Questions

1. Let $B(x)$ be the local average function from the lecture. Recall that

$$\mathbf{F}\ B(x) = \frac{1}{2}(1 + \cos(\frac{2\pi}{N}k)).$$

   (a) What is the Fourier transform of $B(x) * B(x)$?

   (b) Note that $\hat{B}(\frac{N}{2}) = 0$. Why? Give an intuitive explanation.

2. (a) What is the Fourier transform of the shifted impulse function ?

$$s(x) = \delta(x - x_0) = \begin{cases} 1, & x = x_0 \\ 0, & \text{otherwise} \end{cases}$$

   (b) What is the Fourier transform of $I(x) * s(x)$? That is, what is the effect in the frequency domain of convolving with a shifted impulse?

3. What is the Fourier transform of

$$f(x) \;=\; \delta(x-1) + \delta(x) + \delta(x+1) \quad ?$$

   This is similar to the function $B(x)$, but the weights on the neighbors are now different. How would you characterize this function? Lowpass? Bandpass? High pass?

4. Consider the following filter:

$$f(x) = \begin{cases} -2, & x = 0 \\ 1, & x = 1, -1 \\ 0, & \text{otherwise} \end{cases}$$

   (a) What is the amplitude spectrum of $f(x)$?

   (b) What is the phase spectrum of $f(x)$?

   (c) How would you characterize this function? Lowpass? Bandpass? High pass?

5. Use the convolution theorem to show that

$$G(x,\sigma) * G(x,\sigma) * \cdots * G(x,\sigma) \;=\; G(x, \sqrt{m}\ \sigma)$$

   where m is the number of $G(\cdot)$'s on the left side.

6. What is the Fourier transform of $sinGabor(x, k_0, \sigma)$?

7. The bandwidth for a non-ideal filter is defined by the frequencies where the filter has half of its maximum height (see lecture notes).

   (a) What is the bandwidth of $sinGabor(x, k_0, \sigma)$?

   (b) What is the octave bandwidth of $sinGabor(x, k_0, \sigma)$?

   (c) Given a Gabor with center frequeuncy k_0 and a desired octave bandwidth β, how can one choose the standard deviation σ of the Gaussian to achieve that bandwidth?

   Note: The formulas that I am asking for here admittedly do not provide much intuition, so don't spend a lot of time on this question. The main reason I am asking the question is to be sure that you understand the definitions. How *would* you answer the question if required to?

8. Let
$$I(x) = I_0 + a\cos(\frac{2\pi}{N}k_0 x)$$
   for a particular frequency k_0.

   What is $I(x) * G(x, \sigma)$?

Solutions

1. (a) Use the convolution theorem.

$$\mathbf{F}\ B * B(x) = \hat{B}(k)^2 = (\frac{1}{2}(1 + \cos(\frac{2\pi}{N}k)))^2$$

(b) Consider a cosine or sine function with frequency of $k = \frac{N}{2}$. Such a function is of the form $(\ldots, -1, 1, -1, 1, -1, \ldots)$, i.e. $k = \frac{N}{2}$ corresponds to a wavelength (or "period") of two pixels. By inspection, convolving such a function with $B(x)$ produces 0 everywhere.

2. (a)

$$\hat{s}(k) = e^{-i\ \frac{2\pi}{N}kx_0}.$$

(b) From the convolution theorem,

$$\mathbf{F}\ (I(x) * s(x)) \ = \ \hat{I}(k)\hat{s}(k) = \hat{I}(k)e^{-i\ \frac{2\pi}{N}kx_0}.$$

Thus, each sine and cosine component of $I(x)$ undergoes a *phase shift* of $\frac{2\pi}{N}kx_0$ radians. Note that phase shifts are modulo 2π radians.

3.

$$\begin{aligned}\hat{f}(k) &= \sum_{x=0}^{N} f(x)e^{-i\frac{2\pi}{N}kx} \\ &= e^{-i\frac{2\pi}{N}k(N-1)} + e^{-i\frac{2\pi}{N}k0} + e^{-i\frac{2\pi}{N}k1} \\ &= e^{i\frac{2\pi}{N}k} + 1 + e^{-i\frac{2\pi}{N}k} \\ &= 1 + 2\cos(\tfrac{2\pi}{N}k)\end{aligned}$$

You might have been tempted to say it was lowpass since it seems to be blurring. However, it turns out to be more complicated. $\hat{f}(k)$ takes its largest value (3) at $k = 0$, and it crosses 0 and then falls to -1 at $k = \frac{N}{2}$. So, the amplitude spectrum $|\hat{f}(k)|$ falls from 3 to 0 and then rises to 1. Thus, this $f(x)$ is not low-pass, bandpass or highpass.

4. (a)

$$\begin{aligned}\hat{f}(k) &= \sum_{x=0}^{N-1} f(x)e^{-i\frac{2\pi}{N}kx} \\ &= e^{-i\frac{2\pi}{N}k(N-1)} - 2e^{-i\frac{2\pi}{N}k0} + e^{-i\frac{2\pi}{N}k\cdot 1} \\ &= 2\cos(\frac{2\pi}{N}k) - 2\end{aligned}$$

which is always negative. So, taking the magnitude gives

$$\mid \hat{f}(k) \mid \ = 2 - 2\cos(\frac{2\pi}{N}k)$$

which is 0 when $k = 0$ and then grows with k up to $k = \frac{N}{2}$.

(b) Because $\hat{f}(k)$ is a negative real number for all k, the phase spectrum of $f(x)$ is $\phi(k) = \pi$ for all k.

If you need some intuition, then how about this? Convolving with $f(x)$ computes a discrete approximation to a continuous second derivative. For a true continuous second derivative,

$$\frac{d^2}{dx^2}\sin(kx) = -k^2\sin(kx), \qquad \frac{d^2}{dx^2}\cos(kx) = -k^2\cos(kx).$$

So the true second derivative has the sign flip (phase equals π) and its amplitude is 0 when $k = 0$ and in general increases with k^2. The difference in the rate of increase ($2 - 2\cos(\frac{2\pi}{N}k)$ versus k^2) is due to the discretization.

(c) It is non-ideal high pass. It has value 0 when $k = 0$, and rises to its maximum value at $k = \frac{N}{2}$.

5.

$$\begin{aligned} \mathbf{F}\; G(x,\sigma) * G(x,\sigma) * \cdots * G(x,\sigma) &= \{(\ \mathbf{F}G(x,\sigma)\ \}^m \\ &= \{e^{-\frac{1}{2}(\frac{2\pi\sigma}{N}k)^2}\}^m \\ &= e^{-\frac{1}{2}(\frac{2\pi\sqrt{m}\sigma}{N}k)^2} \end{aligned}$$

6. Follow the derivation of the Fourier transform of a cos Gabor given in the lecture.

$$\begin{aligned} \mathbf{F}\; sinGabor(x,k_0,\sigma) &= \mathbf{F}\;\{\ G(x,\sigma)\sin(\frac{2\pi}{N}k_0x)\ \} \\ &= \frac{1}{N}\ e^{-\frac{1}{2}(\frac{2\pi\sigma\ k}{N})^2} * \frac{N}{2i}(\delta(k_0-k)-\delta(k_0+k)) \\ &= \frac{1}{2i}\ \{\ e^{-\frac{1}{2}(\frac{2\pi\sigma\ (k-k_0)}{N})^2} - e^{-\frac{1}{2}(\frac{2\pi\sigma\ (k+k_0)}{N})^2}\ \} \end{aligned}$$

7. From the previous question, we have that the amplitude spectrum of a sine Gabor is:

$$|\ \mathbf{F}\; sinGabor(x,k_0,\sigma)\ | = \frac{1}{2}|\ e^{-\frac{1}{2}(\frac{2\pi\sigma\ (k-k_0)}{N})^2} - e^{-\frac{1}{2}(\frac{2\pi\sigma\ (k+k_0)}{N})^2}\ |$$

The two Gaussians overlap to some extent, but for simplicity one usually ignores this overlap and defines the 'half height' by just considering one of the Gaussians. Since the exponential function $e^{-\frac{1}{2}(\frac{2\pi\sigma\ (k-k_0)}{N})^2}$ has its maximum value of 1 at $k = k_0$, we want to know the k value(s) such that:

$$\frac{1}{2} = e^{-\frac{1}{2}(\frac{2\pi\sigma\ (k-k_0)}{N})^2}$$

Solving for k:

$$k = k_0 \pm \frac{N\sqrt{2\ln(2)}}{2\pi\sigma}.$$

This half-height occurs at two values of k, one greater than k_0 and one less than k_0.

Note that the σ here is from the Gaussian in the spatial domain, so we should think of it as σ_x. Since

$$\sigma_k = \frac{N}{2\pi\sigma_x}$$

(see lecture notes) we have

$$k = k_0 \pm \sigma_k\sqrt{2\ln(2)}.$$

(a) The bandwidth is the difference between these two k values, namely $2\sqrt{2\ln(2)}\ \sigma_k$

(b) The octave bandwidth β is $\log_2$ of the ratio of these two k values:

$$\beta = \log_2\left(\frac{k_0 + \sigma_k\sqrt{2\ln(2)}}{k_0 - \sigma_k\sqrt{2\ln(2)}}\right)$$

(c) Rewriting the above gives

$$\sigma_k = \frac{k_0}{\sqrt{2\ln(2)}}\left(\frac{2^\beta - 1}{2^\beta + 1}\right)$$

which can be converted to σ_x using the formula above.

8. Before turning the crank on the math, make sure you get the intuition here. You are blurring a function that is the sum of a constant and a cosine. Blurring the function blurs each of the components of the sum. Blurring a constant does nothing. Blurring a cosine will give a cosine of the same frequency and possibly different phase. That's what we need to calculate.

   We would expect the amplitude of each frequency to decrease since blurring takes a local average. From the convolution theorem, the amplitude reduction is given by the value of the Fourier transform of the blur function (Gaussian) at the frequency of this cosine.

   Now turn the crank. We saw the Fourier transforms of a cosine and a Gaussian in the lecture. So we apply the convolution theorem.

$$\begin{aligned}
\mathbf{F}\{I(x) * G(x,\sigma)\} &= \mathbf{F}I(x)\ \mathbf{F}G(x,\sigma) \\
&= \{I_0 N\delta(k) + \frac{N}{2}\{\delta(k+k_0) + \delta(k-k_0)\}\}\ \ e^{-\frac{1}{2}(\frac{2\pi}{N}\sigma k)^2} \\
&= I_0 N\delta(k) +\ e^{-\frac{1}{2}(\frac{2\pi}{N}\sigma k_0)^2}\ \frac{N}{2}\{\delta(k+k_0) - \delta(k-k_0)\}\}
\end{aligned}$$

   since $e^{-\frac{1}{2}(\frac{2\pi}{N}\sigma k)^2} = 1$ when $k = 0$ and $e^{-\frac{1}{2}(\frac{2\pi}{N}\sigma k)^2} = e^{-\frac{1}{2}(\frac{2\pi}{N}\sigma k_0)^2}$ when $k = \pm k_0$.

   Noticing that $e^{-\frac{1}{2}(\frac{2\pi}{N}\sigma k_0)^2}$ is a constant, we can take the inverse Fourier transform to get

$$I(x) * G(x,\sigma) = I_0 +\ e^{-\frac{1}{2}(\frac{2\pi}{N}\sigma k_0)^2}\cos(\frac{2\pi}{N}k_0 x).$$

   The only change is that the cosine part of the original function is reduced by a factor $e^{-\frac{1}{2}(\frac{2\pi}{N}\sigma k_0)^2}$ which was the value of the Fourier transform of the Gaussian at the particular frequency k_0 of the cosine.