrix} C_0 \\ C_1\mathbf{i} \\ C_2\mathbf{j} \\ C_3\mathbf{k} \end{bmatrix} + \frac{\epsilon}{2}\begin{bmatrix} D_0 \\ D_1\mathbf{i} \\ D_2\mathbf{j} \\ D_3\mathbf{k} \end{bmatrix},\quad \mathbf{q} = \mathbf{q}_r + \mathbf{q}_d = \begin{bmatrix} c_0 \\ c_1\mathbf{i} \\ c_2\mathbf{j} \\ c_3\mathbf{k} \end{bmatrix} + \frac{\epsilon}{2}\begin{bmatrix} d_0 \\ d_1\mathbf{i} \\ d_2\mathbf{j} \\ d_3\mathbf{k} \end{bmatrix} \tag{7}$$

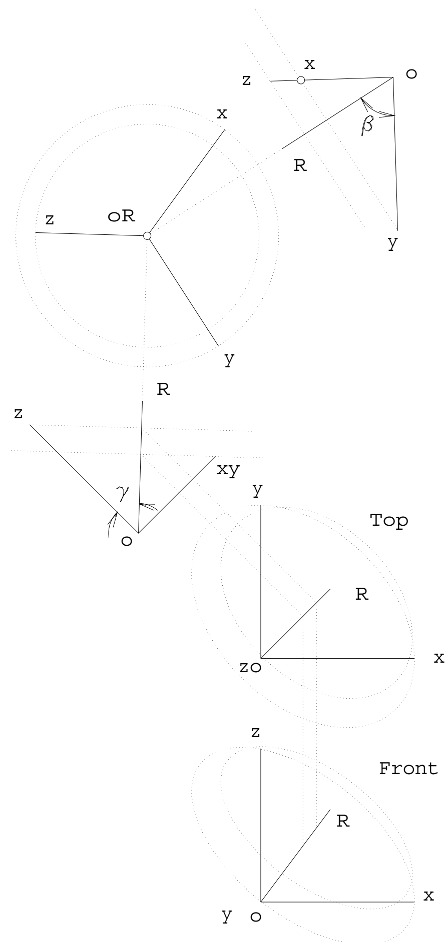

Figure 1: Rotation Axis oR with Direction Numbers $\{3:3:4\}$ Defining α, β, γ

where

$$\mathbf{q}_d = \frac{\epsilon}{2}\begin{bmatrix} d_0 \\ d_1\mathbf{i} \\ d_2\mathbf{j} \\ d_3\mathbf{k} \end{bmatrix} = \frac{\epsilon}{2}\begin{bmatrix} 0 \\ a_1\mathbf{i} \\ a_2\mathbf{j} \\ a_3\mathbf{k} \end{bmatrix}\begin{bmatrix} c_0 \\ c_1\mathbf{i} \\ c_2\mathbf{j} \\ c_3\mathbf{k} \end{bmatrix} = \frac{\epsilon}{2}\begin{bmatrix} -a_1c_1 - a_2c_2 - a_3c_3 \\ (a_1c_0 + a_2c_3 - a_3c_2)\mathbf{i} \\ (a_2c_0 + a_3c_1 - a_1c_3)\mathbf{j} \\ (a_3c_0 + a_1c_2 - a_2c_1)\mathbf{k} \end{bmatrix} \tag{8}$$

Pre-multiplying $\mathbf{q}$ by $\mathbf{Q}$ produces the product $\mathbf{Qq}$.

$$\mathbf{Qq} = \begin{bmatrix} C_0c_0 - C_1c_1 - C_2c_2 - C_3c_3 \\ (C_0c_1 + C_1c_0 + C_2c_3 - C_3c_2)\mathbf{i} \\ (C_0c_2 + C_2c_0 + C_3c_1 - C_1c_3)\mathbf{j} \\ (C_0c_3 + C_3c_0 + C_1c_2 - C_2c_1)\mathbf{k} \end{bmatrix}$$

$$+\frac{\epsilon}{2}\begin{bmatrix} C_0d_0 - C_1d_1 - C_2d_2 - C_3d_3 + D_0c_0 - D_1c_1 - D_2c_2 - D_3c_3 \\ (C_0d_1 + C_1d_0 + C_2d_3 - C_3d_2 + D_0c_1 + D_1c_0 + D_2c_3 - D_3c_2)\mathbf{i} \\ (C_0d_2 + C_2d_0 + C_3d_1 - C_1d_3 + D_0c_2 + D_2c_0 + D_3c_1 - D_1c_3)\mathbf{j} \\ (C_0d_3 + C_3d_0 + C_1d_2 - C_2d_1 + D_0c_3 + D_3c_0 + D_1c_2 - D_2c_1)\mathbf{k} \end{bmatrix} \tag{9}$$

3 General Point Transformation

Before trying to reconcile dual quaternion multiplication with octonian transformation let us examine the ordinary point transformation, $[\mathbf{q}]_4$, populated with elements of $\mathbf{q}$, in Eq. 7.

$$[\mathbf{q}]_4 = \begin{bmatrix} c_0^2+c_1^2+c_2^2+c_3^2 & 0 & 0 & 0 \\ c_0d_1-c_1d_0+c_2d_3-c_3d_2 & c_0^2+c_1^2-c_2^2-c_3^2 & 2(c_1c_2-c_0c_3) & 2(c_1c_3+c_0c_2) \\ c_0d_2-c_2d_0+c_3d_1-c_1d_3 & 2(c_2c_1+c_0c_3) & c_0^2-c_1+c_2^2-c_3^2 & 2(c_2c_3-c_0c_1) \\ c_0d_3-c_3d_0+c_1d_2-c_2d_1 & 2(c_3c_1-c_0c_2) & 2(c_3c_2+c_0c_1) & c_0^2-c_1^2-c_2^2+c_3^2 \end{bmatrix} \tag{10}$$

The general point transformation matrix Eq. 10 is derived in the same way as the planar displacement transformations described in [11]. Using the origin and the absolute points that close the *three* principal axes provides the necessary *four* convenient points.

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix} : \lambda \begin{bmatrix}1\\a_1\\a_2\\a_3\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix} : \begin{bmatrix}0\\c_0^2+c_1^2-c_2^2-c_3^2\\2(c_2c_3+c_0c_3)\\2(c_3c_1-c_0c_2)\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix} : \begin{bmatrix}0\\2(c_1c_2-c_0c_3)\\c_0^2-c_1^2+c_2^2-c_3^2\\2(c_3c_2+c_0c_1)\end{bmatrix} \begin{bmatrix}0\\0\\0\\1\end{bmatrix} : \begin{bmatrix}0\\2(c_1c_3+c_0c_2)\\2(c_2c_3-c_0c_1)\\c_0^2-c_1^2-c_2^2+c_3^2\end{bmatrix}$$

With these four known transformations, 16 equations in 17 unknowns can be set up and solved homogeneously. The unknowns are λ and the 16 matrix elements. It is not hard to show that Eqs. 10 and 11 are identical.

$$[\mathbf{q}] = \begin{bmatrix} c_0^2+c_1^2+c_2^2+c_3^2 & 0 & 0 & 0 \\ a_1(c_0^2+c_1^2+c_2^2+c_3^2) & c_0^2+c_1^2-c_2^2-c_3^2 & 2(c_1c_2-c_0c_3) & 2(c_1c_3+c_0c_2) \\ a_2(c_0^2+c_1^2+c_2^2+c_3^2) & 2(c_2c_1+c_0c_3) & c_0^2-c_1+c_2^2-c_3^2 & 2(c_2c_3-c_0c_1) \\ a_3(c_0^2+c_1^2+c_2^2+c_3^2) & 2(c_3c_1-c_0c_2) & 2(c_3c_2+c_0c_1) & c_0^2-c_1^2-c_2^2+c_3^2 \end{bmatrix} \tag{11}$$

Now the reason Eq. 10, containing all eight dual quaternion elements, is preferable to Eq. 11, using origin displacement vector $\mathbf{a}$, becomes evident. To transform *planes* one uses the adjoint $[\mathbf{q}]_4^A$ of Eq. 10. Dividing it by $(c_0^2+c_1^2+c_2^2+c_3^2)^2$ yields Eq. 12.

$$\begin{bmatrix} c_0^2+c_1^2+c_2^2+c_3^2 & d_0c_1-d_1c_0+c_2d_3-c_3d_2 & d_0c_2-d_2c_0+c_3d_1-c_1d_3 & d_0c_3-d_3c_0+c_1d_2-c_2d_1 \\ 0 & c_0^2+c_1^2-c_2^2-c_3^2 & 2(c_1c_2-c_0c_3) & 2(c_1c_3+c_0c_2) \\ 0 & 2(c_2c_1+c_0c_3) & c_0^2-c_1^2+c_2^2-c_3^2 & 2(c_2c_3-c_0c_1) \\ 0 & 2(c_3c_1-c_0c_2) & 2(c_3c_2+c_0c_1) & c_0^2-c_1^2-c_2^2+c_3^3 \end{bmatrix} \tag{12}$$

The transformed matrix retains the self-adjoint rotation core, matrix (4), the column $j=0$ is transposed to become the row $i=0$ and the elements $j \neq 0$ in this new row have had a sign reversal on their first two products, *i.e.*, $c_0d_i - c_id_0$ has become $d_0c_j - d_jc_0$. Of course, respective inverses are immediately available by transposing Eq. 10 to get an inverse plane transformation and transposing Eq. 12 to get points to go back to where they came from. The adjoint of Eq. 11, on the other hand, would sustain first row elements composed of ugly triple products, containing some a_i and elements of the 3×3 rotation array.

3.1 Octonian Transformation

Putting Eq. 10 and 12 together into the following single 8×8 array, matrix (13), provides the pre-multiplier to transform the following column vector.

$$[x_0 \;\; x_1 \;\; x_2 \;\; x_3 \;\; X_0 \;\; X_1 \;\; X_2 \;\; X_3]^T$$

It contains four homogeneous point coordinates followed by four homogeneous plane coordinates, all the elements of an entire dual quaternion that represents a displaced 3-space.

$$\left[\begin{matrix} c_0^2+c_1^2+c_2^2+c_3^2 & 0 & 0 & 0 \\ c_0d_1-c_1d_0+c_2d_3-c_3d_2 & c_0^2+c_1^2-c_2^2-c_3^2 & 2(c_1c_2-c_0c_3) & 2(c_1c_3+c_0c_2) \\ c_0d_2-c_2d_0+c_3d_1-c_1d_3 & 2(c_2c_1+c_0c_3) & c_0^2-c_1^2+c_2^2-c_3^2 & 2(c_2c_3-c_0c_1) \\ c_0d_3-c_3d_0+c_1d_2-c_2d_1 & 2(c_3c_1-c_0c_2) & 2(c_3c_2+c_0c_1) & c_0^2-c_1^2-c_2^2+c_3^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}\right.$$

$$\left.\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c_0^2+c_1^2+c_2^2+c_3^2 & d_0c_1-d_1c_0+c_2d_3-c_3d_2 & d_0c_2-d_2c_0+c_3d_1-c_1d_3 & d_0c_3-d_3c_0+c_1d_2-c_2d_1 \\ 0 & c_0^2+c_1^2-c_2^2-c_3^2 & 2(c_1c_2-c_0c_3) & 2(c_1c_3+c_0c_2) \\ 0 & 2(c_2c_1+c_0c_3) & c_0^2-c_1^2+c_2^2-c_3^2 & 2(c_2c_3-c_0c_1) \\ 0 & 2(c_3c_1-c_0c_2) & 2(c_3c_2+c_0c_1) & c_0^2-c_1^2-c_2^2+c_3^2 \end{matrix}\right] \quad (13)$$

3.2 Reduction to the Planar Case

A planar transformation is obtained by setting $c_1 = c_2 = d_0 = d_3 = 0$. It uses displaced origin coordinates (a_1, a_2), not pole position (a, b) as in [11]. There, for example, in Eq. 14, $c_0d_1 - c_3d_2$ becomes $2(X_0X_2 + X_1X_3)$ and $c_0d_2 + c_3d_1$ becomes $-2(X_0X_1 - X_2X_3)$.

$$\begin{bmatrix} c_0^2+c_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_0d_1-c_3d_2 & c_0^2-c_3^2 & -2c_0c_3 & 0 & 0 & 0 & 0 & 0 \\ c_0d_2+c_3d_1 & 2c_0c_3 & c_0^2-c_3^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_0^2+c_3^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_0^2+c_3^2 & -c_0d_1-c_3d_2 & -c_0d_2+c_3d_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_0^2-c_3^2 & -2c_0c_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c_0c_3 & c_0^2-c_3^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_0^2+c_3^2 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ 0 \\ X_0 \\ X_1 \\ X_2 \\ 0 \end{bmatrix} \quad (14)$$

4 Conclusion

The transformation for planes was obtained, after taking some liberty, by computing and simplifying the adjoint of the well-documented, [2,6,7], point transformation, Eq. 10. Note that in [6] $d = -d/2$ for, apparently, cosmetic reasons. Then an 8×8 array was assembled with zero blocks to separate operations that act exclusively upon point and planar parts, respectively, when multiplying an eight element vector. The adjoint of the entire array was carried out symbolically and large common factors were eliminated. The inverse transformation was then obtained by simple transposition. Detailed structure of these three versions is described explicitly in Fig. 3 and Fig. 4. Although considerable investigation remains to be done it is contended that these operators are useful not only to write constraint equations wherein points and planes are displaced but in the displacement of lines and in the extraction of so called reciprocal screws. *E.g.*, the auto-transformation of the line leads to itself with a reversed sense of moment; obviously the reciprocal if one considers the line to be a concentrated force with component magnitudes equal to its direction numbers.

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5 APPENDIX

5.1 The Adjoint Matrix Is the Dual of Its Transformation

Three linearly independent points A, B, C form vertices of a triangle with respectively opposite sides on lines a, b, c. These are specified by planar homogeneous coordinates, thus.

$$A\{a_0 : a_1 : a_2\},\ \ B\{b_0 : b_1 : b_2\},\ \ C\{c_0 : c_1 : c_2\},\ \ a\{A_0 : A_1 : A_2\},\ \ b\{B_0 : B_1 : B_2\},\ \ c\{C_0 : C_1 : C_2\}$$

This is shown in Fig. 2. Using a singular, dummy point $X\{x_0 : x_1 : x_2\}$ successively on a, b, c, the following Grassmannian top row determinant minor expansions produce the following homogeneous planar line coordinates.

$$a : \begin{vmatrix} x_0 & x_1 & x_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{vmatrix} \Rightarrow \{b_1c_2 - b_2c_1 : b_2c_0 - b_0c_2 : b_0c_1 - b_1c_0\} \equiv \{A_0 : A_1 : A_2\}$$

$$b : \begin{vmatrix} x_0 & x_1 & x_2 \\ c_0 & c_1 & c_2 \\ a_0 & a_1 & a_2 \end{vmatrix} \Rightarrow \{c_1a_2 - c_2a_1 : c_2a_0 - c_0a_2 : c_0a_1 - c_1a_0\} \equiv \{B_0 : B_1 : B_2\}$$

$$c : \begin{vmatrix} x_0 & x_1 & x_2 \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{vmatrix} \Rightarrow \{a_1b_2 - a_2b_1 : a_2b_0 - a_0b_2 : a_0b_1 - a_1b_0\} \equiv \{C_0 : C_1 : C_2\}$$

Now examine the nonsingular matrices containing rows of these point and line coordinates. They are duals of the same figure; a given triangle. The superscripts A and D stand for adjoint and dual, respectively. The following sequence states this equivalence.

$$\begin{bmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{bmatrix}^A \equiv \begin{bmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{bmatrix}^D \equiv \begin{bmatrix} A_0 & A_1 & A_2 \\ B_0 & B_1 & B_2 \\ C_0 & C_1 & C_2 \end{bmatrix} \equiv \begin{bmatrix} b_1c_2 - b_2c_1 & b_2c_0 - b_0c_2 & b_0c_1 - b_1c_0 \\ c_1a_2 - c_2a_1 & c_2a_0 - c_0a_2 & c_0a_1 - c_1a_0 \\ a_1b_2 - a_2b_1 & a_2b_0 - a_0b_2 & a_0b_1 - a_1b_0 \end{bmatrix}$$

Clearly, the last matrix in the sequence is the adjoint in terms of the two on the left populated by homogeneous point coordinates a_j, b_j, c_j where $j = 0, 1, 2$, is the column index. A word of caution. An entirely different triangle is produced if one plots

$$\left(\frac{A_1}{A_0}, \frac{A_2}{A_0}\right),\ \ \left(\frac{B_1}{B_0}, \frac{B_2}{B_0}\right),\ \ \left(\frac{C_1}{C_0}, \frac{C_2}{C_0}\right)\ \ \text{as point coordinates in the plane.}$$

5.2 Adjoint Transformations

Below one sees some special auto-transformative cases. The point, plane and line undergo premultiplication by their respective reciprocal dual quaternion, *i.e.*, second matrices shown in Figs. 3 and 4. For the line $c_0 = d_0 = 0$, $c_j = p_{0j}$ and $d_i = p_{i+1,i+2}$, i taken modulo 3.

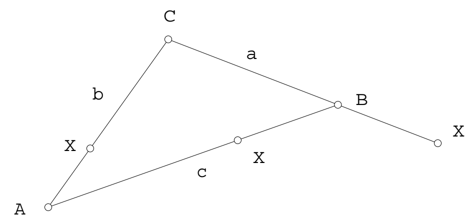

Figure 2: Planar Simplex with Points and Lines

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_0^2 - x_1^2 - x_2^2 - x_3^2 \\ x_0 x_1 \\ x_0 x_2 \\ x_0 x_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ (X_1^2 + X_2^2 + X_3^2) X_0 \\ (X_1^2 + X_2^2 + X_3^2 - X_0^2) X_1 \\ (X_1^2 + X_2^2 + X_3^2 - X_0^2) X_2 \\ (X_1^2 + X_2^2 + X_3^2 - X_0^2) X_3 \end{bmatrix}, \begin{bmatrix} 0 \\ p_{01} \\ p_{02} \\ p_{03} \\ 0 \\ p_{23} \\ p_{31} \\ p_{12} \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ p_{01} \\ p_{02} \\ p_{03} \\ 0 \\ -p_{23} \\ -p_{31} \\ -p_{12} \end{bmatrix}$$

To obtain the line transformation in its simplified form above one must substitute from the Plücker condition. *E.g.*, in the transformation of p_{23}, $-(p_{02}p_{31} + p_{03}p_{12})$ replaces $p_{01}p_{23}$.

5.3 Matrix Structure

In Fig. 3 "R"s refer to rotation matrix elements. These appear upside-down where that matrix has been inverted (transposed). "T"s are the translation elements containing a sum of two differences of products. These are written sideways where the first difference of products was negated (had its sign reversed). All this is shown explicitly in Fig. 4.

5.4 Notes

5.4.1 8-Vector and Dual Quaternion

The five 8-element vectors in expression (1) on p.2 are set up so that they may be operated upon by pre-multiplication with the general 8×8 transformations represented in Figs. 3 and 4 and that embed all elements of a dual quaternion displacement operator. The first contains the homogeneous coordinates of a *point*. Blaschke [1] represents it as the Euclidean point

$$P \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \epsilon \begin{bmatrix} 0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

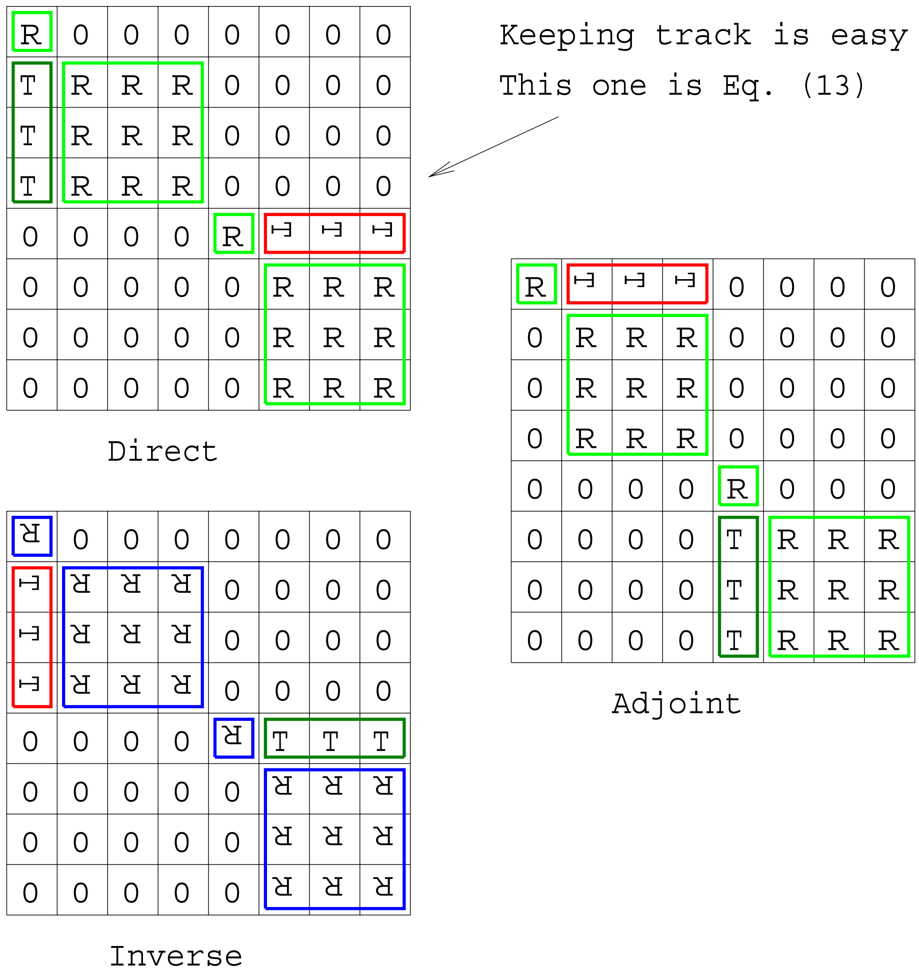

Figure 3: Block Structure of the Three Forms

Similarly, he represents the plane and line in homogeneous, radial Plücker in the case of the line, coordinates. Note that the line in (1) on p.2 is in axial form.

$$\Pi \equiv \begin{bmatrix} 0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} + \epsilon \begin{bmatrix} X_0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{L} \equiv \begin{bmatrix} 0 \\ p_{01} \\ p_{02} \\ p_{03} \end{bmatrix} + \epsilon \begin{bmatrix} 0 \\ p_{23} \\ p_{31} \\ p_{12} \end{bmatrix}$$

The fourth vector contains all eight non-zero elements to represent the pose of some entire rigid body displaced from "home" position by some screw, admittedly a six-parameter entity. However this is a homogeneous 8-dimensional vector representation of a 7-dimensional projective space in which only those points on the quadric

$$x_0X_0 + x_1X_1 + x_2X_2 + x_3X_3 = 0$$

represent valid screw displacements. This reduces the 8-element dual quaternion to the necessary six degrees of freedom. The fifth vector shows how the c_i and d_i in the right hand expression in

Eq. 7 populate the 8-vector. Recall that the dual quaternion represents not only an operator -like our 8×8 matrices- but the pose of a displaced frame containing some rigid body.

5.4.2 Miscellany

- Indeed Blaschke [1] covers the principle of dual quaternion operators quite thoroughly but the intention of this article is to make this accessible to English speakers and those who prefer the familiarity of matrix-vector multiplications to the rules of manipulation implied by dual quaternion algebra.

- Base vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are (redundantly) included in the quaternion 4-vector, Eq. 2, to emphasize orthogonality among the last three elements often referred to as the *vector part*. Furthermore the inner part of a quaternion product that includes $\mathbf{ii} = -1 \neq 1$ implies that these base vectors must be of *imaginary* -not just unit- magnitude.

- Since a quaternion represents a homogeneous 4-vector, only the ratio among element magnitudes is preserved. Though $c_0^2 + c_1^2 + c_2^2 + c^2 = 1$ represents a *normed* or unit quaternion this condition is unnecessary in treating the projective space of pure rotations.

- When rotation angle is represented by a vector, radiating from a chosen point origin, one may choose its magnitude to be proportional to the angle of rotation it represents. Furthermore its direction gives the sense of rotation as that of a right-handed screw advancing in the direction of the vector although the screw of rotation has no lead; does not advance. It is senseless to ask about the hand of screw rotation of a car wheel without choosing an origin somewhere on the axis.

- Of course quaternion multiplication is non-Abelian. This can be quickly demonstrated by computing the difference between two quaternion binary products, one in the reverse order to the other, thus. Notice that the offensive base vectors have been removed.

$$\mathbf{q} - \mathbf{q}' = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} \begin{bmatrix} c_0' \\ c_1' \\ c_2' \\ c_3' \end{bmatrix} - \begin{bmatrix} c_0' \\ c_1' \\ c_2' \\ c_3' \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2(c_2c_3' - c_3c_2') \\ 2(c_3c_1' - c_1c_3') \\ 2(c_1c_2' - c_2c_1') \end{bmatrix}$$

- The nature of the dual unit ϵ, where $\epsilon^2 = 0$, has received a number of simplistic "explanations", not repeated here, however a rigorous treatment was done by Study [10].

$$
\left[\begin{array}{c|c|c|c|c}
c_0^2+c_1^2+c_2^2+c_3^2 & 0 & 0 & 0 & 0\;\;0\;\;0\;\;0 \\
c_0d_1-c_1d_0+c_2d_3-c_3d_2 & c_0^2+c_1^2-c_2^2-c_3^2 & 2(c_1c_2-c_0c_3) & 2(c_1c_3+c_0c_2) & 0\;\;0\;\;0\;\;0 \\
c_0d_2-c_2d_0+c_3d_1-c_1d_3 & 2(c_2c_1+c_0c_3) & c_0^2-c_1^2+c_2^2-c_3^2 & 2(c_2c_3-c_0c_1) & 0\;\;0\;\;0\;\;0 \\
c_0d_3-c_3d_0+c_1d_2-c_2d_1 & 2(c_3c_1-c_0c_2) & 2(c_3c_2+c_0c_1) & c_0^2-c_1^2-c_2^2+c_3^2 & 0\;\;0\;\;0\;\;0 \\
0\;\;0\;\;0\;\;0 & c_0^2+c_1^2+c_2^2+c_3^2 & c_1d_0-c_0d_1+c_2d_3-c_3d_2 & c_2d_0-c_0d_2+c_3d_1-c_1d_3 & c_3d_0-c_0d_3+c_1d_2-c_2d_1 \\
0\;\;0\;\;0\;\;0 & 0 & c_0^2+c_1^2-c_2^2-c_3^2 & 2(c_1c_2-c_0c_3) & 2(c_1c_3+c_0c_2) \\
0\;\;0\;\;0\;\;0 & 0 & 2(c_2c_1+c_0c_3) & c_0^2-c_1^2+c_2^2-c_3^2 & 2(c_2c_3-c_0c_1) \\
0\;\;0\;\;0\;\;0 & 0 & 2(c_3c_1-c_0c_2) & 2(c_3c_2+c_0c_1) & c_0^2-c_1^2-c_2^2+c_3^2
\end{array}\right]
*
\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix}
=
\begin{bmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \\ X'_0 \\ X'_1 \\ X'_2 \\ X'_3 \end{bmatrix}
$$

Dual Quaternion Transformation

$$
\left[\begin{array}{c|c|c|c|c}
c_0^2+c_1^2+c_2^2+c_3^2 & c_1d_0-c_0d_1+c_2d_3-c_3d_2 & c_2d_0-c_0d_2+c_3d_1-c_1d_3 & c_3d_0-c_0d_3+c_1d_2-c_2d_1 & 0\;\;0\;\;0\;\;0 \\
0 & c_0^2+c_1^2-c_2^2-c_3^2 & 2(c_1c_2-c_0c_3) & 2(c_1c_3+c_0c_2) & 0\;\;0\;\;0\;\;0 \\
0 & 2(c_2c_1+c_0c_3) & c_0^2-c_1^2+c_2^2-c_3^2 & 2(c_2c_3-c_0c_1) & 0\;\;0\;\;0\;\;0 \\
0 & 2(c_3c_1-c_0c_2) & 2(c_3c_2+c_0c_1) & c_0^2-c_1^2-c_2^2+c_3^2 & 0\;\;0\;\;0\;\;0 \\
0\;\;0\;\;0\;\;0 & c_0^2+c_1^2+c_2^2+c_3^2 & 0 & 0 & 0 \\
0\;\;0\;\;0\;\;0 & c_0d_1-c_1d_0+c_2d_3-c_3d_2 & c_0^2+c_1^2-c_2^2-c_3^2 & 2(c_1c_2-c_0c_3) & 2(c_1c_3+c_0c_2) \\
0\;\;0\;\;0\;\;0 & c_0d_2-c_2d_0+c_3d_1-c_1d_3 & 2(c_2c_1+c_0c_3) & c_0^2-c_1^2+c_2^2-c_3^2 & 2(c_2c_3-c_0c_1) \\
0\;\;0\;\;0\;\;0 & c_0d_3-c_3d_0+c_1d_2-c_2d_1 & 2(c_3c_1-c_0c_2) & 2(c_3c_2+c_0c_1) & c_0^2-c_1^2-c_2^2+c_3^2
\end{array}\right]
*
\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix}
=
\begin{bmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \\ X'_0 \\ X'_1 \\ X'_2 \\ X'_3 \end{bmatrix}
$$

Adjoint Transformation

$$
\left[\begin{array}{c|c|c|c|c}
c_0^2+c_1^2+c_2^2+c_3^2 & 0 & 0 & 0 & 0\;\;0\;\;0\;\;0 \\
c_1d_0-c_0d_1+c_2d_3-c_3d_2 & c_0^2+c_1^2-c_2^2-c_3^2 & 2(c_1c_2+c_0c_3) & 2(c_1c_3-c_0c_2) & 0\;\;0\;\;0\;\;0 \\
c_2d_0-c_0d_2+c_3d_1-c_1d_3 & 2(c_2c_1-c_0c_3) & c_0^2-c_1^2+c_2^2-c_3^2 & 2(c_2c_3+c_0c_1) & 0\;\;0\;\;0\;\;0 \\
c_3d_0-c_0d_3+c_1d_2-c_2d_1 & 2(c_3c_1+c_0c_2) & 2(c_3c_2-c_0c_1) & c_0^2-c_1^2-c_2^2+c_3^2 & 0\;\;0\;\;0\;\;0 \\
0\;\;0\;\;0\;\;0 & c_0^2+c_1^2+c_2^2+c_3^2 & c_0d_1-c_1d_0+c_2d_3-c_3d_2 & c_0d_2-c_2d_0+c_3d_1-c_1d_3 & c_0d_3-c_3d_0+c_1d_2-c_2d_1 \\
0\;\;0\;\;0\;\;0 & 0 & c_0^2+c_1^2-c_2^2-c_3^2 & 2(c_1c_2+c_0c_3) & 2(c_1c_3-c_0c_2) \\
0\;\;0\;\;0\;\;0 & 0 & 2(c_2c_1-c_0c_3) & c_0^2-c_1^2+c_2^2-c_3^2 & 2(c_2c_3+c_0c_1) \\
0\;\;0\;\;0\;\;0 & 0 & 2(c_3c_1+c_0c_2) & 2(c_3c_2-c_0c_1) & c_0^2-c_1^2-c_2^2+c_3^2
\end{array}\right]
*
\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix}
=
\begin{bmatrix} x''_0 \\ x''_1 \\ x''_2 \\ x''_3 \\ X''_0 \\ X''_1 \\ X''_2 \\ X''_3 \end{bmatrix}
$$

Inverse Transformation

Figure 4: Three Forms of Dual Quaternion Vector Transformation